Behavioral Equilibrium in Economies with Adverse Selection*

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Abstract

I propose a new solution concept, behavioral equilibrium, to study environments with players who are naive in the sense that they fail to account for the informational content of other players’ actions. A behavioral equilibrium requires that: (i) players have no incentives to deviate given their beliefs about the consequences of deviating, (ii) these beliefs are consistent with the information obtained from the actual equilibrium play of all players, and (iii) when processing this information, naive players fail to account for the correlation between other players’ actions and their own payoff uncertainty. I apply the concept to a class of games with monotone selection, which includes several standard adverse selection settings. Contrary to what may be expected without an appropriate equilibrium framework, the presence of naive players (who fail to account for the selection problem) actually exacerbates the adverse selection problem. For example, in a bilateral trading game with one-sided asymmetric information à la Akerlof, the quantity and quality of objects traded in equilibrium are lower when traders are naive. More generally, the proposed equilibrium approach may be applied to study other behavioral biases.

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1 Introduction

There is a large body of evidence documenting psychological and cognitive biases that lead people to make mistakes in the face of uncertainty.\(^1\) A question that has received less attention is how outcomes are affected in economic settings where people who suffer from these biases interact with each other. Of particular interest is whether such mistakes can persist in an equilibrium setting, where people learn from previous experience. In this paper I introduce a game-theoretic equilibrium framework to study these questions in the context of a particular bias: people’s failure to take into account the informational content of other people’s actions.

A prominent example where this bias might arise is a common value auction. Here, a bidder makes a mistake if she ignores the fact that other bidders choose their bids based on private information that should affect her own assessment of the value of the object. Bidders who commit this mistake fail to realize that their bids affect not just the probability of winning but also the expected value of the object conditional on winning. This bias is often offered as an explanation of the winner’s curse phenomenon, where the winner of an auction realizes after winning that she has bid too aggressively and will make negative profits. However, to the extent that bidders obtain feedback about the value of objects that they win, a reasonable hypothesis is that experienced bidders eventually learn to play strategies that do not result in negative expected profits.\(^2\)

While the hypothesis that players do not consistently make negative profits may seem natural, it does not necessarily imply that players learn to correct their biases. Intuitively, players’ behavior determines the sample of observed outcomes from which players learn; in the auction example, the sample may be the set of objects that are actually won, and not the set of objects that could be potentially won. While players may adjust their behavior in the face of negative profits, they may still not realize that the feedback obtained about the value of the object may have been different had they chosen to behave differently.

The failure to account for the informational content of other peoples’ actions leads people to ignore a potential selection effect. This problem may arise in several well-known settings with adverse selection, where the terms of the contract often select the type of people with whom trade will be conducted. For example, in a typical lemons market (Akerlof, 1970), the average quality of cars available for sale depends on the price that is offered. More generally, this selection effect may be present whenever there is a common value component in payoffs and players act based on privately held information.

\(^1\)Some of these biases include the tendency to consider very small samples as largely representative of a population, perseverance in beliefs despite contradictory information, overconfidence, and the failure to perceive correlations. See Rabin (1998) and Gilovich, Griffin, and Kahneman (2002) for a review of these and other biases.

\(^2\)This hypothesis finds support both in experimental settings (Kagel and Roth, 1995) and in the field (Hendricks and Porter, 1988; Hendricks, Pinkse, and Porter, 2003). More generally, the hypothesis that people have correct beliefs about their expected payoff from the actions they choose to follow underlies most equilibrium thinking in economics. There are of course many interesting settings where this hypothesis may not be reasonable, such as when players are inexperienced, don’t receive feedback, or do receive feedback but somehow fail to learn from it.
While most documented evidence for the bias that I study in this paper comes from experiments in auction-like environments, it has been claimed that this bias may be present in other settings as well. The failure of an individual to account for the correlation between the actions of other players and payoff-relevant uncertainty may arise from cognitive limitations and from the complexity of the environment. In addition, this bias may also arise under certain organizational structures. For example, a firm may have one division, say the research department, that produces estimates about uncertain demand conditions using past data, and a different division, say the pricing division, that keeps track of its competitors’ prices. If competitors choose prices based on their own estimates of demand conditions, then these two pieces of information are likely to be correlated, but if these divisions do not communicate with each other this correlation is likely to be ignored. Finally, given that selection problems often pose a challenge for empirical researchers, it seems plausible that economic agents may not always account for them.

In this paper, I propose a steady-state solution concept, behavioral equilibrium, to study settings where some players suffer from this bias. I distinguish between two types of players, each having a different and exogenously given model of the world: those who are not aware of the potential selection effect (naive players), and those who are aware (sophisticated players). Within their constrained model of the world, both types of players: (1) use available data to form beliefs about the consequences of their actions; and (2) choose actions that maximize utility subject to these beliefs.

A behavioral equilibrium is based on the idea of a self-confirming equilibrium (Dekel, Fudenberg, and Levine, 2004), which is a static, steady-state solution concept requiring, in equilibrium, that players have no incentives to deviate given their beliefs about the consequences of deviating, and that these beliefs be consistent with the experience that results from equilibrium behavior. To illustrate the meaning of consistency, suppose that players obtain feedback about their own payoffs. Consistency then requires them to have correct beliefs about their expected payoffs from following the equilibrium action, but not necessarily to have correct beliefs about the payoff from deviating to another strategy. This is less restrictive than Nash equilibrium, which requires players to have correct beliefs about the consequences of deviating to any strategy. In a behavioral equilibrium, sophisticated players behave as in a self-confirming equilibrium. One of the contributions of this paper is to extend the definition of consistency to naive players.

In a behavioral equilibrium, the beliefs of naive players are restricted to be naive-consistent: while

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3See Kagel and Levin (2002) for a review of the experimental literature. Some tentative evidence also appears in field settings including the oil industry (Capen, Clapp, and Campbell, 1971), professional baseball’s free agency market (Cassing and Douglas, 1980), and corporate takeovers (Roll, 1986).

4Aragones, Gilboa, Postlewaite, and Schmeidler (2005) provide a formal justification for why people may fail to see regularities in the data (in the current paper, correlations) due to computational complexity. For other approaches to modeling unawareness see the review by Dekel, Lipman, and Rustichini (1998).

5Previous versions of the notion of a self-confirming (or conjectural) equilibrium in games of complete information appear in Hahn (1978), Battigalli (1987), and Fudenberg and Levine (1993a). A self-confirming equilibrium is often interpreted as the outcome of a learning process, in which players revise their beliefs using observations of previous play. Explicit learning-theoretic foundations have been provided by Fudenberg and Levine (1993b).
it is still the case that the information obtained from actual equilibrium behavior constrains their beliefs, when processing this information naive players ignore the potential correlation between other players’ actions and their own payoff-relevant uncertainty. Whether this bias (or incorrect model of the world) can persist in equilibrium depends on assumptions about the feedback that players obtain and about what players know a priori about the game.

I demonstrate the usefulness of the framework by applying it to a class of games with monotone selection that includes games that satisfy two properties: (i) a monotone selection property (MSP), which requires “lower” actions to result in a “worse” selection of outcomes, and (ii) complementarity between beliefs and actions, which in turn requires beliefs about a “worse” selection of outcomes to encourage “lower” actions.6 These two properties are present in many standard settings with adverse selection. For example, in a lemons market a lower price selects a worse quality of cars, which itself provides buyers with incentives to offer an even lower price. It is precisely this argument that is often provided as an intuition for why markets are thinner and gains from trade are lower in the presence of informational asymmetries between buyers and sellers. Given this intuition, it may be expected that the presence of players who fail to account for this selection effect mitigates this adverse selection problem. One purpose of this paper is to show that this conclusion is incorrect in an equilibrium framework.

Within the class of games with monotone selection, I compare equilibrium outcomes when all players are either naive, sophisticated, or have correct beliefs, as in a Nash equilibrium. I use Nash equilibrium as a benchmark since in certain situations it may be reasonable for a sophisticated player with incentives to experiment to end up having correct beliefs about the consequences of all possible strategies.

Under reasonable assumptions on information feedback and on players’ a priori knowledge of the game, I find that naive-consistent beliefs can be supported in equilibrium and that the presence of players who are unaware of the (adverse) selection problem actually exacerbates this problem. For example, in a bilateral trading game with one-sided asymmetric information à la Akerlof (1970), the quantity and quality of objects traded in equilibrium are lower with a naive buyer than with a sophisticated buyer. Similarly, in a symmetric equilibrium of a symmetric first-price common value auction, bidding is lower when all players are naive than when all players are sophisticated.

To understand this result, consider first the difference between a naive player and a player who has correct beliefs (as in a Nash equilibrium) facing the same profile of opponent strategies. Suppose that a naive player receives feedback about her own payoffs. Then naive-consistency requires her to have correct beliefs about the payoff from playing her (perceived) naive-optimal action. Furthermore, a naive player does not realize that decreasing her action would result (because of the monotone selection assumption) in a worse selection of outcomes. Hence, a naive player believes that choosing an action lower than her naive-optimal action would yield more profits than what

6The results extend to the case where “higher” actions select “worse” outcomes, such as in an insurance context where higher prices select a worse pool of insurees. The essential property is that the selection effect is monotone.
is actually true. Since by definition the naive-optimal action is preferred by the naive player to a lower action, then such must also be the case for a player with correct beliefs, who does know that decreasing her action would lead to a worse selection of outcomes. Therefore, the best response of a player with correct beliefs cannot be lower than the (perceived) best response of a naive player.

Next, compare the behavior of a naive and a sophisticated player facing the same profile of opponent strategies. Suppose that a sophisticated player observes her own payoffs and knows that the selection effect is monotone nondecreasing (i.e. MSP is satisfied). Consider a naive and a sophisticated player who choose the same action which is below the (unique) naive-optimal action. Since payoffs are observed, both players must have correct beliefs about the expected payoff from playing this lower action. By MSP and complementarity between beliefs and actions, the lower the action that is chosen, the lower the action that is optimal for the implied beliefs. From this complementarity in own action, the naive player that chooses an action below the unique naive-optimal action must prefer to deviate to a higher action (otherwise, it is possible to show that there would eventually exist a lower action at which the naive player has no incentives to deviate, contradicting the existence of a unique naive-optimal action). But if the naive player would like to deviate to a higher action, then so would the sophisticated player who knows that there is a monotone selection effect that further increases the desirability of a higher action. Therefore, the sophisticated player never chooses an action which is lower than the unique naive-optimal action.

The previous results compare best responses of players who are naive, sophisticated, or have correct beliefs for a given profile of other players’ strategies. This comparison is of interest in itself, since in monopoly games where only one player acts strategically (e.g. an insurance firm facing a given demand for insurance) best responses correspond to equilibrium behavior. In addition, the previous characterization of best responses can also be used to compare equilibrium behavior in three important classes of games (symmetric games with no private information on one side of the market, games with strategic complementarities, and certain symmetric games with private information such as first-price auctions) and to conclude that the presence of naive players exacerbates the adverse selection problem in equilibrium. The proofs are in the spirit of the literature on monotone comparative statics (e.g. Milgrom and Roberts, 1990).

The result that in some settings markets are thinner in the presence of naive players implies that informational asymmetries may be of even greater concern for the functioning of markets than previously thought. When players are aware of these informational asymmetries, some institutions may naturally arise to mitigate this problem, initiated either by the party having more information (Spence, 1973) or the party which is less informed (Rothschild and Stiglitz, 1976). However, when players fail to account for selection, the adverse selection problem will not only be more severe but these institutions may be less likely to arise.

The work most closely related to this one is a paper by Eyster and Rabin (2005). They provide the first systematic equilibrium analysis of the same bias that I study in this paper by introducing the notion of a cursed equilibrium. In a (fully) cursed equilibrium, players have a priori correct
beliefs about the distribution of payoff uncertainty (such as the common valuation of an object in an auction) and correct beliefs about the distribution of other players’ actions. However, players ignore the relationship between other players’ private information and actions. This ignorance is modeled by assuming that players incorrectly believe that each type profile of the other players plays the same profile of mixed actions – which coincides with the true average distribution of actions.\(^7\) As I show in Section 2, a player in a cursed equilibrium may have incorrect beliefs about the expected payoff she receives from playing her equilibrium strategy. In contrast, in a behavioral equilibrium beliefs about opponents’ actions and payoff uncertainty are restricted by the amount of feedback that players receive in equilibrium and by a priori assumptions on players’ knowledge of the game, but not necessarily by restrictions on players’ beliefs about their opponents’ type-contingent strategies.\(^8\) As a result of this difference, the set of equilibria when players are either naive, sophisticated, or have correct beliefs can be unambiguously compared for a general class of settings where cursed equilibria predicts either ambiguous results or results in the opposite direction (e.g. mitigation of the adverse selection problem).\(^9\)

A different strand of the literature postulates non-equilibrium models of behavior where players follow particular decision rules characterized by a finite depth of reasoning about players’ beliefs about each other (Stahl and Wilson, 1994; Nagel, 1995; Camerer, Ho and Chong, 2004). Crawford and Iriberri (2005) show that behavior that arises from some of these decision rules matches the experimental evidence of overbidding in both private and common value auctions. Both Eyster and Rabin (2005) and Crawford and Iriberri (2005) are motivated to a large extent by experimental evidence and do a very good job at matching this evidence. In contrast, this paper focuses mostly on settings where players repeatedly face similar strategic environments and learn from this experience based on the feedback they receive.

While there is indeed strong evidence that experimental subjects overbid in common value auctions (e.g. Kagel and Levin, 2002), most of these experimental results hold either for inexperienced players or under the assumption that players are told a priori the correct, unconditional expectation of the value of the object. Under the latter assumption, in a behavioral equilibrium naive beliefs cannot persist in equilibrium. The reason is that knowing the correct expected quality of all objects but consistently obtaining objects of lower quality is a fact that cannot be reconciled with a naive model of the world. A behavioral equilibrium is a steady state concept that does not postulate how

\(^7\)Eyster and Rabin (2005) also consider intermediate cases where players “underappreciate” the selection problem: with some probability beliefs are as in a cursed equilibrium and with the remaining probability beliefs about opponents’ type-contingent strategies are correct (as in a Bayesian Nash equilibrium). In contrast, in a behavioral equilibrium players are either aware or unaware of the selection problem, and it is only players who are aware (i.e. sophisticated) who might now either under or overappreciate the informational content of other players’ actions.

\(^8\)For example, a naive player may recognize that other players do act based on private information, but may still ignore that such private information may be of interest to her.

\(^9\)Like Eyster and Rabin (2005), Jehiel and Koessler (2005) also assume that players (i) have a priori correct beliefs over the states of the world representing uncertainty and (ii) make mistakes when forecasting the type-contingent strategies of their opponents by bundling types into analogy classes and playing a best response to the other players’ average strategy in each analogy class. A (fully) cursed equilibrium arises in their framework for a particular analogy class. The same remarks that motivate the alternative approach that I follow in this paper apply to their framework.
or whether the model of the world will be revised, but rather limits itself to answering whether such a model of the world can persist in equilibrium or not. I provide further discussion of these issues in the conclusion.

In Section 2, I present an example that illustrates the selection problem, the approach taken by Eyster and Rabin (2005), the alternative approach that I propose, and the result that naive players exacerbate the adverse selection problem. Section 3 includes the formal definition of behavioral equilibrium in general games. A useful feature of a behavioral equilibrium is that it can be characterized as a fixed point of an appropriate generalization of the best-response correspondence. I exploit this characterization in Section 4, where I apply the framework to games with monotone selection and characterize the set of equilibria when players are naive, sophisticated, or have correct beliefs. Section 5 presents additional examples to which the results obtained in the paper apply. I conclude with suggestions for further research and with broader implications of the proposed framework.

2 An Illustrative Example

Consider a trading game with one-sided asymmetric information of the sort introduced by Akerlof (1970). The seller values the object at $s$ while the buyer values the object at $v = s + x$, where $s$ is the realization of a random variable $\tilde{s}$ that is uniformly distributed on the interval $[0, 1]$ and $x \in (0, 1]$ is a parameter that captures gains from trade. The seller knows her valuation, but the buyer has no private information either about $s$ or $v$. The buyer and seller simultaneously make offers to buy at price $p$ and to sell at price $ask$, respectively. If $ask > p$ there is no trade, the seller keeps the object, and the buyer obtains her reservation utility of zero. If $ask \leq p$, the object is traded and the buyer pays her offered price $p$ to the seller and obtains utility $u(p, v) = v - p$.

Throughout this section I restrict attention to equilibria where the seller plays his weakly dominant strategy, which here consists of choosing an ask price equal to his valuation of the object, $s$.

2.1 Nash equilibrium, the selection problem, and cursed equilibrium

In a standard (Bayesian) Nash equilibrium, the buyer offers a price $p$ to maximize her expected profits

$$
\pi^{NE}(p) = \Pr(\tilde{s} \leq p) \times [E(\tilde{v} \mid \tilde{s} \leq p) - p].
$$

10 Alternatively, the buyer could make an offer to buy at $p$ that is either rejected or accepted by the seller, as in the original version of the game described by Samuelson and Bazerman (1985). The differences between these two trading protocols is discussed in footnote 17.

11 By standard I mean a Bayesian Nash equilibrium under the assumption that there is common knowledge of the joint distribution of $(\tilde{s}, \tilde{v})$. 
Under the current assumptions, equation (1) becomes \( \pi^{NE}(p) = p \times (x - \frac{1}{2}p) \) and the optimal (i.e. Nash equilibrium) price is \( p^{NE} = x \).

In this example, the buyer faces a selection problem: the price that she offers selects the average quality of the objects that are traded in equilibrium. In a Nash equilibrium, buyers account for this problem by conditioning \( E(\bar{v}) \) on the information that trade takes place. As discussed in the introduction, Eyster and Rabin (2005) introduce the concept of cursed equilibrium to study the behavior of players who fail to account for this problem in a general class of games. I now apply their approach to the trading game in order to highlight the differences with the alternative approach proposed in this paper.

A cursed buyer does not realize that the value of the object depends on the price she offers but rather believes that it is given by the unconditional expectation of the object, \( E(\bar{v}) \).\(^{12}\) The perceived profits of a cursed buyer are then

\[
\pi^{Cursed}(p) = \Pr(\bar{s} \leq p) \times [E(\bar{v}) - p],
\]

and the optimal (i.e. cursed equilibrium) price is \( p^{Cursed} = \frac{1}{2}(x + \frac{1}{2}) \). Hence, relative to the Nash equilibrium, a cursed buyer over-prices for \( x < 1/2 \) and under-prices for \( x > 1/2 \).

The following intuition for the under/over-pricing result serves as a starting point for the intuition behind one of the main results in the paper. In this example (since \( \bar{v} \) and \( \bar{s} \) are affiliated), a cursed buyer believes the expected value of the object to be higher than what a non-cursed buyer believes, i.e. \( E(\bar{v}) \geq E(\bar{v} \mid \bar{s} \leq p) \) for all \( p \). This level effect increases a cursed buyer’s willingness to offer higher prices in order to obtain the object. However, there is a second effect working in the opposite direction. A cursed buyer does not realize that increasing her offer would also increase the (average) quality of objects she receives. This slope effect provides a cursed buyer with a weaker incentive to increase her bid relative to a buyer who has correct beliefs, as in a Nash equilibrium. In all other respects (in particular, regarding their beliefs about the probability of trade at different prices), a cursed buyer and a buyer with correct beliefs are identical. Depending on whether the level or slope effect dominates, in equilibrium a cursed buyer can either over-price or under-price relative to a buyer in a Nash equilibrium.\(^{13}\)

2.2 Behavioral Equilibrium: naive and sophisticated buyers

A cursed buyer makes her pricing decision under the belief that the expected value of the object is \( E(\bar{v}) \), but if she repeatedly follows her cursed strategy, the average value of those objects that she

\(^{12}\)This case corresponds to what Eyster and Rabin (2005) call a fully cursed buyer and was originally discussed by Kagel and Levin (1986) and Holt and Sherman (1994).

\(^{13}\)Holt and Sherman (1994) were the first to show that in this setting both underpricing and overpricing were possible in (what would later be called) a fully cursed equilibrium. They referred to what I call the slope effect as the “loser’s curse”.

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actually consumes is lower, $E(\tilde{v} \mid \bar{s} \leq p^{Cursed})$. If she repeatedly obtains feedback about the value of traded objects, then it may be reasonable to expect a cursed buyer to revise her beliefs about the expected value of the object and therefore offer a different price. In addition, a question arises as to how a cursed buyer may know the true unconditional expected value of the object to begin with.

A behavioral equilibrium requires beliefs to be consistent with the feedback obtained from actual equilibrium play. By emphasizing the role of information feedback, it captures the essence of a selection problem: a buyer’s pricing decision affects the average quality of objects that are traded and therefore the sample that she uses to form beliefs about quality. I consider two types of buyers: a naive buyer ignores the selection effect, while a sophisticated buyer is aware of its potential existence. However, a sophisticated buyer may still have incorrect beliefs about the quality of objects that would be traded at prices that, for example, she has not tried out in the past. This is in contrast to a buyer with correct beliefs (as in a Nash equilibrium), who knows the exact price-quality schedule in equilibrium.

A behavioral equilibrium depends on restrictions on players’ beliefs that the modeler wishes to impose, which are in turn motivated by the feedback that players obtain from repeatedly playing the equilibrium strategies. In the context of the trading game, I assume that both naive and sophisticated players observe their own payoffs and that the auctioneer reveals the seller’s ask price at the end of each encounter. According to the formal definition of equilibrium in Section 3, these assumptions in turn imply that in equilibrium the buyer has correct beliefs both about her expected payoffs from following the equilibrium action and about the probability of trade at any possible price.

**Equilibrium with a naive buyer.** The following function (and its appropriate generalization) plays an important role in developing the results in this paper:

$$\pi^N(p, \tilde{p}) = \Pr(\tilde{s} \leq p) \times [E(\tilde{v} \mid \tilde{s} \leq p) - p].$$

Equation (3) represents a naive buyer’s equilibrium belief about her expected payoff from deviating to a price $p$ given that in (a hypothetical) equilibrium she repeatedly chooses $\tilde{p}$.

14 Beliefs about the probability of trade at each price are correct because of the assumption that in equilibrium the distribution of ask prices is known. Beliefs about the expected value of the object: (i) are determined by the price she chooses in equilibrium, $\tilde{p}$ (rather than just being the unconditional expected value of the object, as in a cursed equilibrium); (ii) do not depend on the price $p$ to which she considers deviating (this is the failure to account for the selection problem); and (iii) are correct for $p = \tilde{p}$.

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14 An implicit assumption is that the buyer does not know that in fact $\tilde{v} = \bar{s} + x$; whatever she learns about her valuation depends on the feedback she obtains. A general version of equation (3) is derived in Proposition 3. This equation actually holds true for $\tilde{p} > 0$. Otherwise, no objects are traded, no feedback about the value of objects is obtained, and therefore a buyer can have (quite) arbitrary beliefs about the expected value of the object.
As defined in Section 3, in a behavioral equilibrium a naive buyer chooses a price \( p^N \) such that given her perceived profit function \( \pi^N(\cdot, p^N) \), it is indeed optimal to choose \( p^N \). Hence, the set of equilibria with a naive buyer is given by the set of fixed points of \( H^N(p) = \arg \max_{\tilde{p}} \pi^N(\tilde{p}, p) \). A straightforward calculation yields \( H^N(p) = \frac{1}{2} x + \frac{1}{4} p \), so that when the buyer is naive there is an (essentially) unique equilibrium price \( p^N = \frac{2}{3} x \) that is lower than the Nash equilibrium price \( p^{NE} = x \) for any parameter value \( x \in (0, 1] \). Hence, a naive buyer offers a lower price than a buyer with correct beliefs, leading to a lower probability of trade and to lower gains from trade.\(^{15}\)

The result that the adverse selection problem is exacerbated in the presence of naive players is not a coincidence of this particular example but rather a more general result. Its intuition can be grasped by observing Figure 1, which compares the perceived profit function \( \pi^N(p, \hat{p}) \) of a naive buyer choosing \( \hat{p} \) to the correct profit function in equation (1), \( \pi^{NE}(p) \). The restriction that requires a buyer to have correct beliefs about her expected payoff from playing the equilibrium price eliminates the desire to over-price (i.e. the level effect) discussed earlier. Now, due to the selection effect, only the slope effect remains: a naive player thinks that profits from deviating to a lower price are higher than what they actually are; while she believes that profits from deviating to a higher price are lower than what they actually are. Since only the incentives to under-price are present, naive buyers always under-price relative to buyers with correct beliefs. Figure 1 also illustrates that the price \( \hat{p} \) in that figure cannot constitute a naive equilibrium price, since a naive player choosing \( \hat{p} \) would rather deviate to a higher price. Figure 1b depicts \( H^N \) and the (essentially) unique naive equilibrium price \( p^N \).

**Equilibrium with a sophisticated buyer.** In contrast, a sophisticated buyer who chooses price \( \hat{p} \) in (a hypothetical) equilibrium perceives her profits from deviating to \( p \) to be

\[
\pi^S(p, \hat{p}) = \Pr(\tilde{s} \leq p) [\rho(p, \hat{p}) - p],
\]

where \( \rho(p, \hat{p}) \) denotes her expectations about the value of objects that would be traded at price \( p \). I assume that a sophisticated buyer not only knows that there might be a potential selection problem (so that \( \rho(p, \hat{p}) \) is not necessarily constant in \( p \)), but in addition also knows that this selection problem is monotone: the quality of objects traded in equilibrium is nondecreasing in the price that she offers. Hence, \( \rho(\cdot, \hat{p}) \) is a nondecreasing function for each \( \hat{p} \). An implication is that when choosing \( \hat{p} \), a sophisticated buyer knows that the expected value of the object conditional on trading at a price higher than \( \hat{p} \) is at least \( E(\tilde{v} | \tilde{s} \leq \hat{p}) \). Hence, the perceived profit function of a naive buyer who offers \( \hat{p} \), \( \pi^N(p, \hat{p}) \), constitutes a lower bound for the perceived profit function of a

\(^{15}\)Both with naive and sophisticated buyers, there is always a no-trade equilibrium where the buyer offers a price of zero and trade occurs with probability zero. The reason is that a buyer obtains no feedback about her valuation, so she may have quite arbitrary beliefs that induce her to offer zero. A few refinements could eliminate this equilibrium when sustained by incorrect beliefs. One possibility is to assume that beliefs are “continuous”, i.e. \( \pi^N(p, 0) = \lim_{\tilde{p} \to 0} \pi^N(p, \tilde{p}) \) for all \( p \), which, for example, can be motivated by adding some infinitesimal trembling to the seller’s strategy. Note also that no-trade is a Nash equilibrium outcome when sellers are not restricted to play their weakly dominant strategy.
sophisticated buyer when \( p \geq \hat{p} \). Similarly, it constitutes an upper bound when \( p \leq \hat{p} \).

Furthermore, the assumption that a buyer jointly observes ask prices and realized valuations when trade occurs implies that in an equilibrium where \( \hat{p} \) is offered, the beliefs of a sophisticated player are correct for prices lower than \( \hat{p} \), i.e. \( \rho(p, \hat{p}) = E(\tilde{v} | \tilde{s} \leq p) \) for \( p \leq \hat{p} \). These restrictions place tight bounds on the behavior of a sophisticated buyer:

1. The (essentially) unique equilibrium price with a naive buyer is a lower bound for the set of equilibrium prices with a sophisticated buyer. To see this, suppose \( p < p^N \) were an equilibrium price with a sophisticated buyer. From Figure 1b, \( H^N(p) > p \), meaning that a naive buyer who offers \( p \) believes that she can do better by offering a price higher than \( p \) rather than by choosing \( p \). Since the beliefs of a naive buyer constitute a lower bound for the beliefs of a sophisticated buyer for prices higher than \( p \), it follows that a sophisticated buyer must also believe that she can do better by choosing a higher price. Hence, \( p \) cannot constitute an equilibrium price with a sophisticated buyer.

2. The unique Nash equilibrium price constitutes an upper bound for the set of equilibrium prices with a sophisticated buyer. Suppose, toward a contradiction, that \( p > p^{NE} \) is an equilibrium price with a sophisticated buyer. Since a sophisticated buyer must have correct beliefs in equilibrium for those prices below \( p \), it follows that she must know that she can do better by deviating to \( p^{NE} \). In fact, for this particular example, the set of equilibrium prices with a sophisticated buyer (excluding the no-trade equilibrium) is given by the interval \([p^N, p^{NE}]\).

In particular, in the trading game studied in this section the adverse selection problem is exacerbated in the presence of naive players not only relative to a buyer with correct beliefs, as in a Nash equilibrium, but also relative to a sophisticated buyer who knows that there is a monotone non-decreasing selection effect but who may still have incorrect beliefs about the correct price-quality schedule. In Section 4 I generalize this result to a wider class of games, including games where more than one player acts strategically. In the context of the monopsony trading game, the following turns out to be the more general result:

**Proposition 1** Consider the monopsony trading game where: i) \( p \in A \cup \{0\} \), where \( A \subseteq \mathbb{R}_+ \) is a finite set, ii) \( \tilde{s} \) and \( \tilde{v} \) are affiliated random variables with real-valued, finite support, and 0 belongs in the support of \( \tilde{s} \), iii) \( u \) is nondecreasing in \( v \) and supermodular in \((p, v)\), iv) sellers choose ask prices equal to their valuation, v) the buyer only observes her payoffs and the ask price of the seller.

\[16\] I assume a finite setting with 0 being a feasible price offer and with 0 in the support of the set of seller’s valuations so that the buyer obtains feedback about the value of the object for any possible offer that she makes. As a result, a no-trade equilibrium need not necessarily exist in this setup, which makes the Proposition interesting; if a no-trade equilibrium were to always exist, some of the statements would be trivial.
after each bilateral encounter,\textsuperscript{17} and vi) a sophisticated buyer knows that the selection problem is monotone nondecreasing. Then

1. Every equilibrium price with a naive buyer is either (weakly) lower than the lowest Nash equilibrium price or is a Nash equilibrium price itself.

2. Every equilibrium price with a sophisticated buyer is (i) (weakly) higher than the lowest naive equilibrium price, and (ii) either (weakly) lower than the lowest Nash equilibrium price or a Nash equilibrium price itself.

\textbf{Proof.} See Appendix A. \blacksquare

\textbf{Dynamics leading to equilibrium.} While a behavioral equilibrium is a static, steady-state solution concept, I have motivated it by informally appealing to some form of learning from past outcomes. A similar motivation has led to the development of formal dynamic models with steady states that correspond to Nash equilibria and to self-confirming equilibria (e.g. Fudenberg and Levine, 1998), where the latter correspond to equilibria with sophisticated players. I now show, in the context of the example in this section, that a dynamic model can also be provided to justify convergence to a (behavioral) equilibrium with a naive buyer.

Suppose the buyer knows the probability of trade at each price (presumably because she has collected information about trades in this market), but does not know the expected value of the object. Every period, the buyer chooses a price to maximize her perceived profits, which in the context of the example in this section are \(\pi^N_t(p) = p \times (y_t - p)\), where \(y_t\) is her expectation of the value of the object at time \(t\). Hence, at time \(t\) she offers price \(p^*_t = \frac{1}{2}y_t\). The buyer starts with a prior about the expected value of the object, \(v_1\), and updates this prior as she obtains additional information about the value of the object, which occurs when she trades. No updating takes place if there is no trade, and a buyer simply offers the same price in the next period. Hence, for simplicity I denote a time period \(t\) as a time where trade (and therefore updating) takes place. The updating rule is given by \(y_t = \frac{1}{t} \sum_{i=1}^{t} v_i\), so that a buyer’s estimate of the expected value of the object is the average of all observed valuations (including her initial prior). Conditional on trade taking place at time \(t\) (i.e. \(\tilde{s}_t \leq p^*_t = \frac{1}{2}y_t\)), the value of the traded object is uniformly distributed in the interval \([x, x + p^*_t]\). These dynamics can be captured by a sequence of random variables \(\{\tilde{v}_t\}\) where \(v_1 > 0\) is the initial condition\textsuperscript{18} and \(\tilde{v}_{t+1} \sim U \left[ x, x + \frac{1}{2} \left( \frac{1}{t} \sum_{i=1}^{t} \tilde{v}_i \right) \right] \), where \(\tilde{v}_{t+1}\) is the value of those objects

\textsuperscript{17}In many examples with adverse selection the assumption that one player learns ex-post the reservation values of other players might be too strong. In these cases, it is natural to replace the assumption that players observe “ask prices” with the assumption that players have correct beliefs about the demand or supply function (e.g. about the probability that trade would take place at each different price). In such a case, all the results continue to hold except that Nash equilibrium is no longer an upper bound to the behavior of sophisticated players.

\textsuperscript{18}If the prior \(v_1 = 0\), then the buyer never trades and therefore never updates her prior. This case corresponds to the no-trade equilibrium.
that are traded at time $t + 1$. While $\{\tilde{\nu}_t\}$ is a correlated sequence, in the limit as $t$ becomes large this dependency disappears and $\text{Var}(\tilde{y}_t) = \frac{1}{n} \sum_{i} \sum_{j} \text{Cov}(\tilde{\nu}_i, \tilde{\nu}_j)$ converges to 0 as $t \rightarrow \infty$. Since $\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{i} E\tilde{\nu}_i = \frac{4}{3} x$ irrespective of the initial condition $v_1 > 0$, it then follows that $\hat{y}_t = \frac{1}{n} \sum_{i=1}^{n} \tilde{\nu}_i$ converges in probability to $\frac{4}{3} x$, so that the price offered by the buyer converges in probability to $p^* = \frac{2}{3} x$, which is the price in a naive equilibrium. While it is possible to generalize these adaptive dynamics to some extent, in the remainder of the paper I abstract from the dynamics leading to equilibrium and focus on a steady state definition of equilibrium.

3 Definition of Behavioral Equilibrium

While the above example and the rest of the paper focus on a class of games with a monotone selection property, I now present the formal definition of a behavioral equilibrium in the context of more general games. The main contribution is to define a notion of equilibrium for naive players, which extends the idea of a self-confirming equilibrium to accommodate the behavior of players who fail to take into account the informational content of other players’ actions.

There is a finite set $N$ of players who simultaneously choose actions. Each player $i \in N$ chooses an action from a finite action space $A_i \subset \mathbb{R}^K$ and obtains payoff $u_i(a, v) \in \mathbb{R}$, where $a = (a_i)_{i \in N} \in A \equiv \times_i A_i$ is the profile of actions, and $v \in V \subset \mathbb{R}^K$ is the realization of a random variable $\tilde{v}$. The finite set $V$ is used to capture players’ uncertainty about payoffs. Before choosing an action, each player receives a signal realization $s_i$ from a random variable $\tilde{s}_i$ with finite support $S_i$. The signals and the parameters of the utility function are jointly drawn according to an objective probability distribution $\gamma \in \Delta(S \times V)$, where $S = \times_i S_i$. A (pure) strategy for player $i$ is a function from the set of signals to the set of actions, $\alpha_i : S_i \rightarrow A_i$. A strategy profile is denoted by $\alpha = \{\alpha_i\}_{i \in N}$ and the set of all strategy profiles is $A$. For future reference, I denote this game by $G(\gamma)$.$^{19}$

Players know their utility function, the set $V$ of parametric uncertainty, and their set of feasible actions. I do not restrict beliefs about opponents’ utility functions (or about their rationality).$^{20}$ Player $i$’s conjecture when she receives signal $s_i$ is a probability distribution $\phi_i(s_i) \in \Delta(A_{-i} \times V)$. Players are rational in the sense that for a signal $s_i$, player $i$ chooses an action $a_i \in A_i$ to maximize

$^{19}$The assumptions that the game is finite and that players choose pure strategies are made for ease of exposition and can be relaxed. Existence of equilibrium is not guaranteed under these assumptions, but I will appeal to complementarity conditions to establish existence in finite games. Allowing for mixed strategies would not guarantee existence of an equilibrium with naive players under the assumption that players can distinguish between the consequences that result from each of the actions in the support of the mixed strategy. The reason is that, contrary to what happens in the standard Nash setting, payoffs need not be continuous in mixed strategies.

$^{20}$Espohna (2005a) extends this framework to allow for a wide range of assumptions about what players know about the game (including about other players), what they know that other players know, and so on. Many of the results in Section 4 continue to hold under these additional refinements, except that an equilibrium may then fail to exist.
her expected utility given such conjectures:

\[ E_{\phi_i(s_i)}u_i(a_i, \overline{a_{-i}}, \overline{v}) = \sum_{(a_{-i}, v)} u_i(a_i, a_{-i}, v)\phi_i(s_i)(a_{-i}, v). \]

A behavioral equilibrium restricts conjectures to be consistent with actual equilibrium play. For sophisticated players, I adopt the definition of consistency from the literature on self-confirming equilibrium. For naive players, on the other hand, I provide a new definition that captures their cognitive bias.

**Consistency of beliefs for sophisticated players.** To understand the idea underlying the notion of consistency it is useful to first examine the particular equilibrium restriction implicit in a Nash equilibrium. Both in games of complete and incomplete information, the standard Nash assumption can be interpreted as requiring that in equilibrium players choose optimal actions based on correct conjectures; i.e. in a Nash equilibrium players have correct beliefs about their expected payoffs from choosing any feasible strategy, given the equilibrium strategies of the other players. However, in certain situations it may be sensible to make less restrictive assumptions on equilibrium beliefs. For example, when players form conjectures based on past experience that includes only a difference with the definition of self-confirming equilibrium in Dekel, Fudenberg, and Levine (2004) is that I do not necessarily restrict these conjectures to come from separate beliefs about opponents’ strategies and the joint distribution of signals and payoff uncertainty.

\[ \eta_i(a_i, a_{-i}, v) \neq \eta_i(a_i', a_{-i}', v') \]

\[ u_i(a_i, a_{-i}, v) \neq u_i(a_i', a_{-i}', v') \]

\[ a_i \neq a_i' \]

\[ v \neq v' \]

\[ \overline{a_{-i}} \neq \overline{a_{-i}}' \]

\[ \overline{v} \neq \overline{v}' \]

\[ \eta_i \text{ only reveals payoffs: } \eta_i(a_i, a_{-i}, v) \neq \eta_i(a_i', a_{-i}', v') \text{ if and only if } u_i(a_i, a_{-i}, v) \neq u_i(a_i', a_{-i}', v'). \]

---

21 A difference with the definition of self-confirming equilibrium in Dekel, Fudenberg, and Levine (2004) is that I do not necessarily restrict these conjectures to come from separate beliefs about opponents’ strategies and the joint distribution of signals and payoff uncertainty.

22 Strictly speaking, in games with incomplete information the standard assumption corresponds to the notion of a Bayesian Nash equilibrium with the correct prior being common knowledge among the players.
Consistency restricts players to have correct beliefs about the probability of observing each of the feedback signals. For example, if $\eta_i$ is fully revealing then consistency requires players to have correct conjectures (as in a Nash equilibrium), while if $\eta_i$ reveals payoffs consistency requires players to have correct beliefs about the expected payoff from following their equilibrium strategy.

The assumption that in equilibrium players know the correct distribution of their information feedback signals is captured in the following way. The correct conditional distribution over $(e_a, e_v)$ given a profile of opponents’ strategies $a_{-i}$ and a signal $s_i$ is given by

$$p_i(s_i, a_{-i}, \gamma)(a_{-i}, v) = \sum_{s_{-i}} \sum_{s'_{-i}, v'} \gamma(s_i, v) \times 1 \{a_{-i}(s_{-i}) = a_{-i}\}$$

where $\gamma \in \Delta(S \times V)$ is the objective probability distribution defined earlier.

Given information feedback function $\eta_i$, both a player’s conjecture $\phi_i$ and the true distribution $p_i$ over $(\bar{a}_{-i}, \bar{v})$ induce a distribution over the set of information feedback signals $H_i$. For all $i$ and all feedback signals $h \in H_i$, let

$$[h]_{i, a_i, \eta_i} \equiv \{(a_{-i}, v) : \eta_i(a_i, a_{-i}, v) = h\}$$

denote the set of outcomes $(a_{-i}, v)$ that player $i$ considers possible when she plays action $a_i$ and obtains feedback $h$. Consistency requires players to have correct beliefs about the distribution over feedback that they obtain as a result of the strategies that they and their opponents play.

**Definition 1 (consistency)** The conjecture of player $i$ when she receives signal $s_i$, $\phi_i(s_i)$, is $\eta_i$-consistent for $(a_i, a_{-i}, \gamma)$ if for all $h \in H_i$, $\phi_i(s_i)[h]_{i, a_i, \eta_i} = p_i(s_i, a_{-i}, \gamma)[h]_{i, a_i, \eta_i}$.

**Consistency of beliefs for naive players.** I now define an alternative notion of consistency, naive consistency, that applies to players who fail to realize that the actions of the other players may be correlated with their own payoff-relevant uncertainty (parameterized by $V$). Each feedback $h \in H_i$ is associated with a set of possible opponents’ actions and a set of possible values of $V$. The conjecture of a naive player is required to be “consistent” with respect to each of these “samples” separately, thus ignoring the potential correlation between them – even if such correlation could be inferred from the joint sample by a sophisticated player. In addition to the above restriction, naive players are required to use knowledge of their own utility function and the feedback they obtain about payoffs to confirm their beliefs about $(\bar{a}_{-i}, \bar{v})$. The formal definition follows.
For all \( i \) and \( h \in H_i \) define the sets

\[
k_i^V(h) = \{ v \in V : \exists a \in A \text{ s.t. } \eta_i(a, v) = h \},
\]

\[
k_i^{A_{-i}}(h) = \{ a_{-i} \in A_{-i} : \exists v \in V, a_i \in A_i \text{ s.t. } \eta_i(a_i, a_{-i}, v) = h \}, \text{ and}
\]

\[
k_i^U(h) = \{ u \in \mathbb{R} : u = u_i(a_i, a_{-i}, v) \text{ for } a \in A, v \in V \text{ s.t. } \eta_i(a_i, a_{-i}, v) = h \}.
\]

The set \( k_i^V(h) \) includes all values of \( V \) that player \( i \) may consider possible when she obtains feedback \( h \). Similarly, \( k_i^{A_{-i}}(h) \) and \( k_i^U(h) \) include all possible values of \( A_{-i} \) and \( U \equiv \cup_{(a_i,a_{-i},v)} \{ u_i(a_i, a_{-i}, v) \} \) under feedback \( h \). The set \( k_i^V(h) \subset 2^V \) can be interpreted as a “marginal feedback” signal that a player obtains about all the possible values of \( v \) that may have been realized. With this interpretation in mind, define for all \( i, a_i \) and all \( f \in 2^V \),

\[
[f]_{i, a_i, \eta_i}^V = \{ (a_{-i}, v) : k_i^V(\eta_i(a_i, a_{-i}, v)) = f \}
\]

to be the set of outcomes \((a_{-i}, v)\) under which player \( i \) who chooses action \( a_i \) receives “marginal feedback signal” \( f \in 2^V \). Similarly, for \( g \in 2^{A_{-i}} \) let

\[
[g]_{i, a_i, \eta_i}^{A_{-i}} = \{ (a_{-i}, v) : k_i^{A_{-i}}(\eta_i(a_i, a_{-i}, v)) = g \},
\]

and for \( u \in 2^U \) let

\[
[u]_{i, a_i, \eta_i}^U = \{ (a_{-i}, v) : k_i^U(\eta_i(a_i, a_{-i}, v)) = u \}.
\]

There is a difference between a “marginal feedback” such as \( f \) and the regular feedback \( h \) defined earlier that does not make it possible to have a simple definition of naive-consistency in terms of correct beliefs over certain feedback signals. Before, each outcome \((a_{-i}, v)\) was associated with a unique feedback \( h \). Now, a “marginal” outcome, such as \( v_1 \in V \), may be associated with more than one marginal feedback, e.g. \( f_1 = \{ v_1 \} \) and \( f_2 = \{ v_1, v_2 \} \). In order to calculate the marginal probability over \( v_1 \), a player must assign some belief to the probability that \( v_1 \) occurs when she obtains feedback \( \{ v_1, v_2 \} \). I assume that a naive player fails to account for the selection problem, so that she believes that the process that generates outcomes is independent from the process that determines the information that she obtains about the realized marginal outcomes.\(^{23}\)

\(^{23}\) An analogy can be drawn to how an empirical researcher deals with missing data. One possibility, which is captured by the definition of naive-consistency provided here, is that the selection problem is completely ignored. Alternatively, a researcher (or naive player) may simply ignore the informational content of other peoples’ actions, but she may take selection into account by having some theory of where selection comes from – other than from correlation between \( a_{-i} \) and \( v \). Or yet another alternative is for a researcher (or player) to ignore (or place less weight on) feedback that is ambiguous, such as \( \{ v_1, v_2 \} \), relative to feedback that is fully revealing, such as \( \{ v_1 \} \). This second alternative introduces a new bias, which is the tendency to overweight information that is not ambiguous (see Rabin (1998) for references to this bias). It is straightforward to provide any of these alternative definitions of naive-consistency. When marginal feedback is either fully revealing or completely uninformative (as in most examples
Definition 2 (naive-consistency) The conjecture of player $i$ when she receives signal $s_i$, $\phi_i(s_i)$, is $\eta_i$-naive-consistent for $(a_i, \alpha_{-i}, \gamma)$ if the following conditions are satisfied:

1. for all $(a_{-i}, v)$, $\phi_i(s_i)(a_{-i}, v) = \phi_i^{A_{-i}}(s_i)(a_{-i}) \times \phi_i^V(s_i)(v)$, where $\phi_i^{A_{-i}}(s_i)$ and $\phi_i^V(s_i)$ are probability distributions over $A_{-i}$ and $V$, respectively, that satisfy:

   (a) for all $a_{-i} \in A_{-i}$,
   
   \[
   \phi_i^{A_{-i}}(s_i)(a_{-i}) = \sum_{g \in \mathcal{H}^{A_{-i}}} p_i(s_i, \alpha_{-i}, \gamma)[g]^{A_{-i}}_{i,a_i,\eta_i} \times \frac{\phi_i^{A_{-i}}(s_i)(a_{-i})}{\sum_{a'_{-i} \in g} \phi_i^{A_{-i}}(s_i)(a'_{-i})}, \tag{4}
   \]

   where summation is over $\mathcal{H}^{A_{-i}} = \{ g \in 2^{A_{-i}} : a_{-i} \in g, \sum_{a'_{-i} \in g} \phi_i^{A_{-i}}(s_i)(a'_{-i}) > 0 \}$;

   (b) for all $v \in V$,
   
   \[
   \phi_i^V(s_i)(v) = \sum_{f \in \mathcal{H}^V} p_i(s_i, \alpha_{-i}, \gamma)[f]^V_{i,a_i,\eta_i} \times \frac{\phi_i^V(s_i)(v)}{\sum_{v' \in f} \phi_i^V(s_i)(v')}, \tag{5}
   \]

   where summation is over $\mathcal{H}^V = \{ f \in 2^V : v \in f, \sum_{v' \in f} \phi_i^V(s_i)(v') > 0 \}$;

2. for all $u \in 2^U$, $\phi_i(s_i)[u]_i^{U,a_i,\eta_i} = p_i(s_i, \alpha_{-i}, \gamma)[u]_i^{U,a_i,\eta_i}$.

The previous definition of naive-consistency requires players to believe that $\tilde{a}_{-i}$ and $\tilde{v}$ are independent and to ignore the selection problem (condition 1). The selection problem is ignored since players believe that there is some process that generates marginal outcomes, e.g. $\phi^V$, and that for marginal feedback that is ambiguous, e.g. $\{v_1, v_2\}$, the probability that one particular outcome is observed, e.g. $v_1$, is simply the probability of that outcome conditional on the set of all possible outcomes, according to her beliefs about the process that generates marginal outcomes to begin with, e.g. $\phi^V(v_1) / (\phi^V(v_1) + \phi^V(v_2))$. Beliefs are then obtained endogenously, and by a fixed point argument there always exists a conjecture that satisfies condition 1. In addition, to the extent that players also receive some feedback about their own payoffs, condition 2 requires their beliefs about observed payoff-feedback to be consistent with their beliefs about the distribution of $\tilde{a}_{-i}$ and $\tilde{v}$ and with knowledge of their utility function. It is then possible that there is no conjecture satisfying both conditions 1 and 2, so that $\eta_i$-naive-consistent conjectures may not exist for some particular $(a_i, \alpha_{-i}, \gamma)$. For a simple example, suppose the utility function of player 1 (for some in the paper), then the three approaches coincide. Finally, it is possible to draw an analogy between a sophisticated player and a researcher who knows that the source of the selection problem is the correlation between $\tilde{a}_{-i}$ and $\tilde{v}$.

\footnote{Formally, the right hand sides of equations 4 and 5 can each be used to define two continuous functions from the unit simplex to itself, so that Brouwer's fixed point theorem implies that there exist solutions $\phi_i^{A_{-i}}$ and $\phi_i^V$ to conditions 1a and 1b, respectively.}
for each of the information feedback signals in Figure 2b, so that, e.g. \(\phi^V(v_1)\) is from some unspecified system of equations (since it is easy to see that \(\phi^V(v_1)\) has a unique solution from the information that she obtains about \(E\tilde{u}_1, E\tilde{a}_2, \) and \(E\tilde{v}\), but if \(\tilde{a}_2\) and \(\tilde{v}\) are actually correlated, then there is no conjecture that satisfies independence as required by condition 1 that is “consistent” with feedback about payoffs as required by condition 2, since \(E\tilde{u} \neq E\tilde{a} \times E\tilde{v}\).

**Example [consistency vs naive consistency]** Suppose that player 1 faces the joint distribution \(p\) (which comes from some unspecified \(a_2\) and \(\gamma\)) over the sets of opponent’s actions \(A_2 = \{a_1, a_2, a_3\}\) and payoff uncertainty \(V = \{v_1, v_2, v_3\}\) given in Figure 2a and obtains feedback according to the information policy shown in Figure 2b (where feedback also depends on the action of player 1, which is left unspecified). For example, with probability 3/18 she observes \(h_1\), i.e. that the outcome is \((a_1, v_1)\). The notion of \(\eta_1\)-consistency then requires her conjecture to satisfy \(\phi^S(a_1, v_1) = 3/18\), where the superscript \(S\) stands for the conjecture of a sophisticated player. Similar restrictions hold for each of the information feedback signals in Figure 2b, so that, e.g. \(\phi^S(a_2, v_2) + \phi^S(a_2, v_3) = 4/18\) and \(\sum_{i=1}^3 \phi^S(a_3, v_i) = 6/18\). Hence, there exist multiple conjectures which are \(\eta_1\)-consistent.

In contrast, a naive player 1 processes the information that she obtains about \(A_2\) independently from the information that she obtains about \(V\). Since she can always distinguish her opponents’ actions, \(\phi^{A_2}(a) = 6/18\) for all \(a \in A_2\). Regarding \(V\), her marginal conjecture solves the following system of equations (since it is easy to see that \(\phi(v_i)\) must be strictly positive for each \(i\)):

\[
\begin{align*}
\phi^V(v_1) &= \frac{5}{18} + \frac{6}{18} \times \frac{\phi^V(v_1)}{\sum_{i=1}^3 \phi^V(v_i)} \\
\phi^V(v_2) &= \frac{2}{18} + \frac{4}{18} \times \frac{\phi^V(v_2)}{\phi^V(v_2) + \phi^V(v_3)} + \frac{6}{18} \times \frac{\phi^V(v_2)}{\sum_{i=1}^3 \phi^V(v_i)} \\
\phi^V(v_3) &= \frac{1}{18} + \frac{4}{18} \times \frac{\phi^V(v_3)}{\phi^V(v_2) + \phi^V(v_3)} + \frac{6}{18} \times \frac{\phi^V(v_3)}{\sum_{i=1}^3 \phi^V(v_i)},
\end{align*}
\]

which has a unique solution \(\phi^V(v_1) = 15/36, \phi^V(v_2) = 14/36, \phi^V(v_3) = 7/36\). Condition 1 in the definition of naive-consistency then requires her conjecture to be given by the product of the marginals \(\phi^V\) and \(\phi^{A_2}\).

Finally, whether the above conjecture satisfies condition 2 depends on the utility function. Figure 2c specifies the utility that player 1 obtains from each of the possible outcomes. The probability that \(u = 1\) is observed (i.e. that \(h_1\) or \(h_4\) is observed) is 5/18, which coincides with the probability that \(u = 1\) according to the product of the previous conjectures, \([\phi^{A_2}(a_1) + \phi^{A_2}(a_2)] \times \phi^V(v_1) = 5/18\). A similar result holds for \(u = 0\) and \(u = 2\). Hence, for this utility function naive-consistent conjectures exist and are given by the product of the marginal probability distributions found above.

**Definition and characterization of Behavioral Equilibrium.** A behavioral equilibrium
requires that players choose strategies that are optimal given their conjectures, where these conjectures are restricted by two sources. The first source is the consistency condition introduced above, which in turn depends on a “behavioral” assumption (i.e. whether the player is naive or sophisticated), and on an informational assumption (i.e. the information policy function). I also allow for a second source by restricting conjectures \( \phi_i(s_i) \) to belong to a set \( M_i(s_i) \subset \Delta(A_{-i} \times V) \). This additional restriction can capture several assumptions about what players believe about the game which cannot necessarily be inferred from actual equilibrium behavior. For example, I appeal to such a restriction when assuming (as in Sections 2 and 4) that sophisticated players know that the economic environment is one of adverse selection.\(^{25}\)

Consider a partition of the set of players into a set of sophisticated and a set of naive players, \( N = N^S \cup N^N \).

**Definition 3 (behavioral equilibrium)** A profile of strategies \( \alpha \) is an \( (\eta, M, N) \) behavioral equilibrium of \( G(\gamma) \) if for every player \( i \in N \) and for every \( s_i \in S_i \) there exists a conjecture \( \phi_i(s_i) \in M_i(s_i) \) such that

i) \( \alpha_i(s_i) \) maximizes expected utility given conjecture \( \phi_i(s_i) \),

ii.a) if \( i \in N^S \), \( \phi_i(s_i) \) is \( \eta_i \)-consistent for \( (\alpha_i(s_i), \alpha_{-i}, \gamma) \)

ii.b) if \( i \in N^N \), \( \phi_i(s_i) \) is \( \eta_i \)-naive-consistent for \( (\alpha_i(s_i), \alpha_{-i}, \gamma) \)

When \( \eta, M^S, \) and \( N \) are understood from the context, I sometimes omit the explicit reference to them. In addition, when considering games where either all players are naive or all players are sophisticated, I refer to a behavioral equilibrium as a naive and sophisticated equilibrium, respectively. The special case of a Nash equilibrium is an \( (\eta, M, N) \) behavioral equilibrium where

(a) \( \eta_i \) is fully revealing for all \( i \in N \); (b) \( M \) does not restrict conjectures, i.e. for all \( i \in N \) and \( s_i \in S_i, M_i(s_i) = \Delta(A_{-i} \times V) \); and (c) all players are sophisticated, i.e. \( N^S = N \).

An attractive feature of behavioral equilibrium is that it can be characterized as the set of fixed points of an appropriate generalization of the best response correspondence. For a player \( i \) who

\(^{25}\)There is an implicit assumption on the type of restrictions that I allow. In particular, I do not allow a conjecture when the signal is \( s_i, \phi_i(s_i) \), to be restricted by a conjecture when the signal is \( s_j \neq s_i, \phi_i(s_j) \); e.g. the assumption that players know a priori that \( \tilde{a}_{-i}, \tilde{v} \) are independent from \( \tilde{s}_i \) cannot be captured, since this requires that \( \phi_j(s_j) = \phi_i(s_j) \) for all \( s_i, s_j \). It would be easy to extend the definition to allow for such restrictions, but I do not do so since in the application given in Section 4 an important condition (to compare the behavior of naive and sophisticated players) is that in equilibrium a player’s beliefs when she receives signal \( s_i \) depends (unidimensionally) only on the action that is played when she receives signal \( s_i \), and not on what she may learn from the actions that she plays when she receives other signals. This assumption may not be so restrictive in games where \( \tilde{s}_i \) is actually correlated with \( \tilde{a}_{-i}, \tilde{v} \), which is the case in games with a common value component where the selection effect studied in this paper arises.
observes signal $s_i \in S_i$, let $\Pi_i(\tilde{a}_i, \alpha_{-i}, s_i, \eta_i, M_i)$ denote the set of all expected profit functions $\pi : A_i \rightarrow \mathbb{R}$ such that for all $a_i \in A_i$, $\pi(a_i) = E_{\phi_i(s_i)}u_i(a_i, \tilde{a}_{-i}, \tilde{v})$ for some conjecture $\phi_i(s_i)$ which (a) belongs to $M_i(s_i)$ and (b) is $\eta_i$-consistent for $(\tilde{a}_i, \alpha_{-i}, \gamma)$ if $i \in N^S$ (and $\eta_i$-naive-consistent for $i \in N^N$). Now let

$$H_i(\alpha_i, \alpha_{-i}, \eta_i, M_i) = \left\{ \tilde{\alpha}_i : \begin{array}{l}
\text{for all } s_i \in S_i, \tilde{\alpha}_i(s_i) \in \arg\max_{a \in A_i} \pi(a) \\
\text{for some } \pi \in \Pi_i(\alpha_i(s_i), \alpha_{-i}, s_i, \eta_i, M_i) \end{array} \right\}$$

be the set of strategies that are optimal given conjectures that (a) belong to $M_i$ and (b) are $\eta_i$-consistent ($\eta_i$-naive-consistent for naive players) for the profile of strategies $\alpha$.

Finally, define the generalized best response correspondence as the set of fixed points of the correspondence $H_i$,

$$BR_i(\alpha_{-i}, \eta_i, M_i) = \{ \alpha_i : \alpha_i \in H_i(\alpha_i, \alpha_{-i}, \eta_i, M_i) \}.$$

**Proposition 2** The following statements are equivalent:

1) $\alpha$ is an $(\eta, M, N)$ behavioral equilibrium.

2) $\alpha \in BR(\alpha, \eta, M, N)$, where $BR = \{ BR_i \}_{i \in N}$.

3) $\alpha \in H(\alpha, \eta, M, N)$, where $H \equiv \{ H_i \}_{i \in N}$.

**Proof.** The proof is immediate from the definitions. 

Proposition 2 implies that the set of equilibria is given by the set of fixed points of a generalized best response correspondence. This characterization is exploited in Section 4, where I first characterize best responses as a function of whether players are naive, sophisticated, or have correct beliefs, and then provide (relatively standard) conditions under which this characterization of best responses may be applied to characterize the set of equilibria. Proposition 2 also implies that the set of equilibria is given by the set of fixed points of $H$. I exploit this more direct approach in the context of some symmetric games introduced in Section 4.

Finally, I introduce two additional pieces of notation that are useful in Section 4. First, an alternative, equivalent definition of consistency directly specifies a true probability distribution $p$ over $A_{-i} \times V$ without any reference to either $\gamma$ or $\alpha_{-i}$. For example, the set $\Pi_i(a_i, \alpha_{-i}, s_i, \eta_i, M_i)$ can be equivalently defined as $\Pi_i(a_i, p, s_i, \eta_i, M_i)$, where $p$ is the distribution of $(\tilde{a}_{-i}, \tilde{v})$ derived from $\gamma$ and $\alpha_{-i}$, conditional on $s_i$, and consistency is now defined with respect to $(a_i, p)$ rather than

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26I explicitly introduce the set $\Pi$ in the definition of $H$ because the class of economies to be introduced in Section 4 can be more naturally defined in terms of properties of $\Pi$ rather than properties of $H$. 

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than \((a_i, \alpha_{-i}, \gamma)\). I use this alternative definition in Section 4 when formalizing the assumption that sophisticated players know that the economic environment is one of monotone selection.

Second, it is useful to consider the naive, sophisticated, and “Nash” versions of a same player \(i\). I use a superscript \(m\) when referring to the sets \(\Pi^m_i, M^m_i, H^m_i, \) and \(BR^m_i\) introduced in this section, where \(m \in \{S, N, NE\}\) stands for the version of player \(i\) whose conjectures are restricted to be consistent, naive-consistent, and correct, respectively.

### 4 Behavioral Equilibrium in Games with Monotone Selection

In this section I apply the equilibrium framework introduced in Section 3 to a class of games with “monotone selection”. I exploit the fixed-point characterization of behavioral equilibrium by first establishing conditions under which generalized best responses can be unambiguously compared (in a direct extension of the trading game in Section 2), and by then using these results to compare the set of equilibria when all players are either naive, sophisticated or have correct beliefs.\(^{27}\)

I begin by stating the main properties that characterize the class of games with monotone selection. These properties are stated in terms of non-primitives for generality and to emphasize the main economic intuition behind the results, which parallels the intuition discussed in Section 2. In Section 4.3, I present a particular sub-class of models which are standard in many settings with adverse selection and provide conditions directly on primitives under which the properties to be stated next are satisfied. But, as shown by some examples in Section 5, the class of “monotone selection” games includes interesting economic environments which do not necessarily fall into this standard sub-class of settings with adverse selection.

For simplicity, I restrict attention to games where the feedback that naive players obtain is such that they have unique (but possibly incorrect) beliefs about their expected utility from following any action.\(^{28}\) Throughout this section, I omit the sets \(M_i^N\) that restrict conjectures of naive players from the notation and let \(A \subset \Lambda\) be a subset of the set of strategy profiles.

**Definition 4** A game has a unique naive profit function given \(\Lambda\) and \(\eta\) if for every \(i \in N\) the set of profit functions \(\Pi_i^N(\tilde{a}_i, \alpha_{-i}, s_i, \eta_i)\) is a singleton for every \(\tilde{a}_i \in A_i, \alpha_{-i} \in \Lambda_{-i}, \) and \(s_i \in S_i\).

Having a unique naive profit function is a property of the game and does not depend on the actual mix of naive and sophisticated players in the game. For a game with a unique naive profit function, I refer to the unique element \(\pi_i^N\) of \(\Pi_i^N\) as the naive profit function. As a reminder,\(^{27}\)

\(^{27}\)For games with strategic complementarities (to be defined later), the results comparing equilibria can be extended to the case where players who are naive, sophisticated, or have correct beliefs coexist. This extension may also provide a parameterization of the level of naiveness in a population that may be exploited in experimental and empirical applications.

\(^{28}\)The results can be extended to the more general case where naive beliefs are not uniquely determined at the expense of additional assumptions.
\( \pi_i^N(a_i; \bar{a}_i, \alpha_{-i}, s_i, \eta_i) \in \mathbb{R} \) is the expected payoff that (a naive version of) player \( i \) with signal \( s_i \in S_i \) expects to obtain from playing \( a_i \in A_i \) when her conjectures (a) belong to \( M_i^N(s_i) \) and (b) are \( \eta \)-naive-consistent for \((\bar{a}_i, \alpha_{-i}, \gamma)\). In contrast to \( \pi_i^N \), let \( \pi_i^{NE}(a_i; \alpha_{-i}, s_i) \) denote the (correct) expected profit of player \( i \) when she obtains signal \( s_i \), plays \( a_i \), and her opponents play the profile of strategies \( \alpha_{-i} \). I refer to \( \pi_i^{NE} \) as the Nash profit function of player \( i \).

The following properties are essential to the results in this section. While the first and second properties are used to compare the sets of Nash and naive equilibria, all four properties are used to compare the sets of naive and sophisticated equilibria. I refer to games that satisfy all of the properties that I define next as games with **monotone selection**.

\[ \eta \text{ reveals payoffs.} \] The information policy \( \eta \) reveals payoffs if \( \eta_i \) reveals payoffs for every \( i \in N \) (see Section 3 for a definition).

**[strict] MSP given \( \Lambda \) and \( \eta \)**. The naive profit function \( \pi_i^N \) satisfies the \( [\text{strict}] \) monotone selection property (MSP) for player \( i \) given \( \Lambda \) and \( \eta \) whenever for every \( s_i \in S_i \), \( \pi_i^N(a_i; \bar{a}_i, \alpha_{-i}, s_i, \eta_i) \) is [increasing] nondecreasing in \( \bar{a}_i \) for every \( a_i \in A_i \) and \( \alpha_{-i} \in \Lambda_{-i} \). A game satisfies the \( [\text{strict}] \) monotone selection property if \( \pi_i^N \) satisfies [strict] MSP for every \( i \in N \).

**kMSP given \( \eta \)**. Sophisticated players \( i \in N^S \) know that \( \pi_i^N \) satisfies MSP. This assumption is formally captured by restricting the set of conjectures of sophisticated players in the following way: for all \( i \in N^S \) and \( s_i \in S_i \), let \( M_i(s_i) \) be the set of all \( \phi \in \Delta(A_{-i} \times V) \) such that \( \Pi_i^N(a_i, \phi, \eta_i) \) is strongly nondecreasing in \( a_i \),\(^{29}\)

**Regularity of \( H^N \) given \( \Lambda \) and \( \eta \)**. A game satisfies regularity of \( H^N \) given \( \Lambda \) and \( \eta \) if for every \( i \in N \) and for every \( \alpha_{-i} \in \Lambda_{-i} \): (i) there is a lowest fixed point of \( H_i^N \), \( \bar{a}_i \in H_i^N(\alpha_i, \alpha_{-i}, \eta_i) \), and (ii) for all \( \alpha_i \neq \bar{a}_i \), there exists \( s_i^* \in S_i \) and \( a_i' \geq a_i \) such that

\[
\pi_i^N(a_i'; \alpha_i(s_i^*), \alpha_{-i}, s_i^*, \eta_i) > \pi_i^N(\bar{a}_i; \alpha_{-i}, s_i^*, \eta_i).
\]

The assumption that \( \eta \) reveals payoffs is a minimal equilibrium requirement that restricts players to have correct beliefs about their expected payoff from playing their equilibrium strategies. The MSP assumption is the main feature of an economy with monotone selection. It requires that higher actions lead to a selection of uncertain outcomes that are "better" for a player. Intuitively, MSP implies that a naive player does not recognize that choosing a higher action would actually provide her with higher payoffs than the ones that she expects.\(^{30}\) The assumption that the game

\(^{29}\)Set \( X \subset \mathbb{R}^K \) is strongly smaller than set \( Y \subset \mathbb{R}^K \) if for every \( x \in X \) and \( y \in Y \), \( x \leq y \). A correspondence \( \Pi \) from \( A_i \) to \( \mathbb{R}^K \) is strongly nondecreasing if for every \( a, a' \in A \), \( a' \leq a \) implies \( \Pi(a') \) is strongly smaller than \( \Pi(a) \).

\(^{30}\)While the MSP property assumes nondecreasing selection, all the results extend (but the direction of the effects is reversed) if selection is nonincreasing.
satisfies kMSP is a refinement on the beliefs of sophisticated players that requires them to know that selection is monotone. Regularity of $H^N$ requires, roughly, that for any strategy $\alpha_i$ being chosen by a naive player that is below the lowest fixed point of $H^N$, there is a signal $s_i^*$ such that she would actually like to choose an action higher than $\alpha_i(s_i^*)$. This is a generalization of the fact that $H^N$ lies above the 45° line for $p < p^N$ in Figure 1b. I show below that there is an economically appealing condition under which this property holds in some environments.

The last three properties are defined for a restricted set of strategies $\Lambda$. In many games, this restriction is inconsequential (e.g. the set $\Lambda$ is the unrestricted set of strategies). However, there are some interesting cases (e.g. games where more than one player has private information and there is a nice ordering on uncertainty, such as affiliation) where MSP usually holds for a restricted set of strategies (e.g. nondecreasing strategies). The results in this section then apply to this restricted set of strategies.\[31\]

I now provide two conditions on $\pi_i^N$ such that regularity of $H^N$ given $\Lambda$ and $\eta$ holds. The first condition requires that there is complementarity between $\hat{a}$ and $a$ in the naive profit function $\pi_i^N(a; \hat{a}, \alpha_{-i}, s_i, \eta_i)$, so that the $H^N$ correspondence is monotone. This is a standard property of some settings with adverse selection. For example, in a lemons market, the lower the price that is offered by the buyer, the worse the quality of objects traded (this is MSP), which in turn induces the buyer to choose even lower prices. Hence, the lower the price offered, the lower the price that is optimal for a naive player whose beliefs are determined by the price that she offers.

As shown in Section 5, there are games that fall outside of the scope of standard settings with adverse selection that satisfy MSP and where the $H^N$ is still nondecreasing, but there are other games (example 6) where this correspondence may not be nondecreasing. For these other cases, it is possible to obtain regularity of $H^N$ by imposing some (technical) assumptions on $\pi_i^N$, and I provide a set of such assumptions in the next lemma.

Both conditions require the action space to be completely ordered, e.g. $A_i \subset \mathbb{R}$, which guarantees, roughly, that any point below the lowest fixed point of $H^N$ lies above the 45 degree line (as in Figure 1 in Section 2). A straightforward way to relax this restriction is to allow a multi-dimensional, partially ordered action space but restrict the selection effect to be “unidimensional”. Such an extension is provided in Appendix B.

Lemma 1 (Regularity of $H^N$) Consider a game with unique naive profit function given $\Lambda$ and $\eta$ such that one of the following properties holds for all $i \in N$, $\alpha_{-i} \in \Lambda_{-i}$, and $s_i \in S_i$:

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31This restriction highlights that the existence or not of monotone selection is in some cases an equilibrium phenomenon. To the extent that the focus of this section is on economies with a monotone selection property, restricting attention to equilibria with such a property may still be relevant. In addition, in the example described above it is common for researchers to restrict attention to equilibria in the class of strategies which are nondecreasing because (i) existence can be established in such a class and little is known outside this class, and (ii) strategies outside this class are sometimes contrived and counterintuitive.

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(i) The set \( A_i \subset \mathbb{R} \) is nonempty and finite, and \( \pi_i^N (a_i; \widehat{a}_i, \alpha_{-i}, s_i, \eta_i) \) is single-crossing\(^{32}\) in \((a_i; \widehat{a}_i)\).

(ii) The set \( A_i \subset \mathbb{R} \) is nonempty, compact, and convex, and \( \pi_i^N (a_i; \widehat{a}_i, \alpha_{-i}, s_i, \eta_i) \) is continuous in \((a_i, \widehat{a}_i)\) and strictly quasiconcave in \(a_i\) for each \( \widehat{a}_i \).\(^{33}\)

Then \( H^N \) is regular given \( \Lambda \) and \( \eta \).

**Proof.** See Appendix A. ■

### 4.1 Comparing Generalized Best Responses

The following result compares generalized best responses of players who are naive, sophisticated, or have correct beliefs. This result is of interest in itself, since in monopoly games (as in Section 2) best responses correspond to equilibrium behavior. In addition, this result is used in the next sub-section to compare equilibrium behavior when there is more than one strategic player.

The partial order \( \geq \) used in this paper is the standard product order, e.g. \( \alpha_i \geq \alpha'_i \) if \( \alpha_i(s_i) \geq \alpha'_i(s_i) \) for all \( s_i \in S_i \).\(^{34}\)

**Lemma 2 (Comparing Generalized Best Responses)** Consider a game with unique naive profit function given \( \Lambda \) and \( \eta \) where \( A_i \) is a nonempty, finite set, \( \eta \) reveals payoffs, and the game satisfies MSP given \( \Lambda \) and \( \eta \). Then the following statements hold for every player \( i \in N \) and every \( \alpha_{-i} \in \Lambda_{-i} \):

1. For any \( \alpha_i^N \in BR_i^N(\alpha_{-i}, \eta_i) \) and \( \alpha_i^{NE} \in BR_i^{NE}(\alpha_{-i}, \eta_i) \), \( \max\{\alpha_i^N, \alpha_i^{NE}\} \in BR_i^{NE}(\alpha_{-i}, \eta_i) \). (If MSP is strict, then \( \alpha_i^N \leq \alpha_i^{NE} \).)

2. Suppose in addition that kMSP and regularity of \( H^N \) given \( \Lambda \) and \( \eta \) hold. Let \( \alpha_i^S \) be the lowest element of \( BR_i^S(\alpha_{-i}, \eta_i) \). Then \( \alpha_i^S \geq \alpha_i^N \) for all \( \alpha_i^S \in BR_i^S(\alpha_{-i}, \eta_i, M_i^S) \).

**Proof.** See Appendix A. ■

The first statement implies that, given a fixed profile of other players’ strategies, for every naive best response there is always a Nash best response that is higher (and if MSP is strict, then any naive best response is lower than any Nash best response). The second statement says that every sophisticated best response is higher than the lowest naive best response.

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\(^{32}\)See Appendix A for the definition of single-crossing.

\(^{33}\)Note that these assumptions fall outside of the class of finite economies introduced in Section 3. As remarked earlier, it is possible to extend the definitions in Section 3 to such non-finite economies.

\(^{34}\)I omit proving the extension of the result in Proposition 1 that Nash equilibrium is an upper bound for the set of sophisticated equilibria, which holds under specific restrictions on the beliefs of a sophisticated player that are better motivated on a case by case basis.
4.2 Comparing Equilibria

In this section I apply the characterization of best responses obtained in Lemma 2 to show that the sets of equilibria where all players are either naive, sophisticated, or have correct beliefs can be unambiguously compared in an important class of settings. I exploit the techniques of the literature on monotone comparative statics (e.g. Milgrom and Roberts, 1990), with only some slight modifications. I compare different equilibrium concepts, rather than a parameterized model under the assumption that a Nash equilibrium is always played. Also, the sets of best responses are not ordered by the standard strong set order relation, so that the standard proofs are slightly modified in some cases. In addition, some of the results require that the assumption of strategic complements is satisfied both by the standard “Nash” game where players have correct beliefs, and by the naive version of the game, where beliefs are only required to be naive-consistent.

Regularity of $H^N$ is obtained by restricting attention to games where the action space (but not the strategy space) is either completely ordered, e.g. $A_i \subseteq \mathbb{R}$ for every $i$, or where it is multidimensional but the selection effect is “unidimensional” in the sense that it is driven by a completely ordered component of the action space. Appendix B provides the formal definition of such a property and proves Theorem 1 in that context.

I establish the relationship between Nash, sophisticated, and naive equilibria for each of the following settings.

**Monopoly.** (i) Player $i$ is the only strategic player and all other players play a fixed profile of strategies $\alpha_{-i} \in \Lambda_{-i}$. In addition, at least one of the following conditions hold: (ii) $A_i \subseteq \mathbb{R}$ is nonempty and finite (alternatively, $A_i \subseteq \mathbb{R}^k$ is a finite lattice but the selection effect is “unidimensional”; see Appendix B), and for all $s_i \in S_i$ and $\alpha_{-i} \in \Lambda_{-i}$, $\pi^N_i(a_i; \tilde{a}_i, \alpha_{-i}, s_i, \eta_i)$ is single-crossing in $(a_i, \tilde{a}_i)$; or (ii’) for all $i$, $A_i \subseteq \mathbb{R}$, where $A_i$ is nonempty, compact and convex, and for all $s_i \in S_i$ and $\alpha_{-i} \in \Lambda_{-i}$, $\pi^N_i(a_i; \tilde{a}_i, \alpha_{-i}, s_i, \eta_i)$ is continuous in $(a_i, \tilde{a}_i)$ and strictly quasiconcave in $a_i$.

**Strategic Complementarities.** (i) For every $i \in N$, $A_i \subseteq \mathbb{R}$ is nonempty and finite (alternatively, $A_i \subseteq \mathbb{R}^k$ is a finite lattice but the selection effect is “unidimensional”; see Appendix B); (ii) for every $i \in N$, for every signal $s_i \in S_i$, and for every $\alpha_{-i} \in \Lambda_{-i}$: a) $\pi^N_i(a_i; \alpha_{-i}, s_i)$ is single-crossing in $(a_i, \alpha_{-i})$; b) $\pi^N_i(a_i; \tilde{a}_i, \alpha_{-i}, s_i, \eta_i)$ is single-crossing both in $(a_i, \tilde{a}_i)$ and in $(a_i, \alpha_{-i})$.

**Symmetric game with completely ordered strategy space.** This class of settings extends the previous monopoly setting to allow for competition between players that do not have private information: i) For every $i \in N$ : $\pi^N_i = \pi^N$, $\pi^{NE}_i = \pi^{NE}$, $\eta_i = \eta$, $S_i = \{s\}$ is a singleton, and $A_i = A \subseteq \mathbb{R}$, where $A$ is nonempty, compact, and convex; ii) $\pi^{NE}(a_i; a_{-i})$ is continuous in $(a_i, a_{-i})$ and strictly quasiconcave in $a_i$; iii) $\pi^N(a_i; \tilde{a_i}, a_{-i})$ is continuous in $(a_i, \tilde{a}_i, a_{-i})$ and strictly quasiconcave in $a_i$. 

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When the economy is a monopoly, the set of equilibria is given by the set of generalized best responses, and therefore the results in Lemma 2 apply directly. For the other two classes of games, the relationship between the set of equilibria when all players are either naive, sophisticated, or have correct beliefs is the following.

**Theorem 1 (Comparing Equilibria)** Consider a game with unique naive profit function given \( \Lambda \) and \( \eta \). Suppose that the following properties hold: i) \( \eta \) reveals payoffs, ii) MSP given \( \Lambda \) and \( \eta \), iii) knowledge of MSP given \( \eta \), iv) \( \Lambda \) is a lattice and \( BR^m(\alpha) \subset \Lambda \) for all \( \alpha \in \Lambda \), \( m \in \{N, NE, S\} \). Then the following statements hold, where the sets of equilibria are restricted to strategy profiles that belong to \( \Lambda \).

1. In a game with strategic complementarities:

   (a) The sets of Nash, naive, and sophisticated equilibria are each nonempty; and the sets of Nash and naive equilibria each have lowest and highest elements.

   (b) The highest naive equilibrium is (weakly) lower than the highest Nash equilibrium. (If MSP is strict, then in addition the lowest naive equilibrium is (weakly) lower than the lowest Nash equilibrium).

   (c) Every sophisticated equilibrium is (weakly) higher than the lowest naive equilibrium.

2. In a symmetric game with completely ordered strategy space:

   (a) The sets of symmetric naive, sophisticated, and Nash equilibria are nonempty.

   (b) The highest symmetric naive equilibrium is (weakly) lower than the highest Nash equilibrium. (If MSP is strict, then in addition the lowest symmetric naive equilibrium is (weakly) lower than the lowest symmetric Nash equilibrium).

   (c) Every symmetric sophisticated equilibrium is (weakly) higher than the lowest symmetric naive equilibrium.

**Proof.** See Appendix A.

In addition, there are some interesting games with monotone selection that do not belong to any of the previous settings. I show in Appendix C that a symmetric first-price auction (and a

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\[ ^{35}\text{The assumption that } BR^m(\alpha) \subset \Lambda \text{ for all } \alpha \in \Lambda \text{ implies that equilibria characterized in Theorem 1, while restricted to a certain subset of strategies, are still equilibria of the unrestricted game.} \]
symmetric $k$-th unit auction to some extent as well) satisfies a property under which the results stated in part 2 of Theorem 1 hold.

4.3 A Class of Games with Monotone (Adverse) Selection

I now present a sub-class of models which are standard in many settings with adverse selection and provide conditions directly on primitives under which the properties of games with monotone selection introduced earlier are satisfied. Consider the framework in Section 3 where the utility function is of the special form

$$u_i(a; \theta, t) = \begin{cases} u_i^*(a_i; \theta) & \text{for } (a_{-i}, t) \in \Phi_i(a_i) \\ 0 & \text{otherwise} \end{cases},$$

where $v = (\theta, t) \in \Theta \times T$ represents payoff uncertainty, $a_i \in A_i \subset \mathbb{R}$ is player $i$'s action, and $A_i$ is nonempty and finite. The interpretation is that there are two possible outcomes, each of which occurs depending on players' actions and a random variable $\tilde{t}$. The utility of the first consequence is $u_i^*(a_i; \theta)$ and the utility of the other consequence is normalized to zero for convenience (in particular, it does not depend on $\theta$).

Define

$$\varphi_i(a_i; \alpha_{-i}, s_i) \equiv \Pr((\alpha_{-i}(\tilde{s}_{-i}), \tilde{t}) \in \Phi_i(a_i) \mid \tilde{s}_i = s_i)$$

and

$$\xi_i(a_i, \tilde{a}_i; \alpha_{-i}, s_i) \equiv E \left( u_i^*(a_i, \tilde{\theta}) \mid (\alpha_{-i}(\tilde{s}_{-i}), \tilde{t}) \in \Phi_i(\tilde{a}_i), \ \tilde{s}_i = s_i \right).$$

The correct (i.e. Nash) expected utility for player $i \in N$ with signal $s_i \in S_i$ when opponents play $\alpha_{-i}$ is then

$$\pi_i^{NE}(a_i; \alpha_{-i}, s_i) = \begin{cases} \varphi_i(a_i; \alpha_{-i}, s_i) \times \xi_i(a_i, a_i; \alpha_{-i}, s_i) & \text{for } a_i \text{ such that } \varphi_i(a_i; \alpha_{-i}, s_i) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Consider the following two sets of assumptions. The first set lists assumptions on the economic environment while the second set places restrictions on players’ equilibrium beliefs.

**Assumptions on fundamentals.**

**F1.** $\tilde{t}$ is independent of $(\tilde{\theta}, \tilde{s})$; **F2.** $(\tilde{\theta}, \tilde{s})$ are affiliated; **F3.** $u_i^*$ is nondecreasing in $\theta$ for all $i \in N$; **F4.** $\Phi_i$ is nondecreasing in the strong set order for all $i \in N$;

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36 The results can be extended to economies with multidimensional action spaces as long as the selection effect is unidimensional, as defined in Appendix B.

37 The setting can be generalized by letting $u_i^*$ depend on opponents’ actions $a_{-i}$ in a separable way from $\theta$, i.e. $u_i^*(a, \theta) = u_i^I(a_i, \theta) + u_i^II(a)$. An example is the $k$-th unit auction presented in Section 5. In addition, the payoff from the second consequence may be a function of players’ actions, as in the effort model in Section 5.

38 Assumption F1 is not actually needed for Proposition 3 – it can be replaced with the assumption that $(\tilde{t}, \tilde{\theta}, \tilde{s})$ are affiliated. The important assumption is B1, but then F1 guarantees that no incorrect assumptions are made on sophisticated players’ beliefs.
Assumptions on beliefs. B1. Players believe (correctly) that $t$ is independent of $(\theta, \tilde{a}_{-i})$, i.e. conjectures are such that $\phi_i(a_{-i}, \theta, t) = \phi_i(a_{-i}, \theta) \times \phi_i(t)$ for all $i \in N$, all $(a_{-i}, \theta, t) \in A_{-i} \times \Theta \times T$; B2. every player $i \in N$ has correct beliefs about the probability that each of two consequences occurs, $\varphi_i(a_i; \alpha_{-i}, s_i)$, for all $a_i, \alpha_{-i}$, and $s_i \in S_i$; B3. $\eta$ only reveals payoffs.\footnote{Assumption B2 can either be incorporated as a restriction through $M$ or as a restriction on the information policy $\eta$. For an example of the latter, B2 holds if there is no random variable $t$ and $\eta$ reveals actions. The assumption that, in addition to B2, only payoffs are revealed is important since when players obtain feedback about $\theta$ irrespective of the outcome, naive-consistent beliefs may not exist in general.}

Under these assumptions, the next result shows that the environment has a unique naive profit function and satisfies MSP and regularity of $H$ once players are restricted to choose nondecreasing strategies. There is one caveat here. Consider the naive beliefs of a player who receives signal $s_i$, plays action $\tilde{a}_i$ and whose opponents play the profile of strategies $\alpha_{-i}$. If $\tilde{a}_i$ is such that $\varphi_i(\tilde{a}_i; \alpha_{-i}, s_i) = 0$, then such a player always obtains a payoff of zero, and never obtains feedback about $\theta$. Hence, unless $\Theta$ is a singleton, there may not be a unique belief about profits from choosing other actions. In some examples, such as in the finite version of the trading game presented in Proposition 1, there is no such action $\tilde{a}_i$ and players always get some feedback. In other examples, such as in the continuous version of the trading game in Section 2, a refinement may be necessary to establish uniqueness of naive-consistent beliefs.\footnote{As discussed in footnote 15, the existence of a “no-trade” equilibrium is not an issue in itself, but makes the results hold trivially, when in fact the results do have important content.}

**Proposition 3** Consider the environment introduced in this subsection. Suppose that assumptions B1-B3 hold and let $\Lambda$ be the set of strategy profiles which are nondecreasing, which is a lattice.

1. For any $i \in N$, $s_i \in S_i$, $\alpha_{-i} \in A_{-i}^{S_{-i}}$, and $\tilde{a}_i \in A_i$ such that $\varphi_i(\tilde{a}_i; \alpha_{-i}, s_i) > 0$, the unique naive profit function is given by
   \[
   \pi^N_i(a_i; \tilde{a}_i, \alpha_{-i}, s, \eta_i) = \varphi_i(a_i; \alpha_{-i}, s_i) \times \xi_i(a_i, \tilde{a}_i; \alpha_{-i}, s_i).
   \]

2. Suppose that in addition to B1-B3, assumptions F1-F4 hold. Then MSP given $\Lambda, \eta$ holds for the naive profit function restricted to $\tilde{a}_i$ such that $\varphi_i(\tilde{a}_i; \alpha_{-i}, s_i) > 0$.

3. Suppose that in addition to F1-F4 and B1-B3, assumptions F5-F6 hold. Then regularity of $H^N$ given $\Lambda$ and $\eta$ holds for the naive profit function restricted to $\tilde{a}_i$ such that $\varphi_i(\tilde{a}_i; \alpha_{-i}, s_i) > 0$.\footnote{\Phi_i is nondecreasing in the inclusion set order if $\Phi_i(a'_i) \subseteq \Phi_i(a_i)$ whenever $a'_i \leq a_i$. This condition says that the probability of the first consequence is nondecreasing in a player’s own action.}
Proof. See Appendix A.

If in addition to the assumptions stated in Proposition 3 sophisticated players know that MSP holds, then generalized best responses can be compared as in Lemma 2. If it is also the case that \( BR^m(\Lambda) \subset \Lambda \) and that the game being studied corresponds to one of the settings defined in Section 4.2, Theorem 1 can then be applied. Several examples are provided in Section 5.

5 Additional examples

In this section I present additional examples of games with monotone selection to illustrate the applicability of the results obtained in the paper. The first four examples correspond to the setup in Section 4.3, example 5 is a slight generalization of that setup, and example 6 does not fit into that setup. Examples 1 to 3 are standard in applications of adverse selection and emphasize that the selection problem is exacerbated in the presence of naive players. In contrast, examples 4 to 6 illustrate that selection effects may be of importance in settings where such effects have usually been ignored, and that it is possible for all players to be better off from being naive (example 5).

(1) Monopoly/monopsony. The type of example in Section 2 is ubiquitous in applications of adverse selection to insurance, labor, financial, credit, and used goods markets, as suggested by substituting the names buyer/seller with insurer/insuree, firm/worker, market maker/informed trader, venture capitalist/entrepreneur, etc. In many of these examples, a firm faces a given supply or demand, and the contract that it offers determines not only the number of actual customers, but also the type of customers (e.g. a monopoly selling insurance to a population of agents that differ in their privately-known risk-types). Two differences may arise in these supply/demand settings with respect to the trading game in Section 2.

First, the assumption that ask prices are revealed is replaced with the assumption that firms have (a priori) knowledge about market demand/supply, which may seem reasonable given that some firms spend considerable resources in obtaining reliable estimates of the number of customers that would conduct trade at different prices. However, even sophisticated firms may have incorrect beliefs about the relationship between willingness to pay and private characteristics of customers. The results in Proposition 1 go through, except that Nash equilibrium is no longer a bound for sophisticated equilibrium. The reason is that a firm may not know how much a customer of a specific type would have been willing to pay to conduct business with the firm, but rather only knows if such a customer wishes to conduct business at the equilibrium price.\(^{42}\)

\(^{42}\)There may be contexts where firms have enough experience with their customers to obtain this information, and the result that Nash is a bound for sophisticated behavior holds. In any case, this assumption is still weaker than assuming that firms also have this knowledge about their non-customers (i.e. correct beliefs), which would lead to the non-existence of naive equilibrium and to equivalence between the sets of sophisticated and Nash equilibria.
Second, the selection problem may be monotone decreasing. For example, in an insurance context the higher the price of insurance, the lower the “quality” of the customers that the firm obtains, which in turn provides incentives to increase prices even further, implying that the $H^N$ correspondence is still monotone nondecreasing in this setting. It can be shown that the results in Proposition 1 are then reversed, e.g. every equilibrium with naive firms is either above the highest NE or is a NE itself, which confirms that the adverse selection problem is exacerbated in the presence of naive players irrespective of whether the selection effect is monotone nondecreasing or nonincreasing.

(2) Duopoly with adverse selection. The previous examples can be extended by incorporating competition in the less-informed side of the market. Whether selection remains monotone now depends on the type of customers that are “stolen” from the competition. I present conditions under which MSP still holds, and show that while the adverse selection problem is still exacerbated with naive players, less-informed players may be better off due to softened competition.

For concreteness, I focus on the case where two firms compete to attract workers by simultaneously offering wages $w_i$. Each worker has private information $s$ about her (home) productivity and prefers to work for firm $i$ rather than stay home whenever $s \leq g_i(w_i)$, where $g$ is increasing. In addition, a worker prefers working for firm 1 rather than firm 2 if and only if $h(w_1, w_2) \geq t$, where $h$ is nondecreasing in $w_1$ and nonincreasing in $w_2$ (so that higher values of $t$ indicate a higher preference for working for firm 2). If the worker works for firm $i$, then firm $i$ makes profits $u_i^*(\theta - w_i)$, where $\theta$ is the worker’s productivity at work. Profits are zero when a worker is not hired. The setup fits the framework in Section 4.3, and I make the same assumptions: (players know that) $\bar{t}$ is independent of $(\bar{s}, \bar{\theta})$ (F1 and B1), $(\bar{\theta}, \bar{s})$ are affiliated (F2), $u_i^*$ is nondecreasing in $\theta$ (F3), $u_i^*$ is supermodular in $(\theta, w_i)$, (F6), $\eta$ only reveals payoffs (B2), and firms know the supply of workers in the economy (B3).

In addition, these assumptions imply that the probability of hiring a worker is nondecreasing in the offered wage (F5) and that $\Phi_i$ is nondecreasing in the strong set order (F4). To see the latter, note that for a fixed $w = (w_1, w_2)$, the set of $(s, t)$ for which a worker accepts employment at firm 1 is a product set. As firm 1 increases its wage from $w_1'$ to $w_1$, the set of workers $(s, t)$ that accept employment is increasing in the strong set order. For firm 2, on the other hand, the reverse holds: the set of workers $(s, t)$ that accept employment at firm 2 is nonincreasing in the strong set order. However, by letting $t' = -t$ and $h'(\cdot) = -h(\cdot)$, firm 2 is preferred to firm 1 whenever $t' \leq h'(w_1, w_2)$, where $h'$ is nondecreasing in $w_2$ and nonincreasing in $w_1$. Therefore the set of workers $(s, t')$ that accept employment is nondecreasing in the strong set order, and the proof of Proposition 3 holds since the fact that $\bar{t}$ is independent of $(\bar{s}, \bar{\theta})$ implies that $\bar{t}'$ is also independent of $(\bar{s}, \bar{\theta})$.

Under the above assumptions, Proposition 3 is applicable and the sets of generalized best responses can be compared as in Lemma 2. Equilibria can be compared as in Theorem 1 if the game is symmetric (so that in the definitions above nothing depends on $i$ and the random variable $\bar{t}$ is
appropriately symmetric). If this is the case and the game is nicely behaved (so that, e.g. there are no discontinuities in best responses), then both wages and the quality of hired workers are (roughly) lower when firms are naive. Alternatively, this result extends to an asymmetric setting as long as the game satisfies strategic complementarities, which may be a natural assumption in this context. A noteworthy aspect of this example is that while on average naive firms hire worse candidates (exacerbating the adverse selection problem), the presence of naive firms also softens competition. There exist cases where the second effect dominates and, unlike the monopoly game, profits are higher when firms are naive.

(3) **Symmetric first price auctions and symmetric $k$th unit auctions.** Consider an auction where bidders have private information about the value of an object that would be relevant for other bidders’ assessment of the value of the object (i.e. the object has a “common value” component). By increasing her bid, a bidder would win objects that she would have otherwise not won, and this event occurs when the highest opponent bid is between her original bid and her new, increased bid. Under standard assumptions these objects are likely to be of a higher average quality than the objects she wins at her original bid. Hence, MSP holds and, in turn, this induces bidders to choose even higher bids. In Appendix C, I show that Proposition 3 is directly applicable to both the symmetric first-price auction and the symmetric $k$-th unit auction under assumptions about fundamentals that are standard in the auction literature and under the assumption that players observe their own payoffs and the auctioneer reveals all bids at the end of each auction. In addition, I show that such auctions satisfy a property that makes the result in Theorem 1 hold: in a symmetric equilibrium bidding is less aggressive when all bidders are naive compared to the case when all bidders are sophisticated or all bidders have correct beliefs.

(4) **Team effort.** When a member of a team decides how much effort to exert, she probably takes into account how her effort level would affect the probability of success. However, she may ignore that, since other players choose effort based on private information about the value of a successful outcome, a change in her effort level would also affect the expected value of a successful outcome. To study the effects of being naive in this context, suppose that each member $i \in N$ of a team simultaneously decides to exert a level of effort $x_i$ at a cost $c_i(x_i)$. There are two possible outcomes: either the team succeeds and each player obtains the realization of a random variable $\theta$, or they fail and obtain nothing. Success occurs whenever $\min_{i \in N} x_i \geq t$, where $t$ is the realization of a random variable $\tilde{t}$ that is independent of $\theta$. Hence, efforts are complementary, so that it is possible to show that the “Nash” game has strategic complementarities. Before choosing effort, players obtain private information $s_i$, which is the realization of $\tilde{s}_i$, and $\tilde{s}_1, \ldots, \tilde{s}_N, \theta$ are assumed

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43 Conditions under which a standard product-differentiated oligopoly game satisfies strategic complementarities are well known (Vives, 1999), but less is known about oligopoly games under adverse selection. In the firm-worker example, if $p(w_1, w_2)$ denotes the probability that a worker prefers to work for firm 1 rather than for firm 2 when the offered wages are $(w_1, w_2)$, then a sufficient condition for both the Nash and naive games to satisfy strategic complementarities is that $p$ and $1 - p$ are log-supermodular in $w$. (This assumption is satisfied, for example, if $p$ is differentiable, $\frac{\partial^2 p}{\partial w_1 \partial w_2} = 0$, $\frac{\partial p}{\partial w_1} \geq 0$, and $\frac{\partial p}{\partial w_2} \leq 0$.)
to be affiliated. I restrict attention to strategies that are nondecreasing. In addition, I assume that players observe their own payoffs and that in equilibrium they know the probability of success for each of the effort levels they may exert. However, there is a potential selection problem since players only obtain feedback about the value of success when a success occurs.

The setup is essentially the one in Section 4.3 and it is possible to show that under the assumption that success occurs whenever \( \min_{i \in N} x_i \geq t \), the set of \((x_{-i}, t)\) such that success occurs is increasing in the strong set order in \(x_i\).\(^{44}\) Hence, due to affiliation and nondecreasing strategies, the expected value of success conditional on success taking place is increasing in own effort \(x_i\). As a result, the game satisfies MSP. Intuitively, since it is the minimum level of effort that matters for success, a player has no influence on success when (the realization of) the minimum of other players’ effort level is lower than her effort, but does increase the likelihood of success when the minimum of other players’ effort is higher. Hence, by increasing effort a player not only increases the probability of success, but also makes it more likely that success occurs for higher values of other players’ efforts, which are indicative of higher realizations of \(\bar{\theta}\).

It is straightforward to show that the other assumptions in Section 4.3 also hold, so that since the Nash game has strategic complementarities, Theorem 1 can be applied to conclude that a team of naive players exerts less effort than in a Nash equilibrium. Finally, sophisticated and naive equilibria can be compared if the game is symmetric and best responses are “nicely” behaved, or if the game is asymmetric but the naive version of the game also has strategic complementarities. The naive game does not necessarily satisfy strategic complementarities without further assumptions. The reason is that when other players exert more effort, the marginal effect of effort on the probability of success may increase, but, on the other hand, naive players fail to realize that increasing effort would also have a positive effect on the rewards from effort.

(5) Preemption Game. Consider a game where the reward goes to the player(s) who move first, but where players would benefit from coordinating to move later, once some uncertainty about the benefits of taking the action is realized. Suppose that players have private information about this uncertainty before having to decide whether to act now or later. A player who decides to act now but evaluates whether she should rather wait must account for two effects: (i) the risk of being preempted, and (ii) the fact that if she waits she will only get to act later if some other firm has not preempted her, which is more likely to happen when the benefits from acting are lower. Naive players ignore the second effect.

To study the effects of being naive in this context, consider two players that simultaneously

\(^{44}\)Two important things should be noticed about the choice of \(\min_{i \in N} x_i\). First, if effort is costly then the lowest feasible effort level is always an equilibrium with players who are either naive, sophisticated, or have correct beliefs, so that Theorem 1 is not very informative in this case. But it is straightforward to slightly modify the functional form assumption to rule out the previous equilibrium. Second, a (sufficiently) different functional form assumption may give rise to either ambiguous results or to results in the opposite direction, implying that whether a team of naive players would put less or more effort in equilibrium depends on the “technology” through which effort is translated into success.
choose whether to act now or to wait. If they act now they pay a cost $c$ and obtain $\theta$, which is the realization of a random variable $\tilde{\theta}$. If both players wait, they get to observe the realization of $\tilde{\theta}$ and choose whether to invest $c$ to get $\theta$ or not. A player who waits gets 0 and does not get to observe the realization of $\tilde{\theta}$ if the other player acts first.\footnote{These extreme assumptions can be relaxed but capture a market where a firm that acts first has a strong first mover advantage and distorts what a firm that waits can learn about $\tilde{\theta}$ in a later period.} Payoffs are discounted by the factor $\delta \leq 1$. Before the decision to act is taken, players observe a private signal $s_i$ about the value of $\tilde{\theta}$. I assume that $\tilde{s}$ and $\tilde{\theta}$ are affiliated, and that players observe their payoffs. In this game, a player’s decision to either act or wait determines the sample of $\theta$’s that will be observed.

When attention is restricted to equilibria where players use cutoff strategies $s^e_i$ (so that player $i$ waits when $s_i \leq s^e_i$ and acts when $s_i > s^e_i$), the naive profit function is

$$\pi^N_i(A, A, s^e_i, s_i) = E(\tilde{\theta} | s_i) - c$$
$$\pi^N_i(W, W, s^e_j, s_i) = E(\max\{\tilde{\theta} - c, 0\} | s_i, \tilde{s}_j \leq s^e_j) \Pr(\tilde{s}_j \leq s^e_j | s_i)$$
$$\pi^N_i(A, W) = E(\tilde{\theta} | s_i, \tilde{s}_j \leq s^e_j) - c$$
$$\pi^N_i(W, A) = E(\max\{\tilde{\theta} - c, 0\} | s_i) \Pr(\tilde{s}_j \leq s^e_j | s_i),$$

where $\pi^N_i(x, y, s^e_j, s_i)$ represents perceived profits from doing $x$ when player $i$ actually plays $y$, her signal is $s_i$, and the opponent plays $s^e_j$. From affiliation of $\tilde{s}, \tilde{\theta}$ it follows that $\pi^N_i(A, A, s^e_j, s_i) \geq \pi^N_i(A, W, s^e_j, s_i)$ and $\pi^N_i(W, A, s^e_j, s_i) \geq \pi^N_i(W, W, s^e_j, s_i)$ for every $i, s_i, s^e_j$. Hence, the economy satisfies the monotone selection property (MSP) when strategies are restricted to be nondecreasing – here, the order of the action space is such that $A$ is higher than $W$. A naive player ignores that the expected value of the sample of realized $\theta$’s is lower when she waits, since in those cases she only observes $\tilde{\theta}$ when the other player receives a low signal and also decides to wait.

In addition, it is possible to show that i) $\pi^N_i$ is single-crossing, and therefore $H$ is regular, ii) the Nash game is one of strategic complementarities, and iii) best responses to nondecreasing strategies are also nondecreasing. It then follows from Theorem 1 that if the game is continuous and symmetric (since $s^e \in \mathbb{R}$ and therefore the class of symmetric games with completely ordered strategy space from Section 4 can be invoked) or if the naive version of the game also satisfies strategic complementarities, then (roughly) in equilibrium the presence of naive players results in players waiting more often.\footnote{Once again, the “naive game” need not have strategic complementarities, despite the Nash game always having this property: When the opponent waits more often, due to the preemption effect, a naive player would like to wait more often. But on the other hand, a naive player who waits now believes that $\tilde{\theta}$ has a more favorable distribution (since opponents with higher signals are deciding to wait). This second effect, which is not present under the Nash assumption that players have correct beliefs, encourages players to act rather than wait. As stated in footnote 52, the comparison between naive and Nash equilibria in Theorem 1 still holds if the game has no naive strategic complementarities as long as MSP is strict.}

(6) Duopoly with demand uncertainty. A firm’s belief about uncertain market conditions
may depend on observable market outcomes, such as its own realized sales, which in turn depend on a firm’s own price and on competitors’ prices. A firm that ignores that competitors choose prices based on their own private information about market conditions will fail to realize that its beliefs about market conditions comes from a selected sample.

For illustrative purposes, consider a duopoly example with product differentiation and price competition where costs are zero and the demand of each firm is given by \( D_i(p) = 1 - (1 - \theta)p_i + \theta p_j \), where \( \theta \) is the realization of a random variable \( \tilde{\theta} \) that represents common demand uncertainty. Firm 2 knows the realization of \( \theta \) and firm 1 has no private information. It is easy to see that in this case the best response of firm 1 is both increasing in \( \theta \) and in firm 1’s strategy \( p_1 \). In particular, attention can be restricted without loss of generality to strategies of firm 2 that are increasing in his private information \( \theta \). Suppose that firm 1 observes payoffs (i.e. she observes quantity demanded at the price she chooses) and the average price offered by firm 2.\(^{47}\) These assumptions imply that firm 1 has correct beliefs about \( ED_1 \) and \( E\tilde{p}_2 \), so that her beliefs about expected demand when she is naive and offers price \( \hat{p}_1 \) in (a hypothetical) equilibrium are:

\[
E_{\hat{p}_1} D_1 = 1 - (1 - E_{\hat{p}_1} \tilde{\theta}) \times \hat{p}_1 + E_{\hat{p}_1} \tilde{\theta} \times E\tilde{p}_2 = \ED_1 = 1 - (1 - E\tilde{\theta})\hat{p}_1 + E\tilde{\theta} \times E\tilde{p}_2 + Cov(\tilde{\theta}, \tilde{p}_2),
\]

where a naive firm 1 incorrectly believes that \( \tilde{\theta} \) and \( \tilde{p}_2 \) are independent random variables. From the above equation it follows that a naive firm 1 that makes an offer \( \hat{p}_1 \) believes the expected value of \( \tilde{\theta} \) to be \( E_{\hat{p}_1} \tilde{\theta} = \frac{Cov(\tilde{\theta}, \tilde{p}_2) + \hat{p}_1 E\tilde{\theta} + E\tilde{\theta} E\tilde{p}_2}{\hat{p}_1 + E\tilde{p}_2} \). Since \( p_2(\theta) \) is increasing in \( \theta \), then \( Cov(\tilde{\theta}, \tilde{p}_2) > 0 \) and therefore \( E_{\hat{p}_1} \tilde{\theta} \) is decreasing in \( \hat{p}_1 \). This implies that \( E_{\hat{p}_1} D_1 \) is decreasing in \( \hat{p}_1 \), which in turn implies that \( \pi_1^N(p_1, \hat{p}_1, \tilde{p}_2) = p_1 \times E_{\hat{p}_1} D_1 \) is decreasing in \( \hat{p}_1 \). Hence, this economy satisfies decreasing MSP: the higher the price chosen by firm 1, the lower is firm 1’s expectation about demand.

Intuitively, a naive player does not realize that firm 2 chooses higher prices when \( \theta \) takes on higher values, so that the part of the demand that corresponds to this effect (the covariance between \( \tilde{\theta} \) and \( \tilde{p}_2 \)) must be attributed to an inflated \( \tilde{\theta} \).\(^{48}\) Since the covariance term does not interact with \( p_1 \), but the effect of \( \theta \) on demand is higher for higher values of \( p_1 \), then inflation of \( \tilde{\theta} \) must be lower for larger values of \( p_1 \), leading to a decreasing MSP.

In addition, since firm 1’s optimal choice of price is increasing in her expectations about demand, it follows that the optimal choice of price is decreasing in \( \hat{p}_1 \). Hence, the \( H \) correspondence defined in Section 3 is decreasing, and (Tarski’s) fixed-point theorem cannot be used to show that \( H \) is regular. However, under continuity assumptions it is possible to show that \( H \) is indeed regular. A

\(^{47}\)The assumption that firms do not observe the joint realization of competitors’ prices and demand uncertainty may be reasonable in settings where either a firm observes all prices charged by its competitors but cannot relate them to a specific period of time, and therefore to a specific realization of \( \theta \), or where a firm only observes some aggregate statistic about the prices of its competitors.

\(^{48}\)This is in contrast to a cursed firm, which has correct beliefs about \( E\tilde{\theta} \) and \( E\tilde{p}_2 \), and ignorance of correlation between \( \tilde{\theta} \) and \( \tilde{p}_2 \) translates into a failure to account for a portion of expected demand, and therefore in lower prices. Of course, a cursed firm then has incorrect beliefs about her expected demand at the equilibrium price.
version of Theorem 1 for decreasing MSP then implies that (roughly) prices are higher when firm 1 is naive compared either to a Nash equilibrium or to an equilibrium where firm 1 is sophisticated.

6 Conclusion and further research

In this paper I present a framework to study equilibrium behavior in the presence of players who fail to account for the informational content of other players’ strategies. I introduce the concept of a behavioral equilibrium and apply it to a class of games with monotone selection, where I obtain unambiguous comparisons of the sets of equilibria when players are naive, sophisticated, or have correct beliefs. Contrary to what may be expected without an appropriate equilibrium framework, the presence of naive players who fail to account for the selection effect actually exacerbates the adverse selection problem.

Several extensions are possible within this framework. An interesting question is under which conditions the presence of naive players in markets with a large number of players (all of which may be naive) has a nonnegligible effect on outcomes. In a standard competitive equilibrium model of adverse selection (Akerlof, 1970), players may also be thought of as being naive in the sense that they do not take into account that deviating to some non-equilibrium offer may affect the quality of objects that would be traded. This is in contrast to a game-theoretic approach with sophisticated players, where deviations (and the potential effect on the quality of objects) are considered in equilibrium. It is well known that a setting with strategic players need not approach every competitive equilibrium outcome as the number of players increases, so that in general the presence of naive players may still affect outcomes in large markets.49

A second objective for future research is to provide better foundations for behavioral equilibrium. One direction is to provide dynamic foundations. While the example at the end of Section 2 provides a step in this direction, incorporating more strategic players in a dynamic setting with uncertainty may prove to be a greater challenge. A second direction is to better understand why and when players can be expected to suffer from the bias studied in this paper. By taking a players’ model of the world as exogenously given, the approach in this paper falls short of answering these questions. For example, if the buyer in Section 2 were to always receive feedback about the value of the object (whether it is traded or not), the approach predicts that a naive equilibrium would not exist (since it is impossible to reconcile a naive view of the world with the feedback obtained), but it does not provide insights into what may occur; e.g. would buyers learn to recognize the selection problem, or will they “update” to a new but still incorrect model of the world?

49A simple example where Bertrand competition refines the set of competitive equilibria in an adverse selection model appears in Mas-Colell, Whinston, and Green (1995, Section 13B). In other settings the results can be more stark. Preliminary results suggest that unless some assumption on the informativeness of private information is made, in both symmetric first-price or k-th unit auctions, bidding in a symmetric equilibrium remains strictly less aggressive when all players are naive compared to when all players have correct beliefs, even as the number of players grows to infinity.
Third, while I have focused on a very specific channel through which selection arises, in general there may be many other channels worth exploring. Economic agents trying to interpret information about their environment face similar selection problems to those that challenge empirical researchers. There may be large gains from more realistically modeling an economic agent as an empirical econometrician rather than simply assuming that she has correct beliefs (see Manski (1993b, 2004) for this view). While not the objective of the present paper, in some cases it may be of interest to fully characterize the behavior of players who are sophisticated but may still have incorrect beliefs in equilibrium. Obtaining bounds on the behavior of sophisticated players (as in Section 2) may lead to more robust estimates in empirical work, and understanding how sophisticated players learn from their experience will aid in tightening those bounds. For example, a future goal is to evaluate to what extent results about the existence and magnitude of asymmetric information obtained from structural empirical models under the assumptions that players play a Nash equilibrium would be robust to the weaker assumption that players are sophisticated.

Modeling the process by which players learn from feedback also highlights an important role of information policy that would be otherwise hard to study in a model where players had correct beliefs. In a different paper (Esponda 2005b), I have applied the concept of a self-confirming equilibrium to study how bidding in symmetric first-price auctions is affected by the information about submitted bids that the auctioneer decides to make public after each auction.

Another related topic for future research is to provide better empirical evidence of peoples’ failure to account for selection problems. While the motivation provided in the introduction suggests that people may indeed behave naively in several settings, most of the evidence in economics comes from auction-like experiments. In addition, the underbidding result obtained in this paper cannot be evaluated in terms of the current experimental results. The challenge is to come up with an experimental set-up that is as conducive to learning as possible and at the same time provides players with feedback only about the value of objects that they win and about their probability of winning at each price.\footnote{A common practice in experimental settings is to provide players with the correct unconditional expected value of the object. In this case, overbidding even with “experienced” players may be a reasonable outcome if players make their decisions based on the known unconditional expected value of the object and either do not keep track of the value of the objects they win or rationalize their lower profits by thinking they are being (very) unlucky. But it seems plausible, given the compelling intuition provided in this paper, that players would underprice if they did not have access to information about the unconditional value of the object to begin with.}

The broader contribution of this paper is to suggest an approach to study equilibrium behavior in the presence of psychological and cognitive biases. This approach assumes that players have a particular model of the world, which arises from some behavioral bias. In the spirit of (self-confirming) equilibrium analysis, players’ beliefs are required to be consistent with what they observe about equilibrium outcomes. In addition, what players learn from their observations is itself restricted by their own particular model of the world. Applying this framework to other behavioral biases may provide insights that are hard to obtain without a proper equilibrium framework.
More generally, the conceptual distinction between a model of the world and equilibrium restrictions stressed in this paper can illuminate behavior that is not characterized by cognitive and psychological biases. In a different paper (Esponda, 2005a), I exploit this distinction to study the robustness of equilibrium predictions to different assumptions about beliefs that players may have about their economic environment, including beliefs about structural uncertainty, beliefs about the rationality of other players, beliefs about other players’ beliefs, and so on.

References


Appendix A: Proofs

This Appendix contains three parts. First, I present some standard terminology and results from the literature on monotone comparative statics. Then, I establish some preliminary results that do not appear in the text. Finally, I use all these results to prove the statements in the text.


A partially ordered set \((X, \geq)\) is a lattice if any two elements \(x\) and \(y\) have a supremum and an infimum in the set. A lattice \((X, \geq)\) is complete if every nonempty subset of \(X\) has a supremum and infimum in \(X\). A finite lattice is complete (all lattices considered in this paper are finite subsets of an Euclidean space for simplicity). An element of \(x \in X\) is the highest element of \(X\) if \(x \geq y\) for every \(y \in X\); it is the lowest element if \(x \leq y\) for all \(y \in X\).

Let \((X, \geq)\) be a lattice and \(T\) a partially ordered set. A correspondence \(\phi : T \rightarrow X\) is increasing in the strong set order if when \(t > t'\), then for each \(x \in \phi(t)\) and \(y \in \phi(t')\), \(\sup(x, y) \in \phi(t)\) and \(\inf(x, y) \in \phi(t')\). A function \(f : X \rightarrow \mathbb{R}\) is supermodular if for all \(x, y \in X\), \(f(\inf(x, y)) + f(\sup(x, y)) \geq f(x) + f(y)\). A function \(f : X \rightarrow \mathbb{R}\) is quasi-supermodular if for all \(x, y \in X\), \(f(x) \geq f(\inf(x, y))\) implies \(f(\sup(x, y)) \geq f(y)\), and \(f(x) > f(\inf(x, y))\) implies \(f(\sup(x, y)) > f(y)\). A function that is supermodular is also quasi-supermodular, but the reverse need not hold. If \(X \subset \mathbb{R}\), then every function is supermodular. A function \(f : X \times T \rightarrow \mathbb{R}\) has increasing differences in its arguments \((x, t)\) if \(f(x, t) - f(x, t')\) is nondecreasing in \(x\) for all \(t \geq t'\). A function \(f : X \times T \rightarrow \mathbb{R}\) satisfies the single-crossing property in \((x, t)\) if for \(x > x'\) and \(t > t'\), \(f(x, t') \leq f(x', t')\) implies \(f(x, t) \geq f(x', t)\) and \(f(x, t') > f(x', t')\) implies \(f(x, t) > f(x', t)\). A function with increasing differences in \((x, t)\) satisfies the single-crossing property in \((x, t)\), but the reverse need not hold.

I use the two following fixed point theorems. The sets \(X\) and \(T\) are assumed to be nonempty.

**FP1.** (Tarski, 1955; Milgrom and Roberts, 1990) Let \((X, \geq)\) be a complete lattice and \(T\) a partially ordered set. Suppose \(f : X \times T \rightarrow X\) is a nondecreasing function for each \(t \in T\). Then for each \(t\), the set of fixed points of \(f\) is nonempty and has a lowest element \(x(t) = \inf\{x \in X : f(x, t) \leq x\}\) and a highest element \(x(t) = \sup\{x \in X : f(x, t) \geq x\}\). If in addition \(f\) is nondecreasing in \(t\) for all \(x \in X\), then \(x(t)\) and \(x(t)\) are nondecreasing.

**FP2.** (Milgrom and Roberts, 1994) Let \(X\) be a compact set in \(\mathbb{R}\) and \(f : X \rightarrow X\) a continuous function. Then the set of fixed points of \(f\) is nonempty and has a lowest element \(x = \inf\{x \in X : f(x) \leq x\}\) and a highest element \(x = \sup\{x \in X : f(x) \geq x\}\).
I use the following three monotone comparative statics results corresponding to the optimization problem: \( h(t) = \arg \max_{x \in X} f(x, t) \).

**MCS1.** (Milgrom and Shannon, 1994) Let \( f : X \times T \rightarrow \mathbb{R} \) be quasi-supermodular on the finite lattice \( X \) for each \( t \) in the partially ordered set \( T \). Then: i) the set \( h(t) \) is nonempty and has a lowest element \( \underline{h}(t) \) and a highest element \( \overline{h}(t) \); ii) if \( f \) is single-crossing in \((x, t)\), then \( \underline{h}(\cdot) \) and \( \overline{h}(\cdot) \) are nondecreasing.

**MCS2.** Let \( X \subset \mathbb{R}^K \) be compact and convex, \( T \) a convex set, \( f : X \times T \rightarrow \mathbb{R} \) a continuous function, and \( f(\cdot, t) \) strictly quasi-concave in \( x \) for each \( t \). Then \( h(t) \) is nonempty, single-valued, and continuous in \( t \).

**MCS3.** (Athey, 1998) Let \( u : A \times S \rightarrow \mathbb{R} \) be a function where \( A \subset \mathbb{R}^K \) and \( S \subset \mathbb{R}^K \) is the support of a vector of affiliated random variables \( \tilde{s} \). Let \( \Phi : X \rightarrow S \) be a correspondence that is nondecreasing in the strong set order. Define \( U(a, x) \equiv E[u(a, \tilde{s}) | \tilde{s} \in \Phi(x)] \). If \( u(a, \cdot) \) is nondecreasing in \( s \), then \( U(a, \cdot) \) is nondecreasing in \( x \). If \( u \) is supermodular in \((a, s)\), then \( U \) has increasing differences in \((a, x)\).

**A2. Preliminary Results**

**PR1.** Suppose \( \eta_i \) reveals payoffs. For any \( a_i \in A_i \), \( \alpha_{-i} \in \Lambda_{-i} \), and \( s_i \in S_i \), let \( \pi_i^N(\cdot; a_i, \alpha_{-i}, s_i) \in \Pi_i^N(a_i, \alpha_{-i}, s_i, \eta_i, M_i^N) \) and \( \pi_i^S(\cdot; a_i, \alpha_{-i}, s_i) \in \Pi_i^S(a_i, \alpha_{-i}, s_i, \eta_i, M_i^S) \). Then
\[
\pi_i^N(a_i; a_i, \alpha_{-i}, s_i) = \pi_i^S(a_i; a_i, \alpha_{-i}, s_i) = \pi_i^{NE}(a_i; \alpha_{-i}, s_i).
\]

**PR2.** Consider a game with unique naive conjectures given \( \Lambda \) and \( \eta \) where kMSP given \( \eta \) holds and \( \eta \) reveals payoffs. For any \( i \in N \), \( \hat{a}_i \in A_i \), \( \alpha_{-i} \in \Lambda_{-i} \), and \( s_i \in S_i \), let \( \pi_i^N(\cdot; \hat{a}_i, \alpha_{-i}, s_i) \in \Pi_i^N(\hat{a}_i, \alpha_{-i}, s_i, \eta_i, M_i^N) \) and \( \pi_i^S(\cdot; \hat{a}_i, \alpha_{-i}, s_i) \in \Pi_i^S(\hat{a}_i, \alpha_{-i}, s_i, \eta_i, M_i^S) \). Then \( \pi_i^N(a_i; \hat{a}_i, \alpha_{-i}, s_i) \leq \pi_i^S(a_i; \hat{a}_i, \alpha_{-i}, s_i) \) for \( a_i \geq \hat{a}_i \) and \( \pi_i^N(a_i; \hat{a}_i, \alpha_{-i}, s_i) \geq \pi_i^S(a_i; \hat{a}_i, \alpha_{-i}, s_i) \) for \( a_i \leq \hat{a}_i \).

**proof of PR1.** Since \( \eta_i \) reveals payoffs, \( h_i^{UL}(h) \) is a singleton for all \( h \in H_i \). Let \( \phi_{ai}^N \) denote the
naive conjecture corresponding to $\pi_i^N(\cdot; a_i, \alpha_{-i}, s_i)$. Then:

$$
\pi_i^{NE}(a_i; \alpha_{-i}, s_i) = \sum_{(a_{-i}, v)} u_i(a_i, a_{-i}, v) \times p_i(s_i, \alpha_{-i}, \gamma)(a_{-i}, v) \\
= \sum_{u \in U} u \times p_i(s_i, \alpha_{-i}, \gamma)[\{u\}]_{i, a_i, \eta_i} \\
= \sum_{u \in U} u \times \phi_{a_i}^N(s_i)[\{u\}]_{i, a_i, \eta_i} \\
= \sum_{(a_{-i}, v)} u_i(a_i, a_{-i}, v) \times \phi_{a_i}^N(s_i)(a_{-i}, v) \\
= \pi_i^N(a_i; a_i, \alpha_{-i}, s_i)
$$

where the first, second and last equalities follow from definitions, the third equality follows since $\phi_{a_i}^N(s_i)$ is $\eta_i$-naive-consistent (see condition 2 in the definition), and the fourth equality follows since $k_i^U(h)$ is a singleton for every $h$. A similar proof shows that $\pi_i^S(a_i; a_i, \alpha_{-i}, s_i) = \pi_i^{NE}(a_i; \alpha_{-i}, s_i)$ (since condition 2 in the definition of naive-consistency is implicitly required by the definition of consistency).

**Proof of PR2.** Let $\phi_{a_i}^S$ be the conjecture corresponding to $\pi_i^S(\cdot, \widehat{a}_i)$ and let $p$ be the joint probability distribution over $A_{-i} \times V$ given $\alpha_{-i}, s_i, \gamma$. Note that $\Pi_i^N(\widehat{a}_i, \phi_{a_i}^S, \eta_i, M_i^N) = \Pi_i^N(\widehat{a}_i, p, \eta_i, M_i^N)$ since by definition naive-consistent beliefs depend only on the probability distribution over feedback $h \in H_i$, rather than outcomes $A_{-i} \times V$, and since $\phi_{a_i}^S$ is $\eta_i$-consistent for $(\widehat{a}_i, \alpha_{-i})$ then such distribution is the same for $\phi_{a_i}^S$ and $p$. Then $\Pi_i^N(\widehat{a}_i, \phi_{a_i}^S, \eta_i, M_i^N) = \{\pi_i^N(\cdot; \widehat{a}_i, \alpha_{-i}, s_i)\}$ by the assumption that there is a unique naive profit function. Now let $a_i \geq \widehat{a}_i$ and consider $\pi_i^N(\cdot; a_i, \alpha_{-i}, s_i) \in \Pi_i^N(a_i, \phi_{a_i}^S, \eta_i, M_i^N)$. Then

$$
\pi_i^N(a_i; \widehat{a}_i, \alpha_{-i}, s_i) \leq \pi_i^N(a_i; a_i, \alpha_{-i}, s_i) = \pi_i^S(a_i; \widehat{a}_i, \alpha_{-i}, s_i),
$$

where the inequality follows from kMSP and $a_i \geq \widehat{a}_i$, and the equality follows from PR1 (i.e. $\pi_i^N(a_i; a_i, \alpha_{-i}, s_i)$ are the “correct” beliefs from playing $a_i$ when the “true” distribution over $(a_{-i}, v)$ is given by $\phi_{a_i}^S$). A similar proof establishes that $\pi_i^N(a_i; \widehat{a}_i, \alpha_{-i}, s_i) \geq \pi_i^S(a_i; \widehat{a}_i, \alpha_{-i}, s_i)$ for $a_i \leq \widehat{a}_i$. ■

**A3. Proof of main results**

The proofs are in order of appearance in the text, except for Proposition 1 which is proved last.

**Proof of Lemma 1.**

The following notation is used in the proof of Lemma 1. Define
where the two equalities follow from the fact that some $s$.

Then proceeds as in part i.

written as:

$$h_i^N(a_i, s_i, \alpha_{-i}, \eta_i) \equiv \arg \max_{a'_i} \pi_i^N(a'_i; a_i, \alpha_{-i}, s_i, \eta_i)$$

to be the naive optimal action of $i \in N$, $s_i \in S_i$ as perceived when playing $a_i$. Also define the set of fixed points of $h_i^N$, $br_i^N(s_i, \alpha_{-i}, \eta_i) \equiv \{ a_i \in A_i : a_i \in h_i^N(a_i, s_i, \alpha_{-i}, \eta_i) \}$, which is the naive best response of $i, s_i$ to $\alpha_{-i}$. It then follows that the previously defined sets $H_i^N$ and $BR_i^N$ can be written as:

$$H_i^N(\alpha_i, \alpha_{-i}, \eta_i) = \{ \alpha'_i : \text{For all } s_i \in S_i, \; \alpha'_i(s_i) \in h_i^N(\alpha_i(s_i), s_i, \alpha_{-i}, \eta_i) \}$$

$$BR_i^N(\alpha_{-i}, \eta_i) = \{ \alpha_i : \text{For all } s_i \in S_i, \; \alpha_i(s_i) \in br_i^N(s_i, \alpha_{-i}, \eta_i) \}.$$

**Part (i).** I drop $\alpha_{-i}$ and $\eta_i$ from the notation. Since the finite set $A_i \subset \mathbb{R}$, then $A_i$ is a finite, complete lattice. Since $\pi_i^N$ is single-crossing in $(a'_i, a_i)$, then by MCS1 $h_i^N(a_i, s_i)$ has a lowest element $h_i^N(a_i, s_i)$ that is nondecreasing in $a_i$. Since $h_i^N : A_i \times S_i \rightarrow A_i$, by FP1 there is a lowest fixed point of $h_i^N(s_i)$, $a_i(s_i) = \inf \{ a_i \in A_i : h_i^N(a_i, s_i) \leq a_i \}$, and $\alpha_i$ defined as $\alpha_i(s_i) = a_i(s_i)$ for all $s_i \in S_i$ is the lowest fixed point of $H_i^N(\alpha_i)$. Now consider $\alpha_i \neq \alpha_i$. Then since $A_i \subset \mathbb{R}$ there exists $s_i^* \in S_i$ such that $\alpha_i(s_i^*) < \alpha_i(s_i^*)$. This implies (again since $A_i \subset \mathbb{R}$) that $h_i^N(\alpha_i(s_i^*), s_i^*) > h_i^N(\alpha_i(s_i^*), s_i^*)$. Letting $a'_i = h_i^N(\alpha_i(s_i^*), s_i^*)$ and noting that $\alpha_i(s_i^*)$ is not a fixed point of $h_i^N(a_i, s_i)$ then regularity of $H_i^N$ follows.

**Part (ii).** From continuity, convexity, compactness, and strict quasi-concavity, MCS2 implies that $h_i^N(a_i, s_i)$ is single-valued and continuous in $a_i$. Since $h_i^N : A_i \times S_i \rightarrow A_i$ and $A_i \subset \mathbb{R}$, then by FP2 there is a lowest fixed point of $h_i^N(a_i, s_i)$, given by $a_i(s_i) = \inf \{ a_i \in A_i : h_i^N(a_i, s_i) \leq a_i \}$. The proof then proceeds as in part i. ■

**Proof of Lemma 2.**

**Part 1.** Let $\alpha_i^N \in BR_i^N(\alpha_{-i}, \eta_i)$ and $\alpha_i^{NE} \in BR_i^{NE}(\alpha_{-i}, \eta_i)$. Suppose that $\alpha_i^N(s_i) > \alpha_i^{NE}(s_i)$ for some $s_i \in S_i$. Then:

$$\pi_i^{NE}(\alpha_i^N(s_i); \alpha_{-i}, s_i) = \pi_i^N(\alpha_i^N(s_i); \alpha_i^N(s_i), \alpha_{-i}, s_i, \eta_i) \geq \pi_i^N(\alpha_i^{NE}(s_i); \alpha_i^N(s_i), \alpha_{-i}, s_i, \eta_i) \geq \pi_i^N(\alpha_i^{NE}(s_i); \alpha_i^{NE}(s_i), \alpha_{-i}, s_i, \eta_i) = \pi_i^{NE}(\alpha_i^{NE}(s_i); \alpha_{-i}, s_i),$$

where the two equalities follow from the fact that $\eta_i$ reveals payoffs (PR1), the first inequality
follows from the definition of a naive best response, and the second inequality follows from MSP and \(\alpha_i^N(s_i) \geq \alpha_i^{NE}(s_i)\). Therefore, \(\alpha_i^N(s_i)\) is also a Nash best response for \(i, s_i\), so that \(\max\{\alpha_i^N, \alpha_i^{NE}\} \in BR_i^{NE}\). Now suppose strict MSP holds: the second inequality is then strict, which contradicts the fact that \(\alpha_i^{NE}(s_i)\) is a Nash best response for \(i, s_i\). Therefore, \(\alpha_i^N \leq \alpha_i^{NE}\).

Part 2. Let \(\alpha_i\) be the lowest naive best response given \(\alpha_{-i}\) (which exists by regularity of \(H^N\)) and consider some \(\alpha_i \not\in \alpha_i\). Then for some \(s_i \in S_i\) and \(a'_i > \alpha_i(s_i)\),

\[
\pi_i^S(a'_i; \alpha_i(s_i^*), \alpha_{-i}, s_i^*, \eta_i) \geq \pi_i^N(a'_i; \alpha_i(s_i^*), \alpha_{-i}, s_i^*, \eta_i)
\]

\[
> \pi_i^N(\alpha_i(s_i^*); \alpha_i(s_i^*), \alpha_{-i}, s_i^*, \eta_i)
\]

\[
= \pi_i^S(\alpha_i(s_i^*); \alpha_i(s_i^*), \alpha_{-i}, s_i^*, \eta_i),
\]

where the first inequality follows from kMSP and \(a'_i > \alpha_i(s_i)\) (PR2), the strict inequality (and existence of such \(s_i^*\) and \(a'_i\)) follows from regularity of \(H^N\), and the equality follows since \(\eta_i\) reveals payoffs (PR1). It then follows that \(\alpha_i \not\in BR_i^S(\alpha_{-i}, \eta_i, M_i^S)\). ■

Proof of Theorem 1.

Part 1. I omit \(\eta_i\) and \(M_i\) from the notation.

(a) Since \(A_i\) is a finite, complete lattice and because of the single-crossing conditions assumed, an application of MCS1 (similar to that in the proof of part i. of Lemma 1) implies that there are lowest and a highest Nash and naive best responses, denoted by \(BR_i^{NE}(\alpha_{-i})\), \(BR_i^{NE}(\alpha_{-i})\) and \(BR_i^N(\alpha_{-i})\), \(BR_i^N(\alpha_{-i})\), respectively, each nondecreasing in \(\alpha_{-i}\). For \(n \in \{N, NE\}\), let \(BR^m(\alpha) = \{BR_i^m(\alpha_{-i})\}_{i \in N}\) be the lowest best response map. Since \(\Lambda\) is a complete lattice and \(BR^m : \Lambda \longrightarrow \Lambda\) is nondecreasing in \(\alpha\), then FP1 implies that there is a lowest fixed point of \(BR^m\), given by \(\alpha^m = \inf\{\alpha \in \Lambda : BR^m(\alpha) \leq \alpha\}\). For any \(\alpha\) that is a fixed point of \(BR^m\), \(\alpha \geq BR^m(\alpha)\). Therefore, \(\alpha^m\) is also the lowest fixed point of \(BR^m\). By Proposition 2, \(\alpha^m\) is the lowest Nash (naive) equilibrium for \(m = NE\) \((m = N)\). A similar proof establishes the existence of a highest equilibrium. A sophisticated equilibrium exists since a Nash equilibrium is always a sophisticated equilibrium (this is because correct beliefs are always \(\eta_i\)-consistent and because correct beliefs belong to \(M_i^S\) – i.e. those beliefs consistent with MSP – given that it is actually true that MSP holds).

(b) By part 1 of Lemma 2, \(BR^N(\alpha) \leq BR^{NE}(\alpha)\) for all \(\alpha \in \Lambda\). Let \(T = \{0, 1\}\) and define \(f : \Lambda \times T \longrightarrow \Lambda\) such that \(f(\cdot, 0) = BR^N(\alpha)\) and \(f(\cdot, 1) = BR^{NE}(\alpha)\). Then \(f\) is nondecreasing in \(t \in T\), and from FP1, the highest fixed point of \(BR^N\) (i.e. the highest naive equilibrium) is (weakly) lower than the highest fixed point of \(BR^{NE}\) (i.e. the highest Nash equilibrium). With the additional assumption that MSP is strict, part 1 of Lemma 2 implies that \(BR^N(\alpha) \leq BR^{NE}(\alpha)\) for all \(\alpha \in \Lambda\), so that a similar application of FP1 yields that the lowest fixed point of \(BR^{NE}\) (i.e.
the lowest Nash equilibrium) is (weakly) higher than the lowest fixed point of $BR^N$, which is itself (weakly) higher than the lowest naive equilibrium.\footnote{When MSP is strict, the proof of part (b) can be established along the lines of the proof of part (c) and does not require that the naive game have strategic complementarities.}

(c) Let $\alpha^S$ be a sophisticated equilibrium, i.e. $\alpha^S \in BR^S(\alpha^S)$. I show that there exists a naive equilibrium $\alpha^N$ such that $\alpha^N \leq \alpha^S$. Consider $\Lambda' = \{\alpha \in \Lambda : \alpha \leq \alpha^S\}$. For any $\alpha \in \Lambda'$, $BR^N(\alpha) \leq BR^N(\alpha^S) \leq \alpha^S$, where the first inequality follows from $BR^N$ being nondecreasing and the second inequality follows from the ordering of best responses established in part 2 of Lemma 2 (i.e. $BR^N(\alpha) \leq \alpha'$ for any $\alpha' \in BR^S(\alpha)$). Hence, $BR^N(\Lambda') \subset \Lambda'$ and, since $\Lambda'$ is a complete lattice, it follows from FP1 that there exists a naive equilibrium $\alpha^N \in \Lambda'$.

Part 2.

(a) Consider a symmetric game without private information. Let $a_{-i}$ denote the strategy profile where every player other than $i$ plays $a$, and let $a$ denote the profile where every player plays $a$. By Proposition 2, a symmetric profile $a$ is a naive equilibrium if and only if $a \in H^N(a) = \{\hat{a} : \hat{a} \in \arg\max_{a'} \pi^N(a', a, a_{-i})\}$. Similarly, $a$ is a Nash equilibrium if and only if $a \in H^{NE}(a) = \{\hat{a} : \hat{a} \in \arg\max_{a'} \pi^{NE}(a', a_{-i})\}$. From continuity, convexity, compactness, and strict quasi-concavity, MCS2 implies that $H^N_i(a_i)$ and $H^{NE}_i(a_i)$ are each single-valued and continuous in $a_i$. Then by FP2, lowest and highest naive and Nash equilibria exist and are given by $a^m = \inf\{a : H^m(a) \leq a\}$ and $\bar{a}^m = \sup\{a : H^m(a) \geq a\}$ for $m \in \{N, NE\}$. A sophisticated equilibrium exists since a Nash equilibrium is always a sophisticated equilibrium.

(b) First, I claim that if $H^N(a) > H^{NE}(a)$, then (i) $H^N(a) \neq a$ and (ii) $H^{NE}(a) \neq a$. Consider (i) and suppose, toward a contradiction, that it does not hold: i.e. $a = H^N(a) > H^{NE}(a)$. Then

$$\pi^{NE}(a, a_{-i}) = \pi^N(a, a, a_{-i})$$
$$> \pi^N(H^{NE}(a), a, a_{-i})$$
$$\geq \pi^N(H^{NE}(a), H^{NE}(a), a_{-i})$$

where the first and last equality follows since $\eta$ reveals payoffs (PR1), the strict inequality follows since $a = H^N(a)$ is the unique element of $H^N(a)$ (i.e. it is a unique maximizer), and the weak inequality follows from MSP and $a > H^{NE}(a)$. Note that these relationships contradict that $H^{NE}(a)$ is indeed a maximizer of $\pi^{NE}$ given $a$. The proof of (ii) is similar: suppose $H^N(a) >
\( H^{NE}(a) = a \). Then

\[
\pi^{NE}(H^N(a), a_{-i}) = \pi^N(H^N(a), H^N(a), a_{-i}) \\
\geq \pi^N(H^N(a), a, a_{-i}) \\
> \pi^N(a, a, a_{-i}) \\
= \pi^{NE}(H^{NE}(a), a_{-i}),
\]

which once again contradicts that \( H^{NE}(a) \) is indeed a maximizer of \( \pi^{NE} \) given \( a \).

Next, I claim that \( \bar{a}^{NE} \geq \bar{a}^N \) (these are the extreme points defined in part a.). Suppose not, so that \( \bar{a}^{NE} < \bar{a}^N \). But then from the fact that \( \bar{a}^{NE} \) is defined as the supremum of a certain set, \( H^{NE}(\bar{a}^N) < \bar{a}^N = H^N(\bar{a}^N) \), which by part i) of the previous claim is not possible. Finally, I claim that \( \bar{a}^{NE} \geq a^N \). Suppose not, so that \( \bar{a}^{NE} < a^N \). But then \( H^N(\bar{a}^{NE}) > \bar{a}^{NE} = H^{NE}(\bar{a}^{NE}) \), which now contradicts part ii) of the first claim.

(c) Suppose \( a < a^N \) is part of a symmetric sophisticated equilibrium. Then \( H^N(a) > a \), meaning that a naive player would prefer to deviate to a higher action. But then so would a sophisticated player; to see this let \( \pi^S \in \Pi^S(a, a_{-i}) \) and note that

\[
\pi^S(H^N(a), a, a_{-i}) \geq \pi^N(H^N(a), a, a_{-i}) \\
> \pi^N(a, a, a_{-i}) \\
= \pi^S(a, a, a_{-i}),
\]

where the first inequality follows from kMSP and \( H^N(a) > a \) (PR2), the strict inequality follows from the definition of \( H^N(a) \), and the equality follows since \( \eta \) reveals payoffs (PR1). Since \( a \) is not part of a symmetric sophisticated equilibrium then any sophisticated symmetric equilibrium \( a^S \) satisfies \( a^S \geq a^N \).

**Proof of Proposition 3.**

Part 1. By B2, beliefs about \( \varphi_i(a_i; \alpha_{-i}, s_i) \) are correct for any \( a_i \), so consider beliefs about \( E u_i^*(a_i, \bar{\theta}) \) when \( \bar{a}_i \) is played. Since only payoffs are revealed by \( \eta_i \) (B3) and \( \varphi_i(\bar{a}_i; \alpha_{-i}, s_i) > 0 \), it follows that players observe the exact realization of \( \bar{\theta} \) if and only if \( (\alpha_{-i}(s_{-i}), t) \in \Phi_i(\bar{a}_i) \), and observe nothing about \( \bar{\theta} \) otherwise. In addition, player \( i \) believes that this conditional expectation does not depend on their action \( \bar{a}_i \) for two reasons: i) players know that \( \bar{t} \) is independent of \( \bar{\theta} \) (B1), and ii) players are naive, so they do not take into account when forming their beliefs that their opponents’ actions might be correlated with \( \bar{\theta} \). Therefore, naive-consistency requires that beliefs about \( E u_i^*(a_i, \bar{\theta}) \) be given by the conditional expectation \( E \left( u_i^*(a_i, \bar{\theta}) \mid (\alpha_{-i}(\bar{s}_{-i}), \bar{t}) \in \Phi_i(\bar{a}_i), \bar{s} = s_i \right) \).
Part 2. Since \((\bar{\theta}, \bar{s}, \bar{t})\) are affiliated random variables (F1 and F2), \(u_i^*\) is nondecreasing in \(\theta\) (F3), and \(\Phi_i\) is nondecreasing in the strong set order, it follows from (MCS3) and from the assumption that \(\alpha\) is nondecreasing that \(E\left(u_i^*(a_i, \bar{\theta}) \mid (\alpha_{-i}(\bar{s}_{-i}), \bar{t}) \in \Phi_i(\hat{a}_i), \bar{s} = s_i\right)\) is nondecreasing in \(\hat{a}_i\).

Part 3. Since \(u_i^*\) is supermodular in \((a_i, \theta)\) (F6), \((\bar{\theta}, \bar{s}, \bar{t})\) are affiliated random variables (F1 and F2), and \(\Phi_i\) is nondecreasing in the strong set order, it follows from (MCS3) and from the assumption that \(\alpha\) is nondecreasing that \(E\left(u_i^*(a_i, \bar{\theta}) \mid (\alpha_{-i}(\bar{s}_{-i}), \bar{t}) \in \Phi_i(\hat{a}_i), \bar{s} = s_i\right)\) has increasing differences in \((a_i, \hat{a}_i)\). Since \(\varphi_i(a_i; \alpha_{-i}, s_i)\) is nonnegative and nondecreasing in \(a_i\) (F5) and since the conditional expectation term is nondecreasing in \(\hat{a}_i\) (because MSP holds, by part 2.), it then follows that \(\pi^N\) has increasing differences in \((a_i, \hat{a}_i)\). Lemma 1 then implies regularity of \(H^N\).

**Proof of Proposition 1.**

The proof is an application of the special framework in Section 4.3. Assumptions F1 and B1 hold since there is no random variable \(\bar{t}\) in this context. Assumptions B2 and B3 hold because the buyer only observes payoffs and ask prices. F2 holds since \(\bar{v}, \bar{s}\) are affiliated, F3 because \(u\) is nondecreasing, F4 because trade occurs when \(S \leq [0, p]\), a set which is nondecreasing in the strong set order in \(p\), F5 since \(\Pr(\bar{s} \leq p)\) is nondecreasing in \(p\), and F6 because \(u\) is supermodular. Hence, Proposition 3 and the assumption that a sophisticated buyer knows that the selection effect is monotone implies that Lemma 2 can be applied to the trading game, so that part 1 and the lower bound in part 2 of Proposition 1 hold. It remains to prove that the highest Nash equilibrium is an upper bound for the set of sophisticated equilibria. The proof is a direct extension of the proof for the particular example in section 2 and is omitted.

**Appendix B. Multidimensional action space**

I extend the result in Theorem 1 to an action space that is partially ordered, as long as the selection effect is still driven by a completely ordered component of the action space. To define the latter, decompose the action \(a_i\) into two components, \(a_i = (a_i^{NS}, a_i^S) \in A_i = A_i^{NS} \times A_i^S \subset \mathbb{R}^{K+1}\), where \(A_i^{NS}\) and \(A_i^S\) are finite lattices and \(A_i^S \subset \mathbb{R}\). Write the naive profit function from action \(a_i = (a_i^{NS}, a_i^S)\) when conjectures are \(\eta\)-naive-consistent for \(((\hat{a}_i^{NS}, \hat{a}_i^S), \alpha_{-i})\) as \(\pi_i^N((a_i^{NS}, a_i^S), (\hat{a}_i^{NS}, \hat{a}_i^S), \alpha_{-i}, s_i, \eta)\). The selection effect is unidimensional given \(\Lambda, \eta\) if for every player \(i\) and signal \(s_i \in S_i\), \(\pi_i^N\) is constant in \(\hat{a}_i^{NS}\) for every \(\alpha_{-i} \in \Lambda_{-i}\). In this case, I omit the dependence of the naive profit function \(\pi_i^N\) on \(\hat{a}_i^{NS}\).

Define a game with strategic complementarities and unidimensional selection effect to be a game with strategic complementarities (as defined in Section 4), but where the assumption on the action space is replaced with the assumption of a unidimensional selection effect and, in
addition, i) \( \pi^N_{i}(a_i, \alpha_{-i}, s_i) \) is supermodular in \( a_i \) and has increasing differences in \( (a_i, \alpha_{-i}) \); ii) \( \pi^N_{i}(a_i, \tilde{\alpha}^S_i, \alpha_{-i}, s_i, \eta_i) \) is supermodular in \( a_i \) and has increasing differences in \( (a_i, \tilde{\alpha}^S_i) \) and increasing differences in \( (a_i, \alpha_{-i}) \).

**Theorem B1.** The statement of part 1 of Theorem 1 holds for games with strategic complementarities and unidimensional selection effect.

**Proof.** The idea is to convert a “multidimensional” problem to a “unidimensional” one, prove that Lemma 2 holds for this unidimensional problem, and finally use complementarity in own action to show that the comparison for the unidimensional component of the strategy space extends to the multidimensional component. I omit \( i, \alpha_{-i}, \) and \( \eta_i \) from the notation for simplicity. The proof shows that the result in Lemma 2 holds in the new setup, so that the proof in Theorem 1 can be applied (where the only change in the latter proof is to use the additional condition of supermodularity – which is stronger than quasi-supermodularity – for the monotone comparative statics results to be applicable). Define \( \rho^N_{NS}(a^S, \tilde{\alpha}^S, s) \equiv \arg \max_{a_{NS}} \pi^N(a^NS, a^S, \tilde{\alpha}^S, s) \) and \( \rho^{NE}_{NS}(a^S, s) \equiv \arg \max_{a_{NS}} \pi^{NE}(a^{NS}, a^S, s) \). By supermodularity and increasing differences, there exist extreme (i.e. lowest and highest) elements of \( \rho^N \) which are nondecreasing in \( (a^S, \tilde{\alpha}^S) \) and extreme elements of \( \rho^{NE} \) which are nondecreasing in \( a^S \). I denote these elements by \( a^N_{NS}, \tilde{a}^S_{NS} \) and \( a^{NE}_{NS}, \tilde{a}^{NE}_{NS} \). Define \( \pi^N_*(a^S, \tilde{\alpha}^S, s) \equiv \pi^N(a^N_{NS}(a^S, \tilde{\alpha}^S), a^S, s) \) and \( h^N_*(\tilde{\alpha}^S, s) = \left\{ a^S_\ast \in A^S : a^S_\ast \in \arg \max_{a_S \in A^S} \pi^N(a^S_\ast, \tilde{\alpha}^S, s) \right\} \).

The following relationships establish that \( \pi^N_*(a^S, \tilde{\alpha}^S, s) \) has increasing differences in \( (a^S, \tilde{\alpha}^S) \). Let \( a^S_1, a^S_0, \tilde{\alpha}^S_1, \tilde{\alpha}^S_0 \in A^S \) such that \( a^S_1 \geq a^S_0 \) and \( \tilde{\alpha}^S_1 \geq \tilde{\alpha}^S_0 \). Then

\[
\begin{align*}
\pi^N(a^N_{NS}(a^S_1, \tilde{\alpha}^S_1), a^S_1, \tilde{\alpha}^S_1) - \pi^N(a^N_{NS}(a^S_0, \tilde{\alpha}^S_0), a^S_0, \tilde{\alpha}^S_0) & \geq \pi^N(\sup(a^N_{NS}(a^S_1, \tilde{\alpha}^S_0), a^N_{NS}(a^S_0, \tilde{\alpha}^S_1), a^S_1, \tilde{\alpha}^S_0) - \pi^N(a^N_{NS}(a^S_0, \tilde{\alpha}^S_0), a^S_0, \tilde{\alpha}^S_0) \\
& \geq \pi^N(\sup(a^N_{NS}(a^S_1, \tilde{\alpha}^S_0), a^N_{NS}(a^S_0, \tilde{\alpha}^S_1), a^S_1, \tilde{\alpha}^S_0) - \pi^N(a^N_{NS}(a^S_0, \tilde{\alpha}^S_0), a^S_0, \tilde{\alpha}^S_0) \\
& \geq \pi^N(\sup(a^N_{NS}(a^S_1, \tilde{\alpha}^S_0), a^S_1, \tilde{\alpha}^S_0) - \pi^N(\inf(a^N_{NS}(a^S_1, \tilde{\alpha}^S_0), a^N_{NS}(a^S_0, \tilde{\alpha}^S_1), a^S_0, \tilde{\alpha}^S_0) \\
& \geq \pi^N(\sup(a^N_{NS}(a^S_1, \tilde{\alpha}^S_0), a^S_1, \tilde{\alpha}^S_0) - \pi^N(a^N_{NS}(a^S_0, \tilde{\alpha}^S_0), a^S_0, \tilde{\alpha}^S_0),
\end{align*}
\]

where the first and last inequalities follow from the optimality of \( a^N_{NS} \) (and the fact that both the infimum and the supremum are in the choice sets due to the lattice structure), the second inequality follows from the assumption that \( \pi^N \) has increasing differences in the arguments \( a = (a^{NS}, a^S) \) and \( \tilde{\alpha}^S \), and the third inequality follows from supermodularity of \( \pi^N \) in \( a = (a^{NS}, a^S) \). Since increasing differences implies the single-crossing property, I can now apply the proof of part 1 of Lemma 1 to show that \( H^N_*(a^S) \) is regular. Hence, Lemma 2 holds for the selection component \( a^S \).

Finally, by PR1 \( a^N_{NS}(a^S, a^S, s) = a^{NE}_{NS}(a^S, s) \) and \( \tilde{a}^N_{NS}(a^S, a^S, s) = \tilde{a}^{NE}_{NS}(a^S, s) \) and since they are nondecreasing in \( a^S \), then the comparison of the selection component extends to the entire strategy \( \alpha = (\alpha^{NS}, a^S) \).
Appendix C. Symmetric auctions

Consider the symmetric auction model of Milgrom and Weber (1982). There are $N$ bidders with demand for at most one unit of identical, indivisible objects that are being sold through an auction. Before submitting their bids, each of the $N$ risk neutral bidders receives a private signal $s_i \in S = [\underline{s}, \bar{s}] \subset \mathbb{R}_+$, which is the realization of a random variable $\tilde{s}_i$, $i = 1, \ldots, N$. The value of the object is a random variable $u_i(\tilde{s}_1, \ldots, \tilde{s}_N, x) = u(s_i, s_{-i}, x)$, nonnegative, continuous, and nondecreasing in all its variables. The random variables $\{\tilde{s}_1, \ldots, \tilde{s}_N\}$ are distributed according to the joint density $\tilde{f}$, which is assumed to be affiliated and symmetric in its arguments. Because of symmetry, I focus on the decision problem of a single bidder. Let $\bar{y}(k)$ represent the $k$-th highest signal among the opponents of that bidder, where $k = 1, \ldots, N - 1$, and denote its distribution conditional on a player’s own signal $s$ by $F_{\bar{y}(k)}(\cdot \mid s)$, with density $f_{\bar{y}(k)}(\cdot \mid s)$. Bidders are risk neutral.

I consider two auction mechanisms. In a first price auction there is one object for sale, players simultaneously submit their bids, and the object is allocated to the bidder with the highest bid (ties are resolved randomly). The winner pays her submitted bid. In a $k$-th unit auction, there are $k = 1, \ldots, N - 1$ objects for sale, players simultaneously submit their bids, and the objects are allocated to the bidders with the $k$-th highest bids (ties are resolved randomly). Those who receive the object pay the $(k + 1)$-th highest bid. In both auction formats, losers pay nothing. (Note than when $k = 1$ the $k$-th unit auction is a second price auction).

A (pure) strategy $\beta^j : S \to [0, \infty)$ is a function mapping signals to bids, where the superscript $j \in \{1st, k\}$ denotes either the first or $k$-th unit auction, respectively. I follow the standard approach in auction theory and restrict analysis to equilibria in symmetric and pure strategies. For the first price auction, I also restrict attention to equilibria in increasing, continuous, and differentiable strategies. Let $\phi$ denote the inverse of the strategy $\beta$. Following Milgrom (1981), define

$$v(s, y, k) \equiv E[u(\tilde{s}_i, \tilde{s}_{-i}) \mid \tilde{s}_i = s, \bar{y}(k) = y].$$

In a similar vein, define

$$v^*(s, y, k) \equiv E[u(\tilde{s}_i, \tilde{s}_{-i}) \mid \tilde{s}_i = s, \bar{y}(k) \leq y].$$

Throughout this section, I also assume that players only observe the value of those objects that they win and that the auctioneer reveals ask prices. The naive profit functions under the assumption that all other players choose strategy $\beta$ are then

$$\pi^{1st,N}(b; \hat{b}, \beta, s) = \left(v^*(s, \phi(\hat{b}), 1) - b\right) \cdot F_{\bar{y}(1)}(\phi(b) \mid s).$$
for the first price auction and
\[ \pi_{k,N}^s(b; \tilde{b}, \beta, s) = \left( v^*(s, \phi(b), k) - E[\beta(Y) \mid \bar{s}_i = s, \bar{y}(k) \leq \phi(b)] \right) \cdot F_{\bar{y}(k)}(\phi(b) \mid s) \]

for the k-th unit auction. For the k-th unit auction, the unique Nash equilibrium is given by the symmetric profile of strategies \( \beta_{k,NE}^s(s) = v(s, s, k) \) (Milgrom (1981); Pesendorfer and Swinkels (1997) prove uniqueness); for the first price auction, it is characterized as the solution of the differential equation
\[ \frac{d}{ds} \beta_{1st,NE}^s(s) = \left[ v(s, s, 1) - \beta_{1st,NE}^s(s) \right] \frac{f(s \mid s)}{F(s \mid s)} \]

with boundary condition \( \beta_{1st,NE}^s(s) = v(s, s, 1) \) (Milgrom and Weber, 1982). The following proposition uses a straightforward extension of the proofs of the previous results to characterize symmetric equilibrium behavior when players are naive.\(^\text{54}\)

**Proposition C1.** In the symmetric k-th unit auction the (essentially) unique symmetric equilibrium when players are naive is \( \beta_{k,N}^s(s) = v^*(s, s, k) \). In the first price auction the (essentially) unique symmetric equilibrium \( \beta_{1st,N}^s \) is the solution to the differential equation
\[ \frac{d}{ds} \beta_{1st,N}^s(s) = \left[ v^*(s, s, 1) - \beta_{1st,N}^s(s) \right] \frac{f(s \mid s)}{F(s \mid s)} \]

with boundary condition \( \beta_{1st,N}^s(s) = v(s, s, 1) \).

**Proof.** k-th unit auction: Note that \( \pi^k(b; \tilde{b}, \beta, s) \) corresponds to an auction with private values, where bidding one’s own valuation is a (weakly) dominant strategy. Hence, \( b^* = E\left[ u(\bar{s}_i, \bar{s}_{-i}) \mid s_i = s, \bar{y}(k) \leq \phi(b) \right] \) maximizes \( \pi^k(b; \tilde{b}, \beta, s) \) for any \( \tilde{b} \) and \( \beta \). A naive best response to \( \beta \) is then given by the fixed point \( \beta_{BR}^s = E\left[ u(\bar{s}_i, \bar{s}_{-i}) \mid s_i = s, \bar{y}(k) \leq \phi(\beta_{BR}^s) \right] \), and for \( \beta \) to be a symmetric equilibrium, \( \beta \) must be a naive best response to \( \beta \). The result then follows.

First price auction: Let \( \beta \) be a symmetric naive equilibrium. Then the naive best response when opponents play \( \beta \) must be \( \beta \). By the definition of a naive best response, for almost every type \( s \),
\[ \beta(s) \in \arg \max_b \pi_{1st,N}^s(b; \beta(s), \beta, s). \] (6)

The derivative of \( \pi_{1st}^s(b; \beta(s), \beta, s) \) with respect to \( b \) can be written as
\[ \frac{f(\phi(b) \mid s)}{d\beta(\phi(b))/ds} \left( E[ u(\bar{s}_i, \bar{s}_{-i}) \mid \bar{s}_i = s, \bar{y}(1) \leq \phi(\beta(s))] \right) - b \frac{d\beta(\phi(b))}{ds} \cdot \frac{F(\phi(b) \mid s)}{f(\phi(b) \mid s)} , \]

where \( \phi \) is the inverse of \( \beta \). Let \( \Pi(b, s; \beta) \) denote the term in parenthesis. Milgrom and Weber

\(^{54}\)I follow Milgrom and Weber (1982) in assuming the nondegeneracy condition that \( v \) and \( v^* \) are strictly increasing in \( s \).
(1982) show that because of affiliation (and nondecreasing utility) \( E[u(s ; \bar{s}_i , \bar{s}_{-i} , \bar{x}) | \bar{s}_i = s , \bar{y}(1) \leq s] \) is nondecreasing in \( s \) and \( \frac{F(y | \bar{s})}{F(y | s)} \) are decreasing in \( s \). Hence, \( \Pi(b , s ; \beta) \) is increasing in \( s \). It follows from (6) that the first order condition must be satisfied for \( b = \beta(s) \), so that \( \Pi(\beta(s) , s ; \beta) = 0 \). To show that \( \beta(s) \) actually solves the problem in equation (6) (i.e. that it is actually a Naive equilibrium), I use the fact that \( \Pi(b , \phi(b) ; \beta) = 0 \). Since \( \Pi \) is increasing in \( s \), it follows that \( \Pi(b , s ; \beta) > 0 \) for \( b < \beta(s) \) and \( \Pi(b , s ; \beta) < 0 \) for \( b > \beta(s) \). To prove the former inequality, note that the first order condition can always be written in term of the inverse bid function, \( \Pi(b , \phi(b) ; \beta) = 0 \). Since \( \Pi \) is increasing in \( s \), it follows that \( \Pi(b , s ; \beta) > 0 \) for \( s > \phi(b) \), that is, for \( b < \beta(s) \). The second inequality is proved in a similar way.

Finally, the boundary conditions follow from the standard argument for the boundary condition in a Nash equilibrium.

The previous result implies that in equilibrium naive bidders underbid relative to bidders that have correct beliefs.

**Corollary C2.** In both the symmetric first price auction and the symmetric \( k \)-th unit auction, equilibrium bidding by Naive bidders is less aggressive than Nash equilibrium bidding.

**Proof.** The result is immediate for \( k \)-th unit auctions since \( v \geq v^* \) for any \((s , y , k)\) follows since \( u \) is nondecreasing and from affiliation. For the first price auction, \( v \geq v^* \) implies that 

\[
\frac{d}{ds} \beta^{1st,N}(s) \leq \frac{d}{ds} \beta^{1st,NE}(s).
\]

The result then follows since \( \beta^{1st,N}(s) = \beta^{1st,NE}(s) \).

While in this case naive equilibria can be explicitly characterized and compared to Nash equilibria, a question remains as to whether the relationships established in Theorem 1 for the set of sophisticated equilibria also hold when in addition to the previous assumption, sophisticated bidders are assumed to know that the selection effect is nondecreasing.\(^{55}\) To see that these hold, first note that both auction formats belong to the class of economies in Section 4.3, so that the comparison of best responses in Lemma 2 holds (under the assumption that \( \mu \). Actually, the following variant of that Lemma holds.

**Lemma C3.** Let \( \beta^{h,S} \) be a (symmetric) sophisticated equilibrium for \( h \in \{ k , 1st \} \). Then

i) for almost every \( s \), \( \beta^{h,S}(s) \in \arg \max_{b \geq \beta^{h,S}(s)} \pi^{h,N}(b ; \beta^{h,S}(s) , \beta^{h,S} , s) \)

ii) for almost every \( s \), \( \beta^{h,S}(s) \in \arg \max_{b \leq \beta^{h,S}(s)} \pi^{h,NE}(b ; \beta^{h,S} , s) \)

**Proof.** Part i) follows since the naive profit is a lower bound for a sophisticated profit function for \( b \geq \beta^{h,S}(s) \) (see PR2 in Appendix A). Part ii) follows since, as argued in the proof of Proposition 3, for \( b \leq \beta^{h,S}(s) \) players observe both ask prices and the valuation of the object, and therefore

\(^{55}\)For example, sophisticated bidders may (correctly) believe that other bidders choose nondecreasing strategies and that their signals are affiliated.

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sophisticated players must have correct beliefs about the expected value of the object conditional on winning with a lower bid.

While these auction games are not games with strategic complementarities, I have shown elsewhere (Esponda, 2005b) that symmetric first-price auctions do possess a property that allows for unambiguous comparative statics results when best responses can be compared as in Lemma C3. In this paper I show that this property extends to the naive version of symmetric first price auctions as well, and to the naive version of symmetric k-th unit auctions. The part of the property needed to characterize the set of sophisticated equilibria is the following (see Esponda, 2005b, for further discussion). Let \( \mu(A) \) denote the measure of (an interval) \( A \subset \mathbb{R} \).

**GSWCP** (Generalized Symmetric Weak Complementarity Property) Let \( \beta^S, \beta^N \), and \( \beta^{NE} \) be a symmetric sophisticated, naive, and Nash equilibrium, respectively. GSWCP is satisfied if the following conditions hold.

a) If \( \beta \not\leq \beta^N \), then \( \beta(s) \notin \{ \text{arg max}_{b \geq \beta(s)} \pi^N(b; \beta(s), \beta, s) \} \) for \( s \in S' \), where \( \mu(S') > 0 \).

b) If \( \beta \not\leq \beta^{NE} \), then \( \beta(s) \notin \{ \text{arg max}_{b \leq \beta(s)} \pi^{NE}(b; \beta, s) \} \) for \( s \in S' \), where \( \mu(S') > 0 \).

**Lemma C4.** The symmetric first price auction satisfies GSWCP when equilibria is restricted to strategies \( \beta \) that are increasing and that satisfy the boundary condition \( \beta(s) = v(\bar{s}, \bar{s}, k) \). (Under the same assumptions, part a. of GSWCP also holds for the k-th unit auction)

**Proof.** k-th unit auction: Part a) follows from the fact that \( d\pi^{k,N}(b; \beta(s), \beta, s)/ds \) evaluated at \( b = \beta(s) \) has the same sign as \( \beta^{k,N}(s) - \beta(s) \). To prove this, differentiate \( \pi^{k,N}(b; \beta(s), \beta, s) \) with respect to \( b \) to obtain

\[
f(\phi(b) \mid s) \frac{d\phi(b)}{db} \left\{ E \left[ u(\bar{s}_i, \bar{s}_{-i}) \mid \bar{s}_i = s, \bar{y}(k) \leq s \right] - \beta(\phi(b)) \right\},
\]

which, when evaluated at \( b = \beta(s) \), becomes

\[
\frac{d\pi^{k,N}(b; \beta(s), \beta, s)}{db} \bigg|_{b=\beta(s)} = \frac{f(s \mid s)}{d\beta(s)/ds} \left\{ \beta^{k,N}(s) - \beta(s) \right\}.
\]

**First price auction:** To see part a), let \( \beta \not\leq \beta^{1st,N} \). Since \( \beta(s) = \beta^{1st,N}(s) \) and \( \beta \) is increasing, there exists \( s_1 < s_2 \) such that, for every \( s \in \tilde{S} = (s_1, s_2) \), a) \( \beta(s) < \beta^{1st,N}(s) \) and b) \( \frac{d}{ds} \beta(s) < \frac{d}{ds} \beta^{1st,N}(s) \) (see Esponda 2005b for the proof). Let \( s^* \in \tilde{S} \). From Proposition C1, a naive equilibrium satisfies \( \Pi(b^*, s^*; \beta^{1st,N}) = 0 \), or

\[
E \left[ u(\bar{s}_i, \bar{s}_{-i}) \mid \bar{s}_i = s^*, \bar{y}(1) \leq s^* \right] - \beta^{1st,N}(s^*) - \frac{d\beta^{1st,N}(s)}{ds} \bigg|_{s=s^*} \cdot F(s^* \mid s^*) = 0.
\]
From the previous relationships it then follows that,

\[ \Pi(b, s; \beta) = E \left[ u(\bar{s}_i, \bar{s}_{-i}) \mid \bar{s}_i = s^*, \bar{y}(1) \leq s^* \right] - \beta(s^*) - \frac{d\beta(s)}{ds} \bigg|_{s=s^*} \frac{F(s^* \mid s^*)}{f(s^* \mid s^*)} > 0. \]

Hence, the profit function of a naive bidder who faces \( \beta \) and chooses action \( \beta(s) \) has a positive slope at \( b = \beta(s) \). The claim follows since \( \tilde{S} \) is a set of strictly positive measure. The proof of part b) is very similar and appears in Esponda (2005b).

**Proposition C5.** For the symmetric first price auction, a symmetric sophisticated equilibrium in increasing strategies is bounded above by the unique Nash equilibrium and bounded below by the (essentially) unique naive equilibrium: \( \beta^{1st,N} \leq \beta^{1st,S} \leq \beta^{1st,NE} \) (for the symmetric \( k \)-th unit auction, \( \beta^{k,N} \leq \beta^{k,S} \)).

**Proof.** The result follows from Lemmas C3 and C4 and the fact (which is similar to the standard proof for Nash equilibrium) that any sophisticated equilibrium satisfies the boundary condition \( \beta^S(\bar{s}) = v(\bar{s}, \bar{s}, k) \).
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Figure 1b:
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