Revenue Management Without Commitment: Dynamic Pricing and Periodic Fire Sales

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Abstract

We consider a market with a profit-maximizing monopolist seller who has \( K \) identical goods to sell before a deadline. At each date, the seller posts a price and the quantity available but cannot commit to future offers. Over time, potential buyers with different reservation values enter the market. Buyers strategically time their purchases, trading off (1) the current price without competition and (2) a possibly lower price in the future with the risk of being rationed. We analyze equilibrium price paths and buyers’ purchase behavior in which prices decline smoothly over the time period between sales and jump up immediately after a transaction. In equilibrium, high-value buyers purchase on arrival. Crucially, before the deadline, the seller may periodically liquidate part of his stock via a fire sale to secure a higher price in the future. Intuitively, these sales allow the seller to ‘commit’ to high prices going forward. The possibility of fire sales before the deadline implies that the allocation may be inefficient. The inefficiency arises from the scarce good being misallocated to low-value buyers, rather than the withholding inefficiency that is normally seen with a monopolist seller.

Keywords: revenue management, commitment power, dynamic pricing, fire sales, inattention frictions.

JEL Classification Codes: D82, D83
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1 Introduction

Many markets share the following characteristics: (1) goods for sale are (almost) identical, and all expire and must be consumed at a certain point of time, (2) the initial number of goods for sale is fixed in advance, and (3) consumers have heterogeneous reservation values and enter the market sequentially over time. Such markets include the airline, cruise-line, hotel and entertainment industries. The revenue management literature studies the pricing of goods in these markets, and these techniques are reported to be quite valuable in many industries, such as airlines (Davis (1994)), retailers (Friend and Walker (2001)), etc. The standard assumptions in this literature are that sellers have perfect commitment power and buyers are impatient. That is, buyers cannot time their purchases and sellers can commit to the future price path or mechanism. In contrast, this paper studies a revenue management problem in which buyers are patient and sellers are endowed with no commitment power. ¹

We consider the profit-maximizing problem faced by a monopolist seller (he) who has \( K \) identical goods to sell before a deadline. At any date, the seller posts a price and the quantity available (capacity control) but cannot commit to future offers. Over time, potential buyers (her) with different reservation values (either high or low) privately enter the market. Each buyer has a single-unit demand and can time her purchase. Goods are consumed at the fixed deadline, and all trades happen before or at that point.

Our goal is to show that the seller can sometimes use fire sales before the deadline to credibly reduce his inventory and so charge higher prices in the future. We accordingly consider settings where the seller does not find it profitable to only sell at the deadline and then only to high-value buyers, with the accompanying possibility of unsold units. In such settings, we explore the properties of a pricing path in which, at the deadline, if the seller still has unsold goods, he sets the price sufficiently low that all remaining goods are sold for sure. For most of the time before the deadline, the seller posts the highest price consistent with high-value buyers purchasing immediately on arrival, and occasionally, he posts a fire sale price that is affordable to low-value buyers. By holding fire sales, the seller reduces his inventory quickly, and therefore, he can induce high-value buyers to accept a higher price in the future. Intuitively, these sales allow the seller to ‘commit’ to high prices going forward. Once the transaction happens, whether at the discount price or not, the seller’s inventory is reduced, and the price jumps up instantaneously. Hence, in

¹In the airline industry, although the computer system based on revenue management algorithms is widely used, revenue managers frequently adjust prices based on their information and personal experience instead of mechanically following the pricing strategy suggested by the algorithm. Hence, it is reasonable to investigate the case where the seller has limited commitment power. See "Confessions of an Airline Revenue Manager" by George Hobica. http://www.foxnews.com/travel/2011/12/08/confessions-airline-revenue-manager/.
general, a highly fluctuating path of realized sales prices will appear, which is in line with the observations in many relevant industries.²

The sub-optimality of only selling at the deadline to high-value buyers could occur for many reasons. For example, at the deadline, the seller may expect that there will be little effective high-value demand in the market. This may be because the arrival rate of high-value buyers is low, or because buyers may also leave the market without making a purchase, or because buyers face inattention frictions and so they may miss the deadline, which we discuss in detail below.

The equilibrium price path relies on the seller’s lack of commitment and buyers’ intertemporal concern. An intuitive explanation is as follows. At the deadline, due to the insufficient effective demand, the seller holding unsold goods sets a low price to clear his inventory, which is known as the last-minute deal.³ Before the deadline, since a last-minute deal is expected to be posted shortly, buyers have the incentive to wait for the discount price.⁴ However, waiting for a deal is risky due to competition at the low price, from both newly arrived high-value buyers and low-value ones who are only willing to pay a low price. By weighing the risk of losing the competition and so the deal, a high-value buyer is willing to make her purchase immediately at a price higher than the discount one. We name the highest price she is willing to pay to avoid the competition as her reservation price. For any such high-value buyer, her reservation price is decreasing in time, since the arrival of competition shrinks as the deadline approaches, and decreasing in the current inventory size, since the probability that she will be rationed at deal time depends on the amount of remaining goods. To maximize his profit, the seller posts the high-value buyer’s reservation price for most of the time and, at certain times before the deadline, may hold fire sales to reduce his inventory and to charge a higher price in the future.

Figure 1 illustrates this idea in the simplest case with only two items for sale at the beginning. Suppose the seller serves high-value buyers only before the deadline, allowing discounts at the deadline only. Conditional on the inventory size, the price declines in time. The high-value buyer’s acceptable price in the two-unit case is lower than the price in the one-unit case, and the price difference indicates the difference in the probability that a high value buyer is rationed at the last minute in different cases. If a high-value buyer enters the market early and buys a unit immediately, the seller can sell it at a relatively high price and earn a higher profit than

² For example, McAfee and te Velde (2008) find that airfares’ fluctuation is too high to be explained by the standard monopoly pricing models.


⁴ In the airline industry, many travelers are learning to expect possible discounts in the future and strategically time their purchase. See the Wall Street Journal, July 2002, “A Holiday for Procrastinators: Booking a Last-Minute Ticket,” by Eleena de Lisser.
he could earn from running fire sales. However, if no such buyer ever shows up, then the time will eventually come when selling one unit via a fire sale and then following the one-unit pricing strategy is more profitable to the seller. To see the intuition, consider the seller’s benefit and cost of liquidating the first unit via a fire sale. The benefit is that, by reducing one unit of stock, the seller can charge the high-value buyer who arrives next a higher price for his last unit. On the other hand, the (opportunity) cost is that, if more than one high-value buyer arrives before the deadline, the seller cannot serve the second one, who is willing to pay a price higher than the fire sale price. Since a new high-value buyer arrives independently, as the deadline approaches, the probability that more than one high-value buyer arrives before the deadline goes to zero much faster than the probability that one high-value buyer arrives. Thus, the opportunity cost is negligible compared to the benefit, and therefore, the seller has the incentive to liquidate the first unit via a fire sale.

Analyzing a dynamic pricing game with private arrivals is difficult for the following reason. Since the seller can choose both the price and quantity available at any time, he may want to sell his inventory one-by-one. Thus, some buyers may be rationed when demand is less than supply before the game ends. Suppose a buyer was rationed at time $t$ and the seller still holds unsold units. The rationed buyer privately learns that demand is greater than supply at time $t$ and
uses the information to update her belief about the number of remaining buyers. Buyers who arrive after this transaction have no such information. As a result, belief heterogeneity among buyers naturally occurs following their private histories, and buyers’ strategies will depend on their private beliefs non-trivially. Such belief heterogeneity evolves over time and becomes more complicated as transactions happen one after another, making the problem intractable.

To overcome this technical challenge, we assume that buyers face inattention frictions. That is, in each “period” with a positive measure of time, instead of assuming that buyers can observe offers all the time, we assume that each buyer notices the seller’s offer and makes her purchase decision at her attention times only. In each “period,” a buyer independently draws one attention time from an atom-less distribution. In addition, buyers’ attention can be attracted by an offer with sufficiently low price, that is, a fire sale. This implies that (1) at any particular time, the probability that a buyer observes a non fire sale offer is zero, (2) the probability that more than one buyer observes a non-fire-sale offer at the same time is zero too, and (3) all buyers observe a fire-sale-offer when it is posted. As a result, high-value buyers would not be rationed except at deal time. Furthermore, we focus on equilibria where high-value buyers make their purchases upon arrivals. Therefore, a high-value buyer being rationed at deal time attributes failure of her purchase to the competition with low-buyer buyers instead of other high-value buyers, so she cannot infer extra information about the number of buyers in the market. As we will show, there is an equilibrium in which buyers’ strategies do not depend on their private histories.

As we described earlier, we are interested in the environment where the seller finds selling only at the deadline and serving only high-value buyers to be suboptimal. In the presence of inattention frictions, the seller cannot guarantee that the high-value buyers will be available at the deadline. Hence, at the deadline, to maximize his profit, the seller has to post a last-minute deal to draw full attention of the market, which naturally leads the seller to start selling early.

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5In the airline ticket example, it is natural to assume each buyer checks the price once or twice per day instead of looking at the airfare website all the time.

6In practice, this extra chance is justified by consumers’ attention being attracted by advertisements of deals sent by a third party: low price alert e-mails from intermediate websites such as http://www.orbitz.com and http://www.fare detective.com.

7The idea that, in a continuous-time environment, decision times arrive randomly is not new. See for example, Perry and Reny (1993) and Ambrus and Lu (2014) in bargaining models. However, none of those papers employ such an assumption to avoid the complexities of private beliefs.

8Notice that our economic prediction on the price path does not depend on the presence of inattention frictions. As we mentioned before, a low arrival rate of buyers or the disappearance of present buyers can also exclude the trivial case where the seller is willing to sell at the deadline only. We explore the possibility of disappearing buyers in the extension.
1.1 The Literature

There is a large revenue management literature that has examined the market with sellers who need to sell finitely many goods before a deadline and impatient buyers who arrive sequentially. Gershkov and Moldovanu (2009) extend the benchmark model to the heterogeneous objects case. The standard assumption maintained in these works is that buyers are impatient, and therefore cannot strategically time their purchases. However, as argued by Besanko and Winston (1990), mistakenly treating forward-looking customers as myopic may have an important impact on sellers’ revenue. Board and Skrzypacz (2010) characterize the revenue-maximizing mechanism in a model where agents arrive in the market over time. In the continuous time limit, the revenue-maximizing mechanism is implemented via a price-posting mechanism, with an auction for the last unit at the deadline. In a framework similar to that of Board and Skrzypacz (2013), Li (2013) considers a similar model and characterizes the allocation policy that maximizes the expected total surplus and its implementation.

In the works mentioned above, perfect commitment of the seller is typically assumed. Little has been done to discuss the case in which a monopolist with scarce supply and no commitment power sells to forward-looking customers. Aviv and Pazgal (2008) consider a two-period case, and so do Jerath, Netessine, and Veeraraghavan (2010). Deb and Said (2012) study a two-period problem where a seller faces buyers who arrive in each period. They show that the seller’s optimal contract pools low-value buyers, separates high-value ones, and induces intermediate ones to delay their purchase.

To the best of our knowledge, Chen (2012) and Hörner and Samuelson (2011) have made the first attempt to address the non-commitment issue in a revenue management environment using a multiple-period game-theoretic model. They assume that the seller faces a fixed number of buyers who strategically time their purchases. They show that the seller either replicates a Dutch auction or posts unacceptable prices up to the very end and charges a static monopoly price at the deadline. However, as argued by McAfee and te Velde (2008), arrival of new buyers seems to be an important driving force of many observed phenomena in a dynamic environment. As we will show, the sequential arrival of buyers plays a critical role in the seller’s optimal pricing and fire sale decision.

Additionally, our model is also related to the durable goods literature in which the seller without capacity constraint sells durable goods to strategic buyers over an infinite horizon. As Hörner and Samuelson (2011) show, the deadline endows the seller with considerable commitment power, and the scarcity of the good changes the issues surrounding price discrimination, with

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9See the book by Talluri and van Ryzin (2004).
the impetus for buying early at a high price now arising out of the fear that another buyer will snatch the good in the meantime. In the standard durable goods literature, the number of buyers is fixed. However, some papers consider the arrival of new buyers. Conlisk et.al (1984) allows a new cohort of buyers with binary valuation to enter the market in each period and show that the seller will vary the price over time. In most periods, he charges a price just to sell immediately to high-value buyers. Periodically, he charges a sales price to sell to accumulated low-value buyers. Even though, similar to Conlisk et.al (1984), new arrivals and heterogeneous valuation are also the driving force of fire sales in our model, the economic channels are very different. In their papers, the seller has discounting a cost, so charges low price to sell to accumulated low-value buyers in order to reap some profit and avoid delay costs. However, in our model, the seller does not discount and can ensure a unit profit as the fire sales income at the deadline for all inventory. Since the buyers face scarcity, the seller liquidates some goods to convince future buyers to accept higher prices. Garret (2013) considers a durable good model where (1) the seller has perfect commitment power, (2) the buyer arrives privately, and (3) the present buyer’s value change stochastically. He shows that in such a stationary environment, fire sales may also periodically appears. In neither Conlisk et.al (1984) nor Garret (2013), the present of fire sale is driven by the lack of commitment of the seller.

The rest of this paper is organized as follows. In Section 2, we present the model setting and define the solution concept we are going to use. In Section 3, we derive an equilibrium in the single-unit case. In Section 4, the multi-units case is studied. In Section 5, we provide two applications of our model. Section 6 discusses some modeling choices and possible extensions of the baseline model. Section 7 concludes. The proofs of main results are in Appendices.

2 Model

Environment. We consider a dynamic pricing game between a single seller who has $K$ identical and indivisible items for sale and many buyers. Goods are consumed at a fixed time that we normalize to 1, and deliver zero value after. Time is continuous. The seller has the interval $[0, 1]$ of time in which to trade with buyers. There is a parameter $\Delta$ such that $1/\Delta \in \mathbb{N}$. The time interval $[0, 1]$ is divided into periods: $[0, \Delta), [\Delta, 2\Delta), ...[1 - \Delta, 1]$. The seller and the buyers do not discount.

Seller. The seller can adjust the price and supply at each moment: at time $t$, the seller posts the price $P(t) \in \mathbb{R}$, and capacity control $Q(t) \in \{1, 2, \ldots, K(t)\}$, where $K(t) \in \mathbb{N}$ represents the amount of goods remaining at time $t$, and $K(0) = K$.\footnote{We assume $Q(t) \neq 0$. However, the seller can post a price sufficiently high to block any transactions.} The seller has a zero reservation value
on each item, so his payoff is the summation of all transaction prices.

**Buyers.** There are two kinds of buyers: low-value buyers (L-buyers, henceforth) and high-value buyers (H-buyers, henceforth). Each buyer has a single unit of demand. Let $v_L$ denote an L-buyer’s reservation value of the unit, and $v_H$ that of an H-buyer, where $v_H > v_L > 0$. A buyer who buys an item at price $p$ gets payoff $v - p$ where $v \in \{v_L, v_H\}$.

**Population Dynamics.** The population structure of buyers changes differently over time. At the beginning, there is no H-buyer in the market. As time goes on, H-buyers arrive privately at a constant rate $\lambda > 0$. Let $N(t)$ be the number of H-buyers. Without loss of generality, we normalize the initial number of H-buyer to be zero: $N(0) = 0$. An H-buyer leaves only if her demand is satisfied.\(^{11}\) For tractability, we assume that the population structure of L-buyers is relatively predictable and stationary. At the beginning of the game $M$ L-buyers arrive in the market, where $M \in \mathbb{N}$ is common knowledge. Once an L-buyer’s demand is satisfied, she leaves the market immediately, and another L-buyer immediately replace her.\(^{12}\) This is to say, at any moment, it is common knowledge that there are $M$ L-buyers in the market. For simplicity, we assume $M \geq K(0)$, which means the seller can liquid all inventory by serving L-buyers at any time.\(^{13}\)

**Transaction Mechanism.** If the amount of demand at price $P(t)$ is less than or equal to $Q(t)$, all demands are satisfied; otherwise, $Q(t)$ randomly selected buyers are able to make purchases, and the rest are rationed. A price lower than $v_L$ is always dominated by $v_L$. Thus, L-buyers do not face non-trivial purchase time decisions. To save notation, we assume that they are non-strategic and will accept any price no higher than $v_L$. We define such a price as a deal.

**Definition 1.** A deal is an offer with $P(t) \leq v_L$.

If $i \leq Q(t)$ goods are sold at time $t$, the seller’s inventory goes down. In other words, $\lim_{t' \downarrow t} K(t') = K(t) - i$. Over time, as buyers make purchases, the inventory decreases after purchases. Hence, the path of the inventory process $K(t)$ is left continuous and non-increasing over time. Once $K(t)$ hits zero or time reaches the deadline, the game ends.

**Inattention Frictions.** We assume that buyers, regardless of their reservation value and arrival times, face inattention frictions. At the beginning of each period, all buyers, regardless

\(^{11}\)Our results continue to hold when H-buyers leave the market at a rate $\rho \geq 0$.

\(^{12}\)An added value of this assumption is that it allows us to highlight our channel to generate fire sales. In Conlisk et al (1984), the presence of periodic sales is driven by the arrival and accumulation of low-value buyers. By assuming that the population structure of low-value buyers is stationary, their classical explanation of a price cycle does not work in our model.

\(^{13}\)We make this assumption to simplify the presentation of the paper. Since the replacement of an left L-buyer takes no time, the seller can sequentially sell all inventory as soon as he wants.
of their value, randomly draw an attention time $\tau$, which is uniformly distributed in the time interval of the current period.\footnote{Our results hold for any atom-less distribution with full support.} For an H-buyer who arrives in the period, her attention time in the current period is her arrival time. In the period where the seller posts a deal at time $\tau$, each buyer has an additional attention time at time $\tau$ in the current period. In the rest of this paper, we call these random attention times exogenously assigned by Nature regular attention times, while we call the additional attention time deal attention times. A buyer observes the offer posted, $P(t), Q(t)$ and the seller’s inventory size, $K(t)$ at her attention time only. At that time, she can decide to accept or reject the offer. Rejection is not observed by the seller or other buyers. Since, without deal announcements, each buyer draws her attention time independently, once a buyer observes and decides to take an available offer $P(t) > v_L$, she will not be rationed. Thus the competition among buyers is always intertemporal when $P(t) > v_L$. At deal times when $P(t) \leq v_L$, buyers observe the offer at the same time, so there is direct competition among buyers. Notice that $\Delta$ capture the inattention frictions of buyers.

**History.** A non-trivial seller history at time $t$, $h^t_S = (P(\tau), Q(\tau), K(\tau))_{0 \leq \tau < t}$, is a history such that the game is not over before $t$ and it summarizes all relevant transactions and information about offers in the past. Let $\mathcal{H}_S$ be the set of all seller’s history. The seller’s strategy $\sigma_S$ determines a price $P(t)$ and capacity control $Q(t)$ given a seller history $h^t_S$. Due to the buyers’ inattention frictions, at any time before the deadline, the seller believes that more than one buyer notices an offer with probability zero. As a result, we focus on the seller’s strategy space in which $Q(t) = 1$ for $P(t) > v_L$ without loss of generality.

Let $a(t)$ be an index function such that it is 1 at an H-buyer’s attention times, and 0 otherwise. Thus, $a^t = \{a(\tau)\}_{\tau=0}^t$ records the history of an H-buyer’s past attention times up to $t$. A non-trivial buyer history, $h^t_B = \{a^t, \{P(\tau), Q(\tau), K(\tau)\}_{\tau:a(\tau)=1 \text{ and } \tau \in [0,t]}\}$. In other words, a buyer remembers the prices, capacity and inventory size she observed at her past attention times. Let $\mathcal{H}_B$ denote the set of all history of an H-buyer. Following Chen (2012) and Hörner and Samuelson (2011), we focus on symmetric equilibria in which an H-buyer’s strategy depends only on her history not on her identity. That is to say, the H-buyer’s strategy $\sigma_B$ determines the probability that she will accept the current price $P(t)$ given a buyer’s history $h^t_B$. We focus on a pure strategy profile, so $\sigma_B \in \{0, 1\}$.

### 2.1 Admissible Strategies

We choose a continuous time model in this project, since it has technical advantages in answering our questions. Specifically, the determination of the optimal timing for fire sales is in
fact an optimal stopping time problem; therefore, the continuous-time properties of this problem make the analysis easier. However, continuous time raises obstacles to the analysis of dynamic games. First, it is well known that, in a continuous time game, a well-defined strategy may not induce a well-defined outcome. This is analyzed by Simon and Stinchcombe (1989) and Bergin and MacLeod (1993). The reason is that there is no well-defined “last” or “next” period in a continuous time game; hence, players’ actions at time $t$ may depend on information arriving instantaneously before $t$.

Therefore, to make this game well-defined, we must impose additional restrictions on the set of strategies. Following Bergin and MacLeod (1993), we restrict the seller’s choices in the admissible strategy space. Specifically, to construct the set of admissible strategies, we first restrict the strategy to the inertia strategy space. Intuitively speaking, an inertia strategy is such that instead of an instantaneous response, a player can change her decision only after a very short time lag; hence, such strategy cannot be conditional on very recent information. The set of all inertia strategies includes strategies with arbitrarily short lags, so it may not be complete. To capture the instantaneous response of players, we complete the set and use the completion as the feasible strategy set of our game. For each instantaneous response strategy, we identify its associated outcome as follows. First, we find a sequence of inertia strategies converging to the instantaneous strategy. In such a sequence, each inertia strategy has a well-defined outcome, which gives us a sequence of outcomes. Second, we identify the limit of the outcome sequence as the outcome of this instantaneous response strategy. Lastly, because of the presence of inattention frictions, multiple buyers observe a price higher than $v_L$ at the same time with zero probability. Hence, without loss of any generality, we can restrict the strategy space such that $Q(t) = 1$ for $P(t) > v_L$. Let $\Sigma^*_S$ as the admissible strategy space of the seller. Since H-buyers face inattention frictions, they cannot revise their decision instantaneously, so we do not need to impose any restriction on their strategy; let $\Sigma^*_B$ denote the set of strategies of H-buyers, and let $\Sigma^* = \Sigma^*_S \times \Sigma^*_B$ be the strategy space we study.

### 2.2 Payoff and Solution Concept

In general, a player’s strategy depends on his or her private history. A perfect Bayesian equilibrium (PBE) in our game is a strategy profile of the seller and the buyers, such that given other players’ strategy, each player has no incentive to deviate, and players update their belief via Bayes’ rule where possible. However, the set of all perfect Bayesian equilibria of this game is hard to characterize since buyers may play private strategies depending on their private histories.

We instead look for simple but intuitive equilibria: (no-waiting) weak Markov perfect equilibria that satisfy the following properties. First, the equilibrium strategy profile must be simple; that
is, buyers’ equilibrium strategies can be described as functions of two public state variables specified later. Second, on the path of play, H-buyers do not delay their trade but make their purchases once they arrive. Third, we impose a restriction on buyers’ beliefs about the underlying history off the path of play: each H-buyer believes that there are no other previous H-buyers presently in the market. Notice that last restriction on buyers’ belief off the equilibrium path implicitly rule out the possibility that seller can manipulate buyers beliefs and therefore their willing to pay by charging high price. The restriction is necessary to obtain weak Markov equilibria. Otherwise, after some histories, buyers who saw different deviating prices may have heterogenous private beliefs about the number of present H-buyers and therefore heterogenous willing to pay so that their strategies have to be non-Markovian.

Note that, off the path of play, H-buyers may wait because of the deviation of the seller: the seller can post an unacceptable price for a time period in which H-buyers have to wait for future offers. However, each buyer can observe offers at her past attention times and, for the rest of time, she has to form a belief about the underlying history. The perfect Bayesian equilibrium concept does not impose any restriction on those beliefs where the Bayes’ rule does not apply. To support a (no-waiting) weak-Markov equilibrium, we assume that each H-buyer believes that the seller follows the equilibrium pricing strategy in such time periods. Since each buyer can only sample finitely many times, she believes the total measure of time period in which the seller deviated is zero. Since each H-buyer draws her attention time independently, the probability that other H-buyers can observe these deviating prices with positive probability. Consequently, each H-buyer believes that no other H-buyers are waiting in the market both on and off the path of play.

2.2.1 Payoff

To define the equilibrium, we need to specify an H-buyer’s payoff given she believes that no previous H-buyers are waiting in the market. Given a seller’s continuation strategy \( \bar{\sigma}_S \in \Sigma^*_S \), other H-buyers’ symmetric continuation strategy \( \bar{\sigma}_B \in \Sigma^*_B \), and a buyer’s history \( h^t_B \), an H-buyer’s expected continuation payoff at time \( t \) from choosing a continuation strategy \( \bar{\sigma}'_B \in \Sigma^*_B \) at her attention time is defined as

\[
U (\bar{\sigma}'_B, \bar{\sigma}_B, \bar{\sigma}_S, h^t_B) = \mathbb{E}_{\tau|t} [v_H - P(\tau)]
\]

whenever \( \tau \) is well-defined where \( \tau \in [t, 1] \cup \{2\} \) is H-buyers’ believed transaction time which is random and depends on the other players’ strategies and the population dynamics of buyers.\(^{15}\)

\(^{15}\)When \( \tau \) is not well-defined, the expression of the continuation payoff of the buyer will be explicitly given.
When \( \tau = 2 \), the buyer does not obtain the good because the seller’s stock is sold out before she decides to place an order. In this case, \( P(2) = v_H \). At his attention time \( t \), an H-buyer’s payoff is
\[
\max \{ v_H - p_t, U (\tilde{\sigma}'_B, \bar{\sigma}_B, \bar{\sigma}_S, h_B^t) \}
\]
Notice that the buyer believes that her continuation payoff does not depend on the current price. Hence, the buyer employs a cutoff strategy where she accepts a price if it is less than or equal to some reservation price \( p \), and this reservation price is pinned down by the buyer’s indifference condition:
\[
v_H - p = U (\tilde{\sigma}'_B, \bar{\sigma}_B, \bar{\sigma}_S, h_B^t)
\]
Suppose all H-buyers play a symmetric \( \tilde{\sigma}_B \in \Sigma_B^* \). The payoff to the seller with stock \( k \) from a strategy \( \tilde{\sigma}_S \in \Sigma_S^* \) is given by
\[
\Pi_k (\tilde{\sigma}_B, \tilde{\sigma}_S, h_S^t) = \mathbb{E}_\tau [P(\tau) + \Pi_{k-1} (\bar{\sigma}_B, \tilde{\sigma}_S, h_S^t)]
\]
where \( h_S^t \) is the seller’s history, \( \Pi_0 = 0 \). Because buyers face inattention frictions, by posting any price \( P(1) > v_L \), the seller expects no buyer notices the offer, and his expected profit is zero; by posting a deal price, the seller can sell as many goods as he wants. Apparently, the seller’s dominant strategy is to sell all of his inventory by charging \( v_L \). Hence, at the deadline, we have
\[
\Pi_k (\tilde{\sigma}_B, \tilde{\sigma}_S, h_S^1) = kv_L
\]
Note that the seller may or may not believe that there are previously arrived H-buyers waiting in the market. His belief about the number of H-buyers depends on the price he posted before.

### 2.2.2 Weak-Markov Perfect Equilibrium

We focus on weak-Markov perfect equilibria (weak-MPE), which is commonly used in dynamic pricing and Coasian bargaining literature. 16 We define a weak-MPE to be a PBE where an H-buyer makes her purchase decision based on the current price \( p_t \) and two state variables: calendar time, \( t \) and inventory size, \( K(t) \) and she makes the purchase on her arrival time on the equilibrium path. The seller’s equilibrium strategy depends on the entire private history, but can be describe as a function of two state variables on the path of play. Nonetheless, note that potential deviations can be either Markovian or non-Markovian.

Notice that in the equilibrium, the H-buyer’s strategy is a function of the calendar time and the seller’s inventory size, but it does not imply that the number of other H-buyers is payoff

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irrelevant to an H-buyer in general. In fact, an H-buyer’s continuation value does depend on her belief about the number of other H-buyers. However, we focus on no-waiting equilibria where each H-buyer believes that no other H-buyer is waiting in the market; thus, her strategy does not depend on her belief about the number of other H-buyers non-trivially. Hence, H-buyer’s strategy is Markovian both on and off the path of play. Since H-buyer’s belief cannot be manipulate by the seller, an H-buyer believes that his continuation value after any history does not depend on the current price. As a result, an H-buyer’s equilibrium strategy can be characterized by a cutoff price, which depends on two state variables. In particular, the reservation price $p_k(t)$ is pinned down by the buyer’s indifference condition:

$$v_H - p_k(t) = U(\tilde{\sigma}_B, \tilde{\sigma}_S, h^t_B).$$

where $U(\tilde{\sigma}_B, \tilde{\sigma}_S, h^t_B)$ denotes the buyer’s equilibrium continuation payoff in equilibrium, $(\tilde{\sigma}_B, \tilde{\sigma}_S)$ is the believed continuation equilibrium strategy profile, and the current inventory $K(t) = k$ is consistent with the current buyer’s history $h^t_B$. However, the seller’s equilibrium strategy is non-Markovian as in the standard Coase conjecture literature.\(^{17}\) Henceforth, we refer a (no-waiting) weak-MPE as an equilibrium unless otherwise mention.

3 Single Unit

We start by analyzing the game where $K(0) = 1$, the seller has one unit to sell: $Q(t) = 1$. Deriving equilibria in this game is the first step forward the analysis of more general games. We first provide an intuitive conjecture on an equilibrium of this game and verify our conjecture. Furthermore, we show that the equilibrium we proposed is the unique equilibrium.

The first observation is that the seller can ensure a profit $v_L$ because there are M L-buyers at any time. An intuitive conjecture of the seller’s strategy is to serve the H-buyers only before the deadline to obtain a profit higher than $v_L$ and charge $v_L$ at the deadline if no H-buyer arrives. Since an H-buyer would like to avoid a competition with (1) L-buyers at the deadline, and (2) other H-buyers who may arrive before the deadline, she is willing to forgo some surplus and accept a price higher than $v_L$. Moreover, as deadline approaches, the competition coming from newly arrived H-buyers becomes less and less intense, and therefore the H-buyer’s reservation price declines.

\(^{17}\)A more natural solution concept is Markov perfect equilibrium. However, it does not exist in general. Off the path of play, the seller’s continuation strategy is history dependent, which is similar to the standard Coase conjecture models. See Fudenberg and Tirole (1983) and Gul et al (1983) for detail.
Specifically, we conjecture that in equilibrium, the seller charges a price such that: (1) H-buyers accept it on arrivals, and (2) low type buyers make their purchases only at the deadline if the good is still available. The optimality of the seller’s pricing rule implies that, before the deadline, an H-buyer is indifferent between purchasing at time \( t \) and waiting: on the one hand, if the H-buyer strictly prefers to purchase the good immediately, the seller can raise the price a little bit to increase his profit; on the other hand, if the price is so high that the H-buyer strictly prefers to wait, the transaction will not happen at time \( t \) and all H-buyers wait in the market. Furthermore, we will show that accumulating H-buyers is suboptimal for the seller because the H-buyers’ reserve prices are declining over time. At the deadline, the seller will charge the price \( v_L \) to clean out his stock since he believes that there are no H-buyers left.

We give a heuristic description of the equilibrium in the main text and leave the formal analysis to the Appendix. At the deadline, the H-buyer’s reservation price is \( v_H \). However, the probability that an H-buyer’s regular attention time is at the deadline is zero; thus, the dominant pricing strategy for the seller is to post a deal price \( v_L \) to obtain a positive profit. As a result, in any equilibrium, \( P(1) = v_L \). For the rest of the time, we denote \( p_1(t) \) as an H-buyer’s reservation price at her attention time \( t < 1 \) and the inventory size \( K(t) = 1 \). Consider an H-buyer with an attention time \( t \in [1 - \Delta, 1) \); thus, the probability that new H-buyers arrive before the deadline is \( 1 - e^{-\lambda(1-t)} \). Suppose this H-buyer understands that on the path of play, no H-buyer who has arrived before her waited. Therefore, she believes that she is the only H-buyer in the market. She then faces the following trade-off:

1. if she accepts the current offer, she gets the good for sure at a price which is higher than \( v_L \);
2. if she does not accept the current offer, the seller will believe that no H-buyer arrived and to obtain a positive profit, he will charge a price \( v_L \) to liquidate the good at the deadline. In the latter situation, the H-buyer has to compete with \( M \) L-buyers for the item, and the probability she is not rationed is \( \frac{1}{M+1} \).

These considerations can pin down an H-buyer’s reservation price, \( p_1(t) \), at which she is indifferent between accepting the offer or not at time \( t \). Specifically, the indifference condition of an H-buyer whose attention time is \( t \) is given as follows:

\[
v_H - p_1(t) = e^{-\lambda(1-t)} \frac{1}{M+1} (v_H - v_L).
\]

The left-hand side represents the H-buyer’s payoff if she purchases the good now; the right-hand side represents the expected payoff if she waits, which is risky because (1) other H-buyers may
arrive in \((t, 1)\) with a probability \(1 - e^{-\lambda (1-t)}\), and (2) she has to compete with \(M\) L-buyers at the deadline. Differentiating equation (1) with respect to \(t\), we have \(\dot{p}_1 (t) = -\lambda [v_H - p_1 (t)]\).

Letting \(t \to 1\), we obtain the limit price right before the deadline,

\[
p_1 (1^-) = \frac{M}{M + 1} v_H + \frac{1}{M + 1} v_L. \tag{2}
\]

Hence, if \(M\) is large, the limit price right before the deadline is very close to \(v_H\). Note that \(p_1 (1^-)\) is different from the H-buyer’s actual reservation price at the deadline, \(v_H\). Let \(U_{1-\Delta}\) denote an H-buyer’s expected utility in the last period.\(^{18}\) Since her attention time, \(\tilde{t}\), is a random variable, we have

\[
U_{1-\Delta} = \int_{1-\Delta}^1 \frac{1}{\Delta} e^{-\lambda (\tilde{t}-1+\Delta)} [v_H - p_1 (\tilde{t})] d\tilde{t} \tag{3}
\]

\[
= \int_{1-\Delta}^1 \frac{1}{\Delta} \left[ e^{-\lambda \Delta} \frac{v_H - v_L}{M + 1} \right] d\tilde{t}.
\]

Notice that, for each \(\tilde{t}\), the H-buyer’s ex ante payoff, by considering the risk of the arrival of new buyers and the price declining until \(\tilde{t}\), is \(e^{-\lambda \Delta} \frac{v_H - v_L}{M + 1}\), which is independent from \(\tilde{t}\). Hence, \(U_{1-\Delta} = v_H - p_1 (1 - \Delta)\).\(^{19}\)

Now, consider the H-buyer’s reservation price at an earlier time. Note that, when \(K (0) = 1\), the seller can ensure a profit \(v_L\) at any time by charging the fire sale price. However, he expects to charge a higher price to H-buyers who arrive early and want to avoid competition with H-buyers who arrive in the future and L-buyers. As a result, the fire sale price \(v_L\) is charged only at the deadline. At any other time \(t\), the seller targets H-buyers only and offers a price \(p_1 (t)\). Consider an H-buyer whose attention time is \(t \in [1 - 2\Delta, 1 - \Delta)\). Her indifference condition is given by

\[
v_H - p_1 (t) = e^{-\lambda (1-\Delta-t)} U_{1-\Delta}, \tag{4}
\]

where the left-hand side represents the H-buyer’s payoff if she purchases the good now; the right-hand side represents the expected payoff if she waits, with probability \(e^{-\lambda (1-\Delta-t)}\), she is still in the market at the beginning of the next period and the good is still available; so she can draw a new attention time in the last period and expect a payoff \(U_{1-\Delta}\). Differentiating equation (4) with respect to \(t\), we have \(\dot{p}_1 (t) = -\lambda [v_H - p_1 (t)]\). As \(t\) goes to \(1 - \Delta\), \(v_H - p_1 (t)\) converges to

\(^{18}\)Notice that \(U_{1-\Delta}\) is not the buyer’s continuation value at time the beginning of the last period, but the expected continuation value in the last period over the buyer’s attention time, which is equal to the left-limit of the buyer’s continuation value at \(1 - \Delta\), i.e. \(U_{1-\Delta} = \lim_{\tilde{\sigma} \to \Delta^-} U (\tilde{\sigma}, \hat{h}_B)\).

\(^{19}\)This implies that (1) an H-buyer at the beginning of the last period, is indifferent between being assigned any attention time in the current period, and (2) the H-buyer’s equilibrium continuation value is continuous with respect to \(t\) at \(1 - \Delta\).
$U_{1-\Delta}$, so the buyer’s equilibrium continuation value is continuation at $1-\Delta$. As a result, $p_1(t)$ is differentiable in $[1-2\Delta, 1)$. Repeating the argument above for $1/\Delta$ times, we have the ordinary differential equation (ODE, henceforth) for the H-buyers’ reservation price $p_1(t)$ such that

$$\dot{p}_1(t) = -\lambda(v_H - p_1(t)) \text{ for } t \in [0, 1),$$  \hfill (5)

with a boundary condition (2). In our conjectured equilibrium, the price the seller charges is $p_1(t)$ for $t \in [0, 1)$ and it jumps down to $v_L$ at the deadline.

Similarly, we can derive the seller’s payoff $\Pi_1(t)$. At the deadline, $\Pi_1(1) = v_L$ since the good is sold for sure at the fire sale price. Before the deadline, for a small $dt > 0$, the profit follows the following recursive equation:

$$\Pi_1(t) = p_1(t) \lambda dt + (1 - \lambda dt) \Pi_1(t + dt) + o(dt),$$

where an H-buyer arrives and purchases the good at time $t$ with probability $\lambda dt$, and no H-buyer arrives with a complementary probability. By taking $dt \to 0$, the seller’s profit must satisfy the following ODE:

$$\dot{\Pi}_1(t) = \lambda(\Pi_1(t) - p_1(t)),$$  \hfill (6)

with a boundary condition $\Pi_1(1) = v_L$. Note that, even though the equilibrium price is not continuous in time at the deadline, the seller’s profit is because the probability that the transaction happens at a price higher than $v_L$ goes to zero as $t$ approaches the deadline.

In short, in our conjectured equilibrium, H-buyers accept a price not higher than their reservation price $p_1(t)$, and the seller posts such price for any $t < 1$, and $v_L$ at the deadline. No H-buyer waits on the path of play. The next question is whether players have the incentive to follow the conjectured equilibrium strategies. A simple observation is that no H-buyer has the incentive to deviate since she is indifferent between taking and leaving the offer at any attention time. What about the seller? Does the seller have the incentive to do so and accumulate H-buyers for a while before the deadline? The answer is again no. This is because each buyer believes that no previous buyers are waiting in the market, and the seller is going to follow the equilibrium pricing rule in the continuation play. Since the H-buyer’s reservation price declines over time, the seller always wants to serve the earliest H-buyer. Formally,

**Proposition 1.** Suppose $K = 1$. There is a unique equilibrium in which,

1. for any non-trivial seller’s history, the seller posts a price, $P(t)$ s.t.

$$P(t) = \begin{cases} p_1(t), & \text{if } t \in [0, 1) \\ v_L, & \text{if } t = 1, \end{cases}$$
Figure 2: The equilibrium price path in the single-unit case, $K = 1$. The parameter values are $v_H = 1, v_L = 0.7, M = 3, \text{ and } \lambda = 2$.

where 

$$p_1 (t) = v_H - \frac{v_H - v_L}{M + 1} e^{-\lambda(1-t)}.$$ 

2. an H-buyer accepts a price at her attention time $t \in [0, 1)$ if and only if it is less than or equal to $p_1 (t)$ and she accepts any price no higher than $v_H$ at the deadline.

A simulated equilibrium price path can be found in Figure 2. Given the equilibrium strategy profile, the seller’s equilibrium expected payoff at $t$ can be calculated as follows.

$$\Pi_1 (t) = \int_t^1 e^{-\lambda(s-t)} \lambda p_1 (s) \, ds + e^{-\lambda(1-t)}v_L.$$ 

Notice that neither $p_1 (t)$ nor $\Pi_1 (t)$ depends on $\Delta!$ This is because each H-buyer is indifferent to purchase the good at any time. Fire sales appear with positive probability at the deadline only, that is, the last-minute deal. With probability $e^{-\lambda}$, no H-buyer arrives in the market and the seller posts the last-minute deal. The good is not allocated to an L-buyer unless no H-buyer arrives. As a result, the allocation rule is efficient.
4 Multiple Units

In this section we consider the general case in which the seller has $K > 1$ units to sell. Since most intuition can be explained for the two-unit case, we provide a heuristic description of the equilibrium in a two-unit case, and we then state the equilibrium for $K > 2$.

4.1 The Two-Unit Case

Consider the case where $K = 2$. A simple observation is that, after the first transaction at time $\tau$, $K(t) \leq 1$ for $t \in (\tau, 1]$, and what happens afterwards is characterized by Proposition 1. The question is how the first transaction happens: what is the sale price and when does the H-buyer accept the offer? Note that the seller has a choice to post a price $v_L$ at any $t$. Since this price is so low that L-buyers can afford it, a transaction will happen for sure and the seller’s stock switches to $K(t^+) = K(t) - 1$. In equilibrium, the earliest time at which the seller is willing to sell the first item at the price $v_L$ is denoted by $t^*_1$. In principle, when $K(t) = 2$, $t^*_1$ can be any time before or at the deadline. As we have shown in Proposition 1, in any continuation game with $K(t) = 1$, on the equilibrium path, the seller charges the price $v_L$ only at the deadline; hence, the last equilibrium fire sale time is always $t^*_0 = 1$. However, it is not clear yet when the first equilibrium fire sale time is. Note that, because of the scarcity of the goods at the price $v_L$, an H-buyer may be rationed at $t^*_1$. Consequently, she is willing to pay a higher price before $t^*_1$.

We conjecture that the equilibrium should satisfy the following properties. Before $t^*_1$, the seller posts a price such that an H-buyer is willing to purchase the good once she arrives. Once an H-buyer buys the good, the amount of stock held by the seller jumps to one. From that moment on, the equilibrium is described by Proposition 1. Similar to the single-unit case, when $K(t) = 2$, an H-buyer’s reservation price at $t \leq t^*_1$, $p_2(t)$, satisfies the following ODE:

$$\dot{p}_2(t) = -\lambda [p_1(t) - p_2(t)] \text{ for } t \in [0, t^*_1)$$

(7)

The intuition is as follows. Suppose, at $t < t^*_1$, an H-buyer sees the price $p_2(t)$. It is risky for her to wait because a new H-buyer arrives at rate $\lambda$ and gets the first good at price $p_2(t)$, in which case the original buyer can get the second good only at price $p_1(t)$. At her attention time $t$, the H-buyer is indifferent between taking the current offer and waiting only if the price declining effect, measured by $\dot{p}_2(t)$, can compensate the possible loss.

Since the seller may obtain a higher unit-profit by selling a good to an H-buyer instead of to an L-buyer, a reasonable conjecture is as follows. In equilibrium, the seller does not run any fire sales prior to the deadline. In other words, the first fire sale time is $t^*_1 = 1$, and the seller’s optimal price path, $P(t)$, is such that (1) $P(t) > v_L$ for $t < 1$, (2) an H-buyer takes the offer
once she arrives, and (3) the seller runs a clearance sale at the deadline. Now that $K(t) = 2$, the equilibrium price satisfies the ODE (7) with $t^*_1 = 1$. At the deadline, the seller has to post $v_L$, and an H-buyer can obtain a good at the deal price with probability $\frac{2}{M+1}$; thus, the boundary condition of the ODE (7) at $t = 1$ is $p_2(1^-) = \frac{2}{M+1}v_H + \frac{M-1}{M+1}v_L$. This strategy profile, however, is not an equilibrium!

**Lemma 1.** *In any equilibrium, $t^*_1 < 1$.*

Lemma 1 rules out the aforementioned conjecture. To see why, first note that $p_2(t) < p_1(t)$ for $t < 1$ since an H-buyer is more likely to get the good when the supply is 2. As $t$ approaches the deadline, the probability that a new H-buyer arrives before the deadline becomes smaller and smaller. The probability that only one H-buyer arrives before the deadline is approximated by $\lambda(1 - t)$. In this case,

1. if the seller naively posts price $p_2(t)$, his profit is $p_2(\tau) + v_L$ where $\tau$ is the H-buyer’s arrival time.

2. Alternatively, if the seller runs a one-unit fire sale before the arrival, he can ensure a payoff of $v_L$ immediately and expect a price $p_1(\tau) > p_2(\tau)$ in future.

When $t$ is close to the deadline, the *benefit* of price cutting is approximated by $p_1(1) - p_2(1)$. On the other hand, there is an *opportunity cost* to holding a fire sale before the deadline. More than one H-buyer may arrive before the deadline and the probability of this event is approximated by $\lambda^2(1 - t)^2$. In this case, if the seller naively posts price $p_2(t)$ and $p_1(t)$ to the end but does not post $v_L$, his profit is approximated by $p_2(1) + p_1(1)$. Thus the opportunity cost of the fire sale is approximated by $p_2(1) - v_L$ when $t$ is close to the deadline. As $t$ goes to 1, $\lambda^2(1 - t)^2$ goes to zero at a higher speed than $\lambda(1 - t)$; thus, the cost is dominated by the benefit for $t$ close enough to 1, and therefore, the seller will post the fire sale price $v_L$ to liquidate one unit at $t^*_1 < 1$ to raise future H-buyers’ reservation price. In other words, the fire sale plays the role of a commitment device.

We leave the formal equilibrium construction to the Appendix but illustrate the idea here to provide intuition. Suppose buyers believe that the fire sale time is $t^*_1$. For $t < t^*_1$, and $K(t) = 2$, an H-buyer’s reservation price satisfies the ODE (7); for $t \in [t^*_1, 1)$ and $K(t) = 2$, H-buyers believe that the seller is going to post $v_L$ immediately, and thus their reservation prices satisfies the following equation

$$v_H - p_2(t) = \frac{1}{M+1}(v_H - v_L) + \frac{M}{M+1}[v_H - p_1(t)],$$

19
where the left-hand side of the equation is the H-buyer’s payoff by accepting her reservation price and obtaining the good now, and the right-hand side is her expected payoff by rejecting the current offer. With probability $\frac{1}{M+1}$, the H-buyer gets the good at the deal price right after time $t$, and with a complementary probability, an L-buyer gets the deal and the H-buyer has to take $p_1(t)$ at her next attention time. Since $\Delta$ is small, one can ignore the arrivals and the time difference between two adjacent attention times of the H-buyer, and therefore, the H-buyer’s reservation price at $t \in [t^*_1, 1)$ is given by

$$p_2(t) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_1(t).$$

(8)

Since $p_1(t)$ is continuous on $[0, 1)$, $p_2(t)$ must be right continuous at $t^*_1$. On the other hand, the incentive-compatible condition of the H-buyer implies that $p_2(t)$ must be left continuous at $t^*_1$, and thus the boundary condition of the ODE (7) is

$$p_2(t^*_1) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_1(t^*_1).$$

(9)

As a result, an H-buyer’s reservation price at $t$ when $K(t) = 2$ critically depends on her belief about $t^*_1$.

Given H-buyers’ common beliefs about $t^*_1$, and their reservation prices when $K(t) = 2$, the seller’s problem is to choose his optimal fire sale time to maximize his profit; i.e.:

$$\Pi_2(t) = \max_{t_1} \int_t^{t_1} e^{-\lambda(s-t)} \lambda [p_2(s) + \Pi_1(s)] ds + e^{-\lambda(t_1-t)} [v_L + \Pi_1(t_1)].$$

In equilibrium, buyers’ belief is correct, so the seller’s optimal fire sale time is $t^*_1$ itself. The first-order-condition of the seller’s problem at $t^*_1$ is:

$$\lambda [p_2(t^*_1) - v_L] + \Pi_1(t^*_1) = 0.$$ 

(10)

At $t^*_1$, a transaction happens at price $v_L$ for sure, so we have

$$\Pi_2(t^*_1) = \Pi_1(t^*_1) + v_L,$$

(11)

which is the well-known value-matching condition.

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20In a continuous time game, one cannot define the "next period" appropriately. Hence, there is no well-defined fire sales time right after time $t$. However, in our model, to make sure the buyer’s strategy is well-defined, we only need to appropriately define the buyer’s continuation value if she decides to wait. Since the strategy space is the completion of the space of inertia strategy, the buyer’s continuation payoff can be uniquely specified by the limit of a sequence of payoff induced by a sequence of associated inertia strategies.
For \( t < t_1^* \), and \( K(t) = 2 \), the seller posts the H-buyer’s reservation price, \( p_2(t) \), and his expected profit is given by

\[
\Pi_2(t) = \lambda dt \left[ p_2(t) + \Pi_1(t + dt) \right] + (1 - \lambda dt) \Pi_2(t + dt) + o(dt).
\]

Taking \( dt \to 0 \), the seller’s profit satisfies the following Hamilton-Jacobi-Bellman (henceforth, HJB) equation

\[
\dot{\Pi}_2(t) = -\lambda \left[ p_2(t) + \Pi_1(t) - \Pi_2(t) \right].
\]

Combining (10), (11) and (12) at \( t_1^* \) yields

\[
\dot{\Pi}_2(t_1^*) = \dot{\Pi}_1(t_1^*),
\]

which is known as the smooth-pasting condition.

As a result, at the equilibrium fire sale time \( t_1^* \), three necessary conditions (9), (11), and (13) must hold. The necessity of the value-matching condition (11) and the smooth-pasting condition (13) comes from the optimal stopping time property of the interior fire sale time, and condition (9) results from the H-buyers’ incentive-compatible condition. When time is arbitrarily close to \( t_1^* \), the probability that new H-buyers arrive before \( t_1^* \) shrinks, and the H-buyer needs to choose between taking the current offer and waiting to compete with the L-buyers for the deal. Therefore, her reservation price must make the H-buyer indifferent between taking it and rejecting it. If \( t \) is not close to \( t_1^* \), the competition from newly arrived H-buyers before \( t_1^* \) is non-trivial, and therefore, to convince an H-buyer to accept the price, it must satisfy the ODE (7) with a boundary condition (9) at \( t_1^* \). Figure 3 shows a simulated equilibrium price path. The following proposition formalizes our heuristic description of the equilibrium.

**Proposition 2.** Suppose \( K(0) = 2 \). There exists a unique equilibrium. The H-buyer’s equilibrium strategy is characterized by a Markove reservation price: \( p_1(t) \) and \( p_2(t) \) when \( t < 1 \), \( K(t) = 1 \) and 2, respectively where \( p_1(t) \) is specified in Proposition 1 and

\[
p_2(t) = \begin{cases} 
  v_H - \frac{v_H - v_L}{M+1} e^{-\lambda(1-t)} \left[ e^{\lambda(t_1^*-t)} + \frac{M}{M+1} + \lambda (t_1^*-t) \right], & t \in [0,t_1^*), \\
  \frac{1}{M+1}v_L + \frac{M}{M+1} p_1(t), & t \in [t_1^*,1),
\end{cases}
\]

On the path of play, the seller posts price

\[
P(t) = \begin{cases} 
  p_1(t), & \text{if } t < 1 \text{ and } K(t) = 1, \\
  p_2(t), & \text{if } t < t^* \text{ and } K(t) = 2, \\
  v_L, & \text{if } t = t_1^* \text{ and } K(t) = 2, \text{ or } t = 1
\end{cases}
\]

where the fire sale time \( t_1^* < 1 \), and the associate quantity is \( Q(t) = 1 \) for each \( t \in [0,1] \).
Figure 3: The equilibrium price path for the two-unit case. The solid line is the equilibrium price when $K(t) = 1$, while the dashed line is that when $K(t) = 2$. The first fire sale time is $t^*_1 = 0.84$. When $t \geq t^*_1$ and $K(t) = 2$, the seller posts the deal price, $v_L$, to liquidate the first unit immediately. The parameter values are $v_H = 1$, $v_L = 0.7$, $M = 3$, and $\lambda = 2$.

On the path of play, the seller’s profit when $K(t) = 2$ is given by

$$
\Pi_2(t) = \begin{cases} 
\Pi_1(t) + v_L, & t \geq t^*_1 \\
\int_{t^*_1}^{t} e^{-\lambda(s-t)} \lambda \left[ p_2(s) + \Pi_1(s) \right] ds + e^{-\lambda(t^*_1-t)} \left[ v_L + \Pi_1(t^*_1) \right], & t < t^*_1
\end{cases}
$$

where $t^*_1$ satisfies conditions (9), (11) and (13), $\Pi_1(t)$ is characterized in Proposition 1, and $p_2(t)$ satisfies ODE (7) with a boundary condition (9).

In the equilibrium, for $t < t^*_1$, the price is $p_2(t)$, and it jumps up to $p_1(t)$ once a transaction happens. If there is no transaction before $t^*_1$, the price jumps down to $v_L$, and one unit is sold immediately; it then jumps up to the path of $p_1(\cdot)$. The first fire sale actually happens at $t^*_1$ with probability $e^{-\lambda(1-t^*_1)}$. Since two or more H-buyers arrive after $t^*_1$ with positive probability, the allocation is inefficient. However, in contrast to the standard monopoly pricing game where the inefficiency results from the seller’s withholding, the inefficiency in this game arises from the scarce good being misallocated to L-buyers when many H-buyers arrive late.\footnote{One way to improve the efficiency is to allow the seller overbook and reallocate goods in the end. See Ely et al. (2013) and Fu et al. (2012) for more analysis in a different environment.}

It is worth noting that our equilibrium prediction on the fire sale critically depends on two assumptions: (1) H-buyers are forward-looking, and (2) the number of L-buyers is finite. First,
suppose each H-buyer can draw at most one attention time, and thus she cannot strategically
time her purchase. As a result, for any \( t \in [0, 1] \) and \( k \in \mathbb{N} \), the H-buyers’ reservation price is always \( p_k(t) = v_H \) for any \( k \). Hence, the optimal price path \( P(t) = v_H \) when \( t < 1 \) and \( P(t) = v_L \) when \( t = 1 \) for any \( k \in \mathbb{N} \). In this particular model, the price is constant until \( t = 1 \). In a more general model, for example, buyers may have a heterogeneous reservation value \( v \in [v_L, v_H] \).

Talluri and van Ryzin (2004) consider many variations of this model. In these models, the result does not depend on the seller’s commitment power. Second, when the number of L-buyers, \( M \), is finite, an H-buyer can get a good at the deal price with positive probability. However, if \( M \) is infinity, the probability that an H-buyer can get a good at the deal price is zero. Hence, the difference between \( p_{1}(t) \) and \( p_{2}(t) \) disappears. In fact, an H-buyer cannot expect any positive surplus and is willing to accept a price \( v_H \) at any time.

4.2 The General Case

In general, the seller has \( K \) units where \( K \in \mathbb{N} \). In the equilibrium, the seller may periodically post a deal price before the deadline. Specifically, there is a sequence of fire sale times, \( \{t^*_k\}_{k=1}^{K-1} \), such that \( t^*_{k+1} \leq t^*_k \) for \( k \in \{1, 2, \ldots, K - 1\} \). Each fire sale time \( t^*_k \) represents the time at which the seller finds optimal to hold at most \( k \) unit of inventory. If \( K(t^*_k) \geq k \), the seller immediately puts \( Q = K(t^*_k) - k \) units of good on sales. See Figure 4 as an illustration. When the initial inventory \( K \) is small, we can show that \( t^*_{K-1} > 0 \) and \( t^*_k \) is strictly decreasing. As a result, on the path of play, when \( t \in [0, t^*_{K-1}] \), the seller serves H-buyers only by charging the H-buyer’s reservation price \( p_k(t) \) if the current inventory \( K(t) = k \in \{1, 2, \ldots, K\} \). If no transaction occurs before \( t^*_{K-1} \), at \( t^*_{K-1} \), the seller holds fire sales to liquidate one unit so that his inventory \( K(t) \leq K - 1 \) after then. By the same logic, for any \( k \in \{2, \ldots, K - 1\} \), when \( t \in [t^*_k, t^*_{k-1}] \), the seller’s equilibrium inventory \( K(t) \) cannot be greater than \( k \). By the hypothesis, \( t^*_k < t^*_{k-1} \), thus at each fire sale time \( t^*_k \), the seller puts at most one unit on sales. However, when the initial inventory size \( K \) is so large that there exists a \( K^* < K \) s.t. \( t^*_k = 0 \) for each \( k \in \{K^*, K^* + 1, \ldots, K\} \), the seller holds multiple-unit fire sales at time \( t = 0 \), i.e. \( Q(0) = K - K^* \) and \( P(0) = v_L \).

We derive the equilibrium by induction. Suppose in the \((K-1)\)-unit case, H-buyers’ reservation price is \( p_k(t) \) for \( k \in \{1, 2, \ldots, K-1\} \), and the seller’s equilibrium strategy is consistent with the
description above. The seller’s equilibrium profit is represented by $\Pi_k(t)$ for $k \in \{1, 2, \ldots, K - 1\}$. Now we construct the H-buyers’ reservation price and the seller’s pricing strategy and payoff in the $K$-units case. To satisfy the H-buyers’ incentive-compatible condition, the equilibrium price at $t$ when $K(t) = K \in \mathbb{N}$ satisfies the following differential equation:

$$\dot{p}_K(t) = -\lambda [p_{K-1}(t) - p_K(t)] \text{ for } t \in [0, t^*_K),$$

(14)

where $t^*_K$ is the first equilibrium fire sale time when $K(t) = K$, and

$$p_K(t) = \frac{i}{M+1} v_L + \frac{M+1-i}{M+1} p_{K-i}(t) \text{ for } t \in [t^*_{K-i}, t^*_K).$$

(15)

where $i = 1, 2, \ldots, K-1$ and $t^*_0 := 1$. Similar to the two-unit case, the incentive-compatible condition of the H-buyer implies that $p_K(t)$ must be continuous at $t^*_K$; thus, the boundary condition of the ODE (14) is given by $p_K(t^*_K) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_{K-1}(t^*_K)$, and therefore, the H-buyer’s best response is specified for any $t \in [0, 1]$ and $k \in \{1, 2, \ldots, K\}$.\(^{22}\)

The seller’s problem is to choose the optimal fire sale time and quantity to maximize his profit. Formally,

$$\Pi_K(t) = \max_{t_{K-i} \in [0, 1]} \int_t^{t_{K-i}} e^{-\lambda(\tau-t)} \lambda [p_K(s) + \Pi_{K-1}(s)] ds + e^{-\lambda(t_{K-i}-t)} [v_L + \Pi_{K-1}(t_{K-i})].$$

In equilibrium, buyers’ beliefs are correct, so the seller’s optimal fire sales time when $K(t) = K$ is $t^*_K$, which satisfies the value-matching and the smooth-pasting conditions.

If there exists an interior solution, $t^*_K$ is pinned down as follows. At $t^*_K$,

$$p_K(t^*_K) = \frac{1}{M+1} v_L + \frac{M}{M+1} p_{K-1}(t^*_K),$$

(16a)

$$\Pi_K(t^*_K) = \Pi_{K-1}(t^*_K) + v_L,$$

(16b)

$$\dot{\Pi}_K(t^*_K) = \dot{\Pi}_{K-1}(t^*_K).$$

(16c)

In equilibrium, we have $t^*_K \leq t^*_K$. The intuition is simple. In a no-waiting equilibrium, no previous arrived H-buyers are waiting in the market; thus, the demand from H-buyers shrinks as the deadline approaches. What is more, the probability that more than $k$ H-buyers arrive before the deadline is approximated by $\lambda^k (1-t)^k$ when the current time $t$ is close to the deadline.

\(^{22}\)Notice that $p_K(t)$ is not left continuous at $t^*_K$ for $i > 1$. At any $t$ which is arbitrarily close but less than $t^*_K$, the H-buyer’s expected continuation value by waiting is $\frac{M+1}{M+1} (v_H - v_L) + \frac{i-1}{M+1} p_{K-i+1}(t)$. Intuitively, at $t < t^*_K$, the H-buyer believes the seller with $K$ units of inventory is going to put $i-1$ units on sales “immediately” , which is before $t^*_K$. On the other hand, at $t^*_K$, the H-buyer believes that the seller is going to put $i$ units on sales “immediately” so that his continuation value by waiting is $\frac{M+1}{M+1} (v_H - v_L) + \frac{i}{M+1} p_{K-i}(t)$. 

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Apparently, the higher $k$ is, the smaller the probability is. Hence, the seller who holds more units has the incentive to liquidate part of his inventory early.

The following proposition formalizes our heuristic equilibrium description.

**Proposition 3.** Suppose $K \in \mathbb{N}$. There is a unique equilibrium in which there is a sequence of fire sale times $\{t^*_k\}_{k=1}^{K-1}$ such that:

1. $0 \leq t^*_{k+1} < t^*_k < 1$ if $t^*_k > 0$ for $k = 1, 2, \ldots, K - 2$,

2. $t^*_{k+1} = 0$ if $t^*_k = 0$ for $k = 1, 2, \ldots, K - 2$,

3. the H-buyers’ reservation price is $p_k(t)$ for $t < 1$ and $K(t) = k \in \{1, 2, \ldots, K(0)\}$ and $v_H$ at $t = 1$,

4. on the path of play when $K(t) = k$, the seller posts price

$$P(t) = \begin{cases} p_k(t), & \text{if } t < t^*_{k-1}, \text{ and } K(t) = k, \\ v_L, & \text{if } t = t^*_{k-1}, \text{ and } K(t) = k \end{cases}$$

with the associated quantity $Q(t)$ s.t.

$$Q(t) = \begin{cases} K - K^*, & \text{if } t = 0 \text{ and } \exists K^* = \min\{k \in \mathbb{N} | t^*_k = 0, k < K\}, \\ 1, & \text{otherwise} \end{cases}$$

In equilibrium, when $K(t) = k$, the price is $p_k(t)$ for $t < t^*_{k-1}$. Without any transaction, the price smoothly declines and jumps up to $p_{k-1}(t)$ once a transaction happens at $t$. If there is no transaction before $t^*_{k-1}$, the seller holds fire sales at time $t^*_{k-1}$, so that the price jumps down to $v_L$ and the transaction takes place immediately. After time $t^*_{k-1}$, the inventory size is reduced to be $k - 1$, and the price path jumps back to $p_{k-1}(\cdot)$. Consequently, a highly fluctuating price path can be generated. In Figure 5, we provide some simulation of equilibrium price path.

Notice that, our model predicts that the price is declining as the deadline is approaching given the inventory size. However, on the equilibrium path, as the deadline approaches, transaction occurs as well, so the inventory size declines, which pushes the price up. Hence, when the initial inventory is not too large and the arrival rate of new H-buyer is not too small, the equilibrium price will statistically rise as the deadline approaches. These implication is consistent with the empirical studies by Escobari (2012). He studies the price pattern in Airline industry and shows that, as the departure time approaches, the unconditional airfare rises. However, once the number of available seats is controlled, the airfare declines as the departure time approaches.
Figure 5: Simulated price path for different realizations of H-buyers’ arrival in the 8-unit case. The upper edge of the shaded area describes the equilibrium list price, and dots indicate transactions. The parameter values are $v_H = 1$, $v_L = 0.7$, $M = 10$, $K = 8$ and $\lambda = 7$.

Given the equilibrium strategy profile, we can calculate the seller’s equilibrium profit when $K(t) = k$ and $t \leq t^*_{k-1}$ which is given by

$$
\Pi_k(t) = \begin{cases} 
\int_{t}^{t_{k-1}^*} e^{-\lambda(s-t)} \lambda [p_k(\tau) + \Pi_{k-1}(\tau)] \, d\tau + e^{-\lambda(t_{k-1}^*-t)} [v_L + \Pi_{k-1}(t_{k-1}^*)], & \text{if } t < t_{k-1}^* \\
v_L + \Pi_{k-1}(t), & \text{if } t = t_{k-1}^*
\end{cases}
$$

where $t_{k-1}^*$ satisfies conditions (16a), (16b) and (16c).

5 Applications

In this section, we consider apply the baseline model to consider to simple applications.

5.1 Optimal Inventory Decision

In the baseline model, we treat the seller’s initial inventory $K$ as a parameter. In short the run, it is a reasonable assumption for most relevant industries. However, in the long run, the seller can adjust his inventory size. To consider the seller’ optimal inventory choice, we assume
Figure 6: The solid line is the profit with BAR, while the dashed line is that without BAR. When \( t \) is close to 0, the profit with BAR is higher than that without BAR. The parameter values are \( v_H = 1, v_L = 0.7, M = 3, \) and \( \lambda = 2. \)

that before time 0, the seller can choose an initial inventory size by paying some cost. For simplicity, the marginal cost of inventory production is assumed to be constant, and greater than \( v_L. \)

The following proposition shows that the seller optimal initial inventory exists.

**Proposition 4.** Suppose that the constant marginal cost of production \( c \) is strictly greater than \( v_L. \) There exists a finite natural number \( K_c = \arg \max_{k \in \mathbb{N}} \Pi_k(0) - kc. \)

The intuition behind the proposition is very simple. The marginal benefit of adding one more unit of the inventory is \( \Pi_{k+1}(t) - \Pi_k(t) \) for \( t = 1 \) for each \( k, \) while the marginal cost is \( c \) which is independent of \( k. \) We can show that the marginal benefit is strictly decreasing in \( k, \) and there is a finite \( K^* \) s.t. the marginal \( \Pi_{K^*+1}(t) - \Pi_{K^*}(t) = v_L. \) Since \( c > v_L, \) we can find a finite \( K_c \leq K^* \) s.t. the marginal surplus of adding one more unit of inventory is positive only if \( k < K_c. \) Apparently, \( K_c \) is non-increasing in \( c: \) a higher production cost induces a lower initial inventory.

\[ \text{**23**Otherwise, the solution of the optimal inventory size is not well-defined in our model since the seller can make profit from producing and selling infinitely many goods to L-buyers.} \]
5.2 Best Available Rate

In the baseline model, we assume the seller has no commitment power. What if the seller has partial commitment power? In practice, sellers in both the airline and the hotel industries sometimes employ a best available rate (BAR) policy and commit to not posting price lower than this best rate in the future. Does the seller have the incentive to do so in our model? Suppose the seller can commit to not posting a deal before the deadline. Then the seller may benefit. The intuition is as follows. An H-buyer’s reservation price depends on the next fire sale time. If there is a deal soon, the reservation price is low, since there is a non-trivial probability that an L-buyer can obtain a good at the fire sale price. At the beginning of the game, if the seller can employ a BAR and commit to not posting $v_L$ before the deadline, he can charge a higher price conditional on the inventory size. To illustrate the idea, we can consider the two-unit case. The seller’s payoff by committing $P(t) > v_L$ for $t < 1$ is

$$\Pi^{BAR}_2 = \int_0^1 e^{-\lambda s} \lambda [p_2(s) + \Pi_1(s)] ds + e^{-\lambda 2} v_L,$$

such that $p_2(t)$ satisfies the ODE (7) with a boundary condition $p_2(1^-) = \frac{M-1}{M+1} v_H + \frac{2}{M+1} v_L$. By committing to no fire sale before the deadline, the seller can ask a higher price when $K(t) = 2$. As a result, $\Pi^{BAR}_2 > \Pi_2(0)$ for certain parameters. In Figure 6, we plot the profit with BAR, $\Pi^{BAR}_2(t)$ and that without it, $\Pi_2(t)$. In the beginning $\Pi^{BAR}_2(t) > \Pi_2(t)$. As time goes on, the difference between them vanishes and becomes negative when the time is very close to the deadline.

6 Discussions

This section discusses the role of each assumption used in our model.

6.1 Inattention Frictions

We assume that buyers face inattention frictions. There are two kinds of attention times: a regular attention is randomly and independently drawn by the Nature in each period, and (2) a deal attention is triggered by the deal alter. Buyers observe the offer posted by the seller only at their attention times. The assumption is very natural in the dynamic pricing setting and also provides some technical convince: it ensures the existence of weak-Markov perfect equilibria. However, a natural question is that whether it is restrictive to assume that the regular attention time is completely exogenous. After all, people can endogenously choose their attention times in
some extent. We believe the exogeneity of regular attention is less restrictive as it looks like in
our model. First, in a no-waiting equilibrium, the equilibrium price is such that the H-buyer is
indifferent between accepting immediately and waiting at almost all the time. As a result, even
though an H-buyer is allowed to choose his regular attention time endogenously in each period,
he has no strong preference over all feasible choices, i.e. an H-buyer has no incentive to obey the
regular attention time assigned by the Nature in any period. Second, in a no-waiting equilibrium,
an H-buyers would make his purchase once he arrives. Thus, the seller has no incentive to draw
extra regular attention.\footnote{Similarly, in a no-waiting equilibrium, the seller has no incentive to manipulate buyers’ deal attention times (or cutoff) as well. The reason to hold fire sale is to quickly “throwing away” some inventory to L-buyers since the seller believes that there is no H-buyer in the market. However, L-buyers would only make their purchases when the price is not greater than \(v_L\). On the one hand, sending a deal alert for a price higher than \(v_L\) is meaningless since L-buyers would not make their purchases. On the other hand, the seller has no incentive to lower the deal alert cutoff price as well since it will delay the transaction and increase the possibility that a newly arrived H-buyer gets the bargain.}

The presence of the deal attention also plays a critical role in our model. It has two implica-
tions: (1) once the seller holds fire sales, the goods can be sold immediately, and (2) H-buyers’
equilibrium reservation price is smooth. The first implication is easy to understand, while the
second deserves more explanation. Briefly speaking, without the extra attention drawn by the
deal alert, the H-buyer’s reservation price is not continuous but jumps up at the beginning of
the fire sales period, which undermines the existence of the no-waiting equilibrium. Specifically,
consider a two-unit model where buyers observes offers only at their regular attention times.
Suppose that the equilibrium fire sales time is in the \(l\)th period: \(t^*_l \in [l\Delta, (l + 1)\Delta)\) for some
\(l = 0, 1, 2, \ldots\). At time \(t^*_l\), the seller posts a fire sale price until the bargain is sold. Notice
that the buyer strictly prefers to draw an attention time at or right after \(t^*_l\) so that his chance
to take the fire sale offer is closed to 1. One can show that the buyer’s continuation value is
discontinuous at \(l\Delta\): \(\lim_{\tau \uparrow l\Delta} U(\tilde{\sigma}_B, \tilde{\sigma}_S, h^*_B) = U_l\Delta > \lim_{\tau \downarrow l\Delta} U(\tilde{\sigma}_B, \tilde{\sigma}_S, h^*_B)\). The idea is that, the
H-buyer’s believed probability of getting the fire sales offer jumps down at \(t = l\Delta\), so is her’
believed continuation value. As a result, the buyer’s reservation price has a jump at \(l\Delta\), which
gives the seller incentive to post unacceptable price to ”save” some buyers arriving in the end
of the \((l - 1)\)th period and serve them in the \(l\)th period. The seller’s incentive to accumulate
buyers will destroy the existence of no-waiting equilibrium, even though we believe the economic
insight we delivered still exists in other PBE. The presence of the deal attention ensures that the
H-buyer’s believed probability of getting the fire sales change smoothly.

An alternative assumption is to eliminate the deal alert assumption but assume that each
buyer has infinitely many regular attention times in each period: the first attention time \(\tau_1\)
is uniformly drawn from the support \([l\Delta, (l + 1)\Delta]\), the second \(\tau_2\) is uniformly drawn from \((\tau_1, (l + 1)\Delta)\), and the third \(\tau_3\) is from \((\tau_2, (l + 1)\Delta)\)... In such a model, a buyer can draw infinitely many attention times, and at each of them, the buyer believes that the conditional probability she takes the fire sales offer is \(\frac{1}{M+1}\), so her continuation value is continuous at the beginning of each period so that the seller has no incentive to accumulate buyers, and therefore, we can construct a no-waiting equilibrium. Due to the lack of deal attentions, it takes time to sale goods at the fire sale price.

While the assumption of inattention frictions is made for technical convince, we believe it is very natural in many dynamic settings. First, we believe it is natural to assume that buyers do not pay attention on the price all the times due to their schedule of other daily living. Each buyer can only spend very limited time on certain shopping activities. Second, in most markets, especially online market, buyers cannot coordinate their purchase times, but make their own purchase decisions respectively. Thus, it is reasonable to assume that buyers independently “draw” their attention times.

6.2 Asymmetry in Population Dynamics

We assume that the number of L-buyers is constant: once an L-buyer leaves, another L-buyer replace him ”immediately”. This assumption together with the deal alert assumption implies that at any time the seller can get ride of “excess” inventory “immediately”. This assumption is critical to the existence of the weak-Markov equilibrium. In fact, what matters here is the H-buyer’s belief about the probability he can get the bargain at the fire sale price, which is determined by the H-buyer’s belief on the number of present L-buyers in a no-waiting equilibrium. In a model where L-buyers also arrive privately, one has to deal with private strategy unless making assumptions on the observability of the history of fire sales. To see the reason, suppose that L-buyers also arrive the market according to a Poisson process. The number of present L-buyers \(M(t)\) will be a random variable, and players form beliefs on \(M(t)\) since it determines the H-buyer’s chance to take the fire sales offer and therefore his continuation value. The law of motion of \(M(t)\) relies on the previous transactions: in a no-waiting equilibrium, if a good is purchased at a fire sales price, it must be purchased by an L-buyer. However, in general, the previous transaction prices are not observed by buyers symmetrically. Consequently, buyers may have heterogenous beliefs about \(M(t)\) and therefore they have to play private strategies after some histories. In particular, suppose that an H-buyer deviates from the no-waiting equilibrium by waiting in the market. The seller holds fire sale at time \(t^*\), but the deviating H-buyer is rationed. She would update her belief on \(M(t^*)\) according to her private experience being rationed. Her continuation strategy would also depend on her private belief on \(M(t')\) for \(t' > t^*\). However,
newly arrived H-buyers do not have such information, so that we cannot sustain a weak Markov equilibrium in the continuation play. To accommodate the assumption of arriving L-buyers, one has to make further assumptions on the observability of previous transaction. One feasible choice is to assume that all previous deal alerts are public observable so that all buyer share the symmetric belief about $M(t)$.

6.3 No-Waiting Equilibrium

We focus on no-waiting weak-Markov perfect equilibrium, which allows us not to keep tracking of the law of motion of the number of H-buyers $N(t)$. A natural question is whether there are other equilibria involving waiting? While the fully characterization of the set of PBE is extremely difficult, we can show that there is no other weak-Markov perfect equilibria (weak-MPE). The key idea is as follows. To sustain a weak-Markov equilibrium, an H-buyer’s equilibrium strategy must be a function of two public state variables after every history, which implicitly implies that the seller cannot manipulate H-buyers’ beliefs on the number of present buyers in the market. In the no-waiting equilibrium, each H-buyer believes that he is the only H-buyer in the market both on and off the path of play. If there are other weak-Markov equilibrium, the H-buyer’s belief on the number of present buyers cannot be affected by the seller’s price non-trivially. i.e., it can only be a function of the state variables. However, such an weak-MPE does not exist. The reason is that, in any weak-MPE, conditional on the inventory size, the H-buyer’s reservation price is non-increasing over time and strictly decreasing in the time interval that the seller is supposed to serve the H-buyers, thus the seller always has the incentive to serve H-buyers when he is supposed to charge unacceptable price to accumulate buyers.

6.4 Observability of Inventory

In our baseline model, the inventory size is observable. The seller’s incentive to hold fire sales critically depends on this assumption. We believe that our model may be too stylized so that it does not perfectly match any real market since in practice, the inventory size may not be perfectly observed by buyers. However, we still believe that our mechanism is illustrative for the following reasons.

First, in many industries, buyers observe some imperfect but informative signal of the real inventory. For example, in the airline industry, the available seats of each flights can be observed online. The number of available seats is not the real inventory itself since the airline seller sometimes blocks some seats for elite passengers, but it is a informative proxy. Escobari (2012) uses the number of available seats as a proxy of the real inventory and empirical studies the
price pattern in airline industry. He finds that the price significantly increases as the number of available seats decreases.

Second, even though the real inventory size is the seller’s private information. However, the seller may be able to use price to signal his inventory. If the inventory size is small, the seller has the incentive to charge high price and not to hold fire sales. However, if his inventory size is large, it is more costly to the seller to charge high price. Hence, we can imagine that some partial separating equilibria may exist. However, fully investigating of such a dynamic signaling game is beyond the goal of this paper.

6.5 Disappearing Buyers and Discounting

In the baseline model, we assume an H-buyer leaves the market only when her demand is satisfied. Our results do not qualitatively change if buyers leave at a non-trivial rate over time. Suppose a buyer leaves the market at a rate $\rho > 0$ at any time, and her payoff by leaving the market without making a purchase is zero. If a buyer chooses to wait in the market, she faces the risk of exogenous leaving. In particular, when $K = 1$, an H-buyer’s reservation price satisfies the following ODE

$$\dot{p}_1(t) = -(\lambda + \rho) [v_H - p_1(t)] \quad \text{for } t \in [0, 1),$$

with the boundary condition (2). By rejecting the current offer, an H-buyer needs to take into account two risks: (1) another H-buyer arrives and purchases the first units before her next attention time, and (2) her exogenous departure. Her payoff is zero if either happens.

In the two-unit case, for $t < t^*_1$, the H-buyer’s reservation price follows

$$\dot{p}_2(t) = -\lambda [p_1(t) - p_2(t)] - \rho [v_H - p_2(t)],$$

and for $t \geq t^*_1$, the form of $p_2(t)$ is identical to that in the baseline model. The intuition behind it is as follows. For $t < t^*_1$, by rejecting a current offer, an H-buyer needs to take into account the risk that (1) another H-buyer arrives before her next attention time, and (2) she exogenously leaves the market. In the former case, she has to pay $p_1(\tilde{t})$ instead of $p_2(\tilde{t})$ at her next attention time $\tilde{t} > t$; in the latter case, she obtains a payoff of zero, which is equivalent to paying a price $v_H$. Since the risk of exogenous departure will only change the H-buyer’s reservation price qualitatively, our main results still hold.

Similarly, the presence of discounting will only qualitatively change our main results as well. As long as players have the same discount rate, one can simply normalize the price to take the discounting into account.
6.6 Multiple Types

In general, considering buyers’ multiple reservation values is complicated in our model. The reason is that on the path of play the seller may accumulate some types of buyers, and players have to form beliefs on the number of such types of buyers so that one cannot construct no-waiting equilibrium. However, we believe the key mechanism in our binary-type model still works in a multi-type model. In a multi-type model, in order to eliminate the information rent of buyers in the future, the seller has the incentive to reduce his inventory quickly as he has in the binary-type model. Thanks to the well-known skimming property, buyers must employ a cutoff strategy. Hence, to quickly reduce his inventory, the seller has to charge a price which is significantly lower than the current price to increase a large amount of demand. Once the reduction of the inventory is accomplished, the seller can charge higher price again since the remaining and future buyers’ willing to pay becomes higher. As a result, we expect price jump down and up over time on the equilibrium path.

7 Conclusion

This paper makes two contributions. First, we highlight a new channel for generating the periodic fire sales. When the deadline is approaching, the seller, if he still has a large inventory, does not expect many arrivals of high-value buyers, so he has the incentive to liquidate part of his stock via a sequence of fire sales to increase future H-buyers’ reservation price. This insight can justify the price fluctuations in industries such as airlines, cruise-lines and hotel services. Second, by introducing the inattention frictions of buyers, we provide a tractable framework to study dynamic pricing problems in which the seller lacks commitment power. We believe that the assumption inattention frictions is very natural in many dynamic pricing settings so that it can be applied in many other environments.
A Appendices

A.1 Proofs for the Single-Unit Case

Equilibria Construction

We construct an equilibrium such that the following conditions hold: (1) the seller posts a price $P(t)$ such that an H-buyer is indifferent between taking and leaving it for $t < 1$, (2) an H-buyer makes the purchase once she arrives, and (3) $P(1) = v_L$ is posted at the deadline.

Consider the last period first. At the deadline, an H-buyers’ reservation price is $v_H$. However, in the presence of inattention friction, it is the seller’s dominating strategy to post $P(1) = v_L$ in order to obtain positive profit. Hence, in any equilibrium, the seller posts $P(1) = v_L$. At $t \in [1 - \Delta, 1)$, the H-buyers’ reservation price is

$$v_H - p_1(t) = e^{-\lambda(1-t)} \frac{v_H - v_L}{M+1},$$

As $t \to 1$, $p_1(t) \to p_1(1^-)$. Differentiating $p_1(t)$ yields

$$\dot{p}_1(t) = -\lambda e^{-\lambda(1-t)} \frac{v_H - v_L}{M+1} = -\lambda [v_H - p_1(t)]$$

with a boundary condition $p_1(1^-)$ at $t = 1$.

By the equilibrium hypothesis, the seller posts H-buyer’s reservation price $p(t)$ at any time $t \in [1 - \Delta, 1)$. Let $U_{1-\Delta}$ be an H-buyer’s expected payoff at the beginning of the last period. The expectation is over the random attention time, and the risk of arrival of new buyers. Hence

$$U_{1-\Delta} = \int_{1-\Delta}^{1} \frac{1}{\Delta} e^{-\lambda(s-1+\Delta)} [v_H - p_1(s)] ds = e^{-\lambda\Delta} \frac{v_H - v_L}{M+1} = v_H - p_1(1 - \Delta).$$

Suppose an H-buyer whose regular attention time in the last period $t$ is smaller than but arbitrarily close to $1 - \Delta$. At time $t$, the H-buyer’s reservation price is

$$v_H - p_1(t) = e^{-\lambda(1-\Delta-t)} U_{1-\Delta},$$

and we also have

$$\dot{p}_1(t) = -\lambda [v_H - p_1(t)].$$

As $t \to 1 - \Delta$, $\lim_{t\to1-\Delta} p_1(t) = p_1(1-\Delta)$, so $p_1(t)$ is differentiable at $1 - \Delta$. Repeating the above argument for $1/\Delta$ times, the reservation price $p_1(t)$ is differentiable in $[0, 1)$ and satisfies the ODE (5) with the boundary condition (2).
By the equilibrium hypothesis, the deal price is posted at the deadline only, and H-buyers do not delay their purchases, so neither the H-buyers’ reservation price nor the seller’s equilibrium profit depends on $\Delta$. The closed-form solution of $p_1(t)$ and $\Pi_1(t)$ are given by

$$p_1(t) = v_H - \frac{v_H - v_L}{M + 1} e^{-\lambda(1-t)},$$

$$\Pi_1(t) = \left[ 1 - e^{-\lambda(1-t)} \right] v_H + e^{-\lambda(1-t)} v_L - \frac{v_H - v_L}{M + 1} e^{-\lambda(1-t)} \lambda (1-t).$$

In sum, the equilibrium strategy profile $(\sigma^*_S, \sigma^*_B)$ is given as follows.

- After any private history, an H-buyer’s reservation price is $p(t)$ for $t < 1$ and $v_H$ at $t = 1$.
- After any seller’s history, the seller posts a price $= p(t)$ for $t < 1$ and $v_L$ at $t = 1$.

**The Proof of Proposition 1**

We prove Proposition 1 step by step. A simple observation is that, given the seller’s equilibrium strategy, H-buyers do not have an incentive to deviate since they are indifferent everywhere. To ensure the existence of the conjecture equilibrium, we only need to rule out deviations by the seller.

First of all, it is obvious that it is the seller’s dominant strategy to post $P(1) = v_L$ at the deadline after any history. Hence, the seller has no profitable deviation at $t = 1$. Second, we need to rule out the profitable deviation at $t < 1$ after any history. Formally, after a seller’s private history $h^t_S$, the seller’s profit by following equilibrium strategy $\sigma_s(t) = p_1(t)$ is given by

$$\Pi_1(\sigma_S, \sigma_B, h^t_S) = \int_t^{t+dt} [g(s|h^t_S)p_1(s)]ds + (1 - G(s|h^t_S))\Pi_1(\sigma_S, \sigma_B, h^{t+dt}_S)$$

where $dt \geq 0$ and the probability measure $G(s|h^t_S)$ is the seller’s subjective belief about the earliest (present or newly arriving) H-buyer’s regular attention time after $t$ being not greater than $s$ given his private history $h^t_S$, and $g(\cdot|h^t_S)$ is the associated Radon-Nikodym derivative. Notice that

$$\Pi_1(\sigma_S, \sigma_B, h^t_S) = \int_t^1 [g(s|h^t_S)p_1(s)]ds + (1 - G(1|h^t_S))v_L$$

and $p_1(s)$ is strictly decreasing, so $\dot{\Pi}_1 < 0$ after any history. We want to show that, after any history, the seller would not be better off by charging non equilibrium price. There are three kinds of “one-shot” deviations we need to rule out: (1) charging $v_L$, (2) charging a price between $v_L$ and $p_1(t)$ in a time period $[t, t + dt)$, and (3) charging unacceptable price higher than $p_1(s)$ for $s \in [t, t + dt)$ for any $dt > 0$. We rule out them one-by-one.
1. $\Pi_1(\sigma_S, \sigma_B, h^t_S) > v_L$ for any $t < 1$, so the seller has no incentive to deviate by charging $v_L$ at any time $t < 1$.

2. Consider a deviation strategy $\hat{\sigma}_S$: charging a price $\hat{p}(s) \in (v_L, p_1(s))$ for $s \in [t, t + dt]$ and then switch back to the equilibrium strategy after $t + dt$. The seller’s profit is given by

$$
\Pi_1(\hat{\sigma}_S, \sigma_B, h^t_S) = \int_t^{t+dt} [g(s|h^t_S)\hat{p}(s)]ds + (1 - G(t + dt|h^t_S))\Pi_1(\sigma_S, \sigma_B, h^{t+dt}_S)
$$

Notice that by charging $\hat{p}(s)$, the seller would not change the continuation history after $t + dt$. Apparently, the deviation is not profitable.

3. Consider another deviation strategy $\tilde{\sigma}_S$: charging a price strictly greater than $p_1(s)$ for $s \in [t, t + dt)$. Since the price is higher than the buyer’s reservation price, there would be no trade even some H-buyers’ attention times are in the interval of $[t + dt)$, so the seller’s profit is given by

$$
\Pi_1(\tilde{\sigma}_S, \sigma_B, h^t_S) = \int_t^{t+dt} [g(s|h^t_S)\tilde{p}(s)]ds + (1 - G(t + dt|h^t_S))\Pi_1(\sigma_S, \sigma_B, h^{t+dt}_S)
$$

where $\Pi(\sigma_S, \sigma_B, h^{t+dt}_S)$ denotes the seller’s equilibrium continuation profit after a deviating history. Since $p_1(t)$ is strictly decreasing, we must have $\Pi(t + dt, \tilde{h}^{t+dt}_S) \leq p_1(t + dt) < p_1(t)$ which implies that charging a price higher than $p_1(s)$ is not profitable for $s \in [t, t + dt)$.

Consequently, it is the seller’s best response to post $p_1(t)$ for $t < 1$, and $v_L$ at $t = 1$, and our conjecture equilibrium is an equilibrium. By construction, $p_1(t)$ is unique, so there is no other equilibrium. Q.E.D.

### A.2 Proofs for the Two-Unit Case

#### The Proof of Lemma 1

Suppose not. Since $v_L$ is posted only at the deadline, the seller’s equilibrium profits at the deadline are given by

$$
\Pi_k(1) = kv_L, k = 1, 2.
$$

and $p_k(t)$, the reservation price at $k = 1, 2$, is post to serve H-buyers only at any $t < 1$. Specifically,

$$
p_2(t) = v_H - \frac{v_H - v_L}{M + 1} e^{-\lambda(1-t)} [2 + \lambda (1 - t)], \text{ and }
$$

$$
p_1(t) = v_H - \frac{v_H - v_L}{M + 1} e^{-\lambda(1-t)}
$$
Define $\tilde{\Pi}_2 (t)$ as the seller’s profit if $p_2 (t)$ is always posted when $t < 1$ and $K(t) = 2$, then

$$
\tilde{\Pi}_2 (t) = \int_t^1 \lambda e^{-\lambda (s-t)} [p_2 (s) + \Pi_1 (s)] \, ds + 2v_L e^{-\lambda (1-t)}
$$

$$
= 2v_H - 2 (v_H - v_L) e^{-\lambda (1-t)}
$$

$$
- \frac{v_H - v_L}{M + 1} e^{-\lambda (1-t)} \left[ \lambda (1-t) (M+3) + \lambda^2 (1-t)^2 \right].
$$

Immediately,

$$
\tilde{\Pi}_2 (t) - [v_L + \Pi_1 (t)]
$$

$$
= (v_H - v_L) \left( 1 - 2e^{-\lambda (1-t)} \right)
$$

$$
+ \frac{v_H - v_L}{M + 1} e^{-\lambda (1-t)} \left[ M + 1 - \lambda (1-t) (M+2) - \lambda^2 (1-t)^2 \right].
$$

Though this difference is not monotone, using a Taylor expansion and algebra, there are two cases: (i) either $\tilde{\Pi}_2 (t) - [v_L + \Pi_1 (t)] < 0$ for all $t < 1$ when $\tilde{\Pi}_2 (0) < v_L + \Pi_1 (0)$, (ii) or, if $\tilde{\Pi}_2 (0) > v_L + \Pi_1 (0)$, $\exists t^*_1 < 1$ s.t. $\tilde{\Pi}_2 (t^*_1) = v_L + \Pi_1 (t^*_1)$ and $\tilde{\Pi}_2 (t) < v_L + \Pi_1 (t)$ for $t \in (t^*_1, 1)$.

Q.E.D.

The Proof of Proposition 2

The proof of proposition 2 is divided into several steps. To make the proof clear to the reader, we note that we will be following this road map:

1. Given the seller’s equilibrium strategy and the H-buyer’s reservation price in the single unit case $p_1 (\cdot)$, we construct the H-buyer’s reservation price $p_2 (\cdot)$ on $t \in [0, 1)$.

2. Given the H-buyer’s equilibrium strategy, we show that the seller has no incentive to charge price that an H-buyer does not accept after any history. Namely, the seller either charge the H-buyer’s reservation price $p_k(t)$ or $v_L$ for $t \in [0, 1]$ regardless his private history $h^t_s$.

3. We solve and characterize the seller’s unique optimal fire sales time $t^*_1$.

H-buyers’ Best Response. By the equilibrium hypothesis, all buyers believe that

- there is no other H-buyers in the market,

- there is a fire sale time $t^*_1 < 1$ s.t. the seller posts a deal offer at the fire sale price $v_L$ with quantity $Q(t) = 1$ for $t \in [t^*_1, 1)$ if $K(t) = 2$,

- the seller posts price $p(t)$ for $t < 1$ when $K(t) = 1$, and
• at the deadline \( t = 1 \), the seller posts a clearance fire sale at a price \( v_L \) and supply \( Q(t) = K(t) \).

Given such a belief, we construct the H-buyers’ reservation price. Apparently, an H-buyer’s reservation price is \( v_L \) at the deadline regardless of the inventory \( K(1) \). Now we consider the H-buyers’ reservation price for \( t < 1 \). Once, \( K(t) \) becomes 1, the H-buyers’ reservation price becomes \( p_1(t) \), which is given in equation (1). For \( K(t) = 2 \) and \( t^*_1 \in [0, 1) \), there are two cases.

**Case 1.** When \( K(t) = 2 \) and \( t \in [t^*, 1) \), an H-buyer believes that (1) she is the only H-buyer in the market, and (2) the seller would post fire sale “immediately” if no transaction occurs currently. As a result, the H-buyer’s reservation price \( p_2(t) \) when \( t \in [t^*, 1) \) satisfies:

\[
v_H - p_2(t) = \frac{1}{M+1} (v_H - v_L) + \frac{M}{M+1} e^{-\lambda(l\Delta - t)} U^1_{l\Delta},
\]

(A.1)

where, as in the single-unit case, \( U^1_{l\Delta} = e^{-\lambda(l\Delta)} \frac{v_H - v_L}{M+1} \) and \( t \in ((l-1)\Delta, l\Delta) \) for some \( l < 1/\Delta \), hence

\[
p_2(t) = \frac{Mv_H + v_L}{M+1} - \frac{M}{(M+1)^2} (v_H - v_L) e^{-\lambda(1-t)}
\]

\[
= \frac{M}{M+1} p_1(t) + \frac{1}{M+1} v_L, \quad t \in [t^*_1, 1).
\]

Observe that \( \hat{p}_2 = M / (M+1) \hat{p}_1 \) for \( t \in [t^*_1, 1) \).

Notice that for \( t > t^*_1 \), \( p_2(t) \) in equation (A.1) denotes the H-buyer willing to pay if the current price is not \( v_L \), and the right-hand-side of equation (A.1) denotes the buyer’s continuation value to wait. We justify the continuation value by arguing that the buyer believes that the seller will hold fire sales right after time \( t \). However, in a continuous-time game, the time right after \( t \) or the “next period” is not well-defined. However, since each outcome in our game is identified by the limit outcome of a sequence of inertia strategy, we can still appropriately define the buyer’s believed continuation value.

**Case 2.** When \( K(t) = 2 \) and \( t \in [0, t^*_1) \), the H-buyer’s reservation price \( p_2(t) \) is pinned down as follows. There are some \( l \in \mathbb{N} \) s.t. \( [(l-1)\Delta, l\Delta) \cap [0, t^*_1) \neq \emptyset \).

In the “fire sales period”, \( l\Delta \geq t^*_1 \geq (l-1)\Delta \), and \( p_2(t) \) for \( t \in [(l-1)\Delta, t^*) \) satisfies:

\[
v_H - p_2(t) = e^{-\lambda(t^*_1-t)} U^2_{t^*_1} + \lambda (t^*_1 - t) e^{-\lambda(t^*_1-t)} e^{-\lambda(l\Delta-t^*_1)} U^1_{l\Delta},
\]

(A.2)

The right-hand-side of the equation (A.2) denotes the buyer’s continuation value if she decides to wait, which is consist of several terms.
1. with probability \( e^{-\lambda(t_1^*-t)} \), there is no new H-buyer arriving before time \( t_1^* \) so that the seller will hold fire sales,

2. \( U_{i1}^2 = \frac{1}{M+1} \left( v_H - v_L \right) + \frac{M}{M+1} e^{-\lambda(t_1^*-t)} U_{i1}^1 \) is the buyer’s expected continuation value at \( t_1^* \). By the definition of \( p_2(\cdot) \) on \([t_1^*, 1)\), \( U_{i1}^2 = v_H - p_2(t_1^*) \) where \( p_2(t_1^*) \) is the H-buyer reservation price at \( t_1^* \) if the seller posts a non fire sales price, and it is pinned down by equation (A.1) in case 1.

3. \( \lambda(t_1^*-t)e^{-\lambda(t_1^*-t)} \) denotes the probability that a new H-buyer arrives before \( t_1^* \) so that the seller does not hold fire sales at time \( t_1^* \). Similarly, the H-buyer has to wait for her regular attention time in the next period to make her purchase.

In a non fire sale period, \( l\Delta < t_1^* \), and the H-buyer’s reservation price is given by

\[
v_H - p_2(t) = e^{-\lambda(l\Delta-t)} U_{i1}^2 + \lambda(l\Delta-t) e^{-\lambda(l\Delta-t)} U_{i1}^1
\]

(A.3)

where \( U_{i1}^2 = v_H - p_2(l\Delta) \). In either of the two cases, we have

\[
p_2(t) = v_H - \frac{v_H - v_L}{M+1} \left[ e^{\lambda(1-t_1^*)} + \frac{M}{M+1} + \lambda(t_1^* - t) \right] < p_1(t),
\]

for \( t \in [0, t_1^*) \), and \( \dot{p}_2(t) = -\lambda(p_1(t) - p_2(t)) \) for \( t \in [0, t_1^*) \).

The H-buyer’s incentive compatible condition implies that \( p_2(t) \) must be left continuous at \( t_1^* \), so \( p_2(\cdot) \) is continuous on \([0, 1)\).

In the hypothetic equilibrium, H-buyers have no incentive to deviate after any history since each of them always believes that the seller has not charge unacceptable price for a positive measure of time, and the continuation price is either her equilibrium reservation price or \( v_L \).

**The Seller’s Problem: No Accumulation Result.** Now we consider the seller’s problem. First, we claim that, given the H-buyer’s best response, after any history with \( K(t) = 2 \) for \( t \in [0, 1] \), the seller’s best response is to post either a price either \( p_2(t) \) or \( v_L \) so that on the equilibrium the seller does not post unacceptable to accumulate H-buyers.

**Lemma A.1.** After any seller’s history \( h_S^t \) with \( K(t) = 2 \) and \( t < 1 \), the seller’s best response given buyer’s equilibrium strategy \( \sigma_B(t, 2) \) is either \( \sigma_S(h_S^t) = (p_2(t), 1) \) or \( \sigma_S(h_S^t) = (v_L, 1) \).

**Proof.** By Proposition 1, once the inventory becomes \( K(t) = 1 \), the seller’s best response is to post \( p_1(t) \) for \( t < 1 \) and \( v_L \) at the deadline. Obviously, any price \( p \in (v_L, p_2(t)) \cup (\infty, v_L) \) is strictly dominated by either posting \( p_2(t) \) or \( v_L \), so we only need to rule out the deviation \( \dot{\sigma}_B \) that the seller posts unacceptable price \( p > p_2(s) \) for \( s \in [t, t+dt) \) and switch back to \( \sigma_B \) after time \( t+dt \) for each \( dt > 0 \). Denote the history after such a deviation as \( h_S^{t+dt} \). There are two cases.
1. Suppose \( \sigma_S(\tilde{h}_S^{t+dt}) = (v_L, 1) \). The seller’s deviation payoff is

\[
\Pi_2(\sigma_S, \sigma_B, h_S^t) = v_L + \int_t^{t+dt} g(s|h_B^t)ds\Pi_1(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt}) + (1 - G(t + dt|h_B^t))\Pi_1(\sigma_S, \sigma_B, h_S^{t+dt})
\]

where \( G(\cdot|h_S^t) \) again denotes the seller’s subjective belief that the first regular attention time of H-buyers after time \( t \) is not greater than \( s \), \( h_S^{t+dt} \) denotes the history until \( t + dt \) in which no H-buyer’s attention time is in \([t, t+dt]\) and the seller holds fire sales at time \( t + dt \) with an associated continuation value \( v_L + \Pi_1(\sigma_S, \sigma_B, h_S^{t+dt}) \), and \( \tilde{h}_S^{t+dt} \) denotes the history in which some (present or newly arrived) H-buyer’s attention time is in \([t, t+dt]\) and the seller holds fire sales at time \( t + dt \), with an associated continuation value \( v_L + \Pi_1(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt}) \).

Now suppose the seller uses the following strategy \( \tilde{\sigma}_S \): posts the fire sale offer at \( t \) with \( Q(t) = 1 \) and switch back the equilibrium strategy then. The seller’s profit is given by

\[
\Pi_2(\tilde{\sigma}_S, \sigma_B, h_S^t) = v_L + \int_t^{t+dt} g(s|h_B^t)p_1(s)ds + (1 - Gt + dt|h_B^t))\Pi_1(\sigma_S, \sigma_B, h_S^{t+dt})
\]

Since \( p_1(s) > \Pi_1(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt}) \) for \( s < t + dt \), \( \tilde{\sigma}_S \) dominates the deviation strategy \( \sigma_S \).

2. Suppose \( \sigma_S(\tilde{h}_S^{t+dt}) = (p_2(t + dt), 1) \). The seller’s deviation payoff is

\[
\Pi_2(\sigma_S, \sigma_B, h_S^t) = \int_t^{t+dt} g(s|h_B^t)ds\Pi_2(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt}) + (1 - G(t + dt|h_B^t))\Pi_2(\sigma_S, \sigma_B, h_S^{t+dt})
\]

where \( \Pi_2(\sigma_S, \sigma_B, h_S^{t+dt}) \) denotes the history until \( t + dt \) in which no H-buyer’s attention time is in \([t, t + dt]\), and \( \Pi_2(\sigma_S, \sigma_B, \tilde{h}_S^{t+dt}) \) denotes the history in which some (present or newly arrived) H-buyer’s attention time is in \([t, t + dt]\). Now suppose the seller uses the following strategy \( \tilde{\sigma}_S \): charging \( p_2(s) \in [t, t + dt) \) and switch back to the equilibrium strategy \( \sigma_S \) if either \( K(s) = 1 \) for \( s \in [t, t + dt) \) or \( s > t + dt \). The seller’s profit is given by

\[
\Pi_2(\tilde{\sigma}_S, \sigma_B, h_S^t) = \int_t^{t+dt} g(s|h_S^t)[p_2(s) + \Pi_1(\sigma_S, \sigma_B, h_S^s)]ds + (1 - G(t + dt|h_B^t))\Pi_2(\sigma_S, \sigma_B, h_S^{t+dt})
\]

where \( \Pi_1(\sigma_S, \sigma_B, h_S^s) \) denotes the seller’s profit by following equilibrium strategy at time \( s \) given the history \( h_S^s \) and the inventory size \( K(s) = 1 \).

By the hypothesis, following a history \( h_S^{t+dt} \), the seller’s equilibrium profit is given by

\[
\Pi_2(\sigma_S, \sigma_B, h_B^{t+dt}) = \int_t^{t+dt} g(s|h_B^{t+dt})[p_2(s) + \Pi_1(\sigma_S, \sigma_B, h_S^s)]ds + (1 - G(\tau|h_S^{t+dt}))ds[v_L + \Pi_1(\sigma_S, \sigma_B, h_S^s)]
\]

\[
\leq p_2(t + dt) + \Pi_1(\sigma_S, \sigma_B, h_S^{t+dt})
\]
where the last inequality comes from the facts that both $p_2(\cdot)$ and $\Pi_1(\sigma_S, \sigma_B, h_S)$ is strictly decreasing over time after any history and $\Pi_2(\sigma_S, \sigma_B, h_S^t) \geq v_L + p_2(s)$ and $\Pi_1(\sigma_S, \sigma_B, h_S^t)$ for $s < \tau$. Consequently, we must have

$$\Pi_2(\hat{\sigma}_S, \sigma_B, h_S^t) \geq \Pi_2(\tilde{\sigma}_S, \sigma_B, h_S^t).$$

In sum, we must have

$$\Pi_2(\sigma_S, \sigma_B, h_S^t) = \max\{\Pi_2(\bar{\sigma}_S, \sigma_B, h_S^t), \Pi_2(\hat{\sigma}_S, \sigma_B, h_S^t)\}$$

so that the seller has no incentive to deviate from $\sigma_S$ to $\tilde{\sigma}_S$.

Last, it is clear that $\Pi_2(\sigma_S, \sigma_B, h_S^t) > 2v_L$ for $t < 1$, so it is never optimal to deviate to post a fire sales with $Q(t) = 2$ before the deadline.

Lemma A.1 implies that after any history, the seller has no incentive to charge an unacceptable price. As a result, on the path of play, at any time, the seller believes that there is no $H$-buyer waiting in the market. The only remaining problem for the seller when $K(t) = 2$ is to decide when to charge $p_2(t)$ and $v_L$.

**The Seller’s Problem: Optimal Fire Sale Time.** Given the buyer’s reservation price $p_2(\cdot)$ based on the belief of $t_1^*$, the seller chooses the actual fire sale time, with $p_2(\cdot)$ forced to be the pricing strategy before the fire sale time. Hence,

$$\Pi_2(t) = \max_{t_1} \int_t^{t_1} e^{-\lambda(s-t)} \lambda [p_2(s) + \Pi_1(s)] ds + e^{-\lambda(t_1-t)} [v_L + \Pi_1(t_1)]. \quad (A.4)$$

In equilibrium, the buyers’ belief is correct, so the seller’s optimal choice is indeed $t_1^*$. The first derivative w.r.t. $t_1$ at $t_1^*$ is

$$e^{-\lambda(t_1^*-t)} \lambda [p_2(t_1^*) - v_L] + e^{-\lambda(t_1^*-t)} \Pi_1(t_1^*) = \lambda e^{-\lambda(t_1^*-t)} [p_2(t_1^*) - v_L - p_1(t_1^*) + \Pi_1(t_1^*)] = 0$$

Or equivalently, if an interior equilibrium fire sales time $t_1^*$ exists, it must satisfied the following first-order-condition (FOC)

$$p_2(t_1^*) - v_L - p_1(t_1^*) + \Pi_1(t_1^*) \leq 0. \quad (A.5)$$

and the inequality is replaced by an equality when $t_1^* > 0$. The remaining is to show that there is a unique $t^*$ s.t equation (A.5) and solves the seller’s problem (A.4).

Define $f(\cdot)$ on $[0, 1]$ as follows:

$$f(t) = p_2(t) - v_L - p_1(t) + \Pi_1(t).$$
For \( t \geq t_1^* \), we have \( p_2(t_1) - p_1(t_1) = \frac{1}{M+1} [v_L - p_1(t_1)] \). Let

\[
\dot{f}^0(t) = \Pi_1(t) - v_L - \frac{p_1(t) - v_L}{M + 1} = \frac{v_H - v_L}{M + 1} \left\{ M - e^{-\lambda(1-t)} \left[ M + \frac{M}{M+1} + \lambda (1-t) \right] \right\}
\]

Obviously, \( \dot{f}^0(t) < 0 \) and \( \dot{f}^0(1) = -M/(M+1) < 0 \). Define \( t_1^* \) as the unique solution to \( \dot{f}^0(t) = 0 \) if it exists, otherwise define \( t_1^* = 0 \). By construction, for \( t \in (t_1^*, 1) \), the optimal solution of (A.4) is \( t \); thus, the seller does not have any incentive to choose a fire sale time later than \( t_1^* \). If \( t_1^* > 0 \) i.e. \( \dot{f}^0(t_1^*) = 0 \), and let \( f(t) = \dot{f}^0(t) \) for \( t \in [t_1^*, 1] \). For \( t < t_1^* \),

\[
\dot{p}_2(t) = -\lambda (p_1(t) - p_2(t)),
\]

hence

\[
f(t) = \frac{1}{\lambda} \dot{p}_2(t) + \Pi_1(t) - v_L,
\]

and \( \dot{f}(t) = \dot{p}_2(t) + \dot{\Pi}_1(t) - \dot{p}_1(t) \) in which \( \dot{\Pi}_1(t) - \dot{p}_1(t) = \lambda e^{-\lambda(1-t)} \frac{v_H - v_L}{M+1} [1 - \lambda (1-t) - M] < 0 \) and \( \dot{p}_2(t) < 0 \), therefore \( \dot{f}(t) < 0 \) for \( t < t_1^* \). Since \( p_2, p_1 \) and \( \Pi_1 \) are all continuous over \([0, 1]\), we have a continuous \( f(t) \) and \( \lim_{t \rightarrow t_1^*} f(t) = f(t_1^*) = 0 \), consequently \( f(t) > 0 \) for \( t < t_1^* \); thus, the FOC (A.5) is not only necessary but also sufficient to establish the equilibrium fire sale time, and therefore the seller does not have any incentive to choose a fire sale time earlier than \( t_1^* \). Since \( t_1^* \) is uniquely constructed, there is no other equilibrium. Q.E.D.

### A.3 Proofs for the Multi-Unit Case

#### Proof of Proposition 3

We construct the equilibrium by induction. Suppose there is a unique equilibrium for the game where \( K(0) = K - 1 \) in which

1. there exists a sequence of \( \{t_k^*\}_{k=1}^{K-2} \), such that \( t_0^* := 1, t_k^* \leq t_{k-1}^* \) and \( t_{K-2}^* > 0 \),

2. the seller posts price \( P(t) = p_k(t) \) if \( K(t) = k \) and \( t < t_{k-1}^* \) where \( p_k(t) \) for \( k = \{1, 2, ...K - 1\} \) such that \( p_{k+1} < p_k \), and \( \dot{p}_k < 0 \) where differentiable, and

3. the seller holds fire sales at \( t = t_{k-1}^* \) for \( K(t) > k \) to liquidate redundant inventory, i.e. \( P(t) = v_L \) and \( Q(t) = K(t) - k \).

Consequently, by the indifference conditions of an H-buyer’s reservation price and uniform distributed attention time in a period, we can define \( U_k^{t_{k-1}^*} = v_H - p_k(l\Delta) \) as the expected utility of an

\( t_{K-2}^* = 0, t_{K-1}^* = 0 \) as well.
H-buyer if her next attention time is in next period starting from \( l \Delta < t_{K-1}^* \) and \( K (l \Delta) = k \), and 
\[
U_{K-1}^k = v_h - p_k (t_{K-1}^*)
\]
the expected utility if the next attention time is \( t_{K-1}^* \) and \( K (t_{K-1}^*) = k \). We construct the unique candidate equilibrium for the game where \( K (0) = K \), which includes: the H-buyers reservation price \( p_K (t) \), the equilibrium first fire sale time \( t_{K-1}^* \), and the seller’ pricing strategy.

**H-buyers’ Best Response: Construction.** When \( t_{K-2}^* = 0, t_{K-1}^* = 0 \) as well. When \( t_{K-2}^* < 0 \), similar to the two-unit case, we can construct a fire sale time \( t_{K-1}^* \in [0, t_{K-2}^*] \). Suppose buyers believe that the seller posts deals at \( 0 \leq t_{K-1}^* < t_{K-2}^* < \ldots < t_1^* \leq 1 \) when \( K (t_k^*) > k \) and posts the H-buyer’s reservation price \( p_k (t) \) when \( K (t) = k, k = 1, \ldots, K \).

Consider the case where \( t \geq t_{K-1}^* \). Trivially, the seller will post \( p_K (1) = v_L \) at the deadline and the reservation price of an H-buyer is \( v_H \). When \( t \in [t_{K-i}^*, t_{K-i-1}^*] \), and \( K (t) = K \), an H-buyer understands that seller deviates by charging the H-buyer’s reservation price when he is supposed to hold fire sales. Hence the buyer believes that no other H-buyers are in the market and the seller believes so too and thus expects the seller to post a fire sales offer \( (P(t), Q(t)) = (v_L, i) \) immediately and to reduce his inventory to \( K - i \), hence
\[
p_K (t) = \frac{i}{M+1} v_L + \frac{M+1-i}{M+1} p_{K-i} (t)
\]
for \( i = 1, \ldots, K-1 \).

Note that, when \( t > t_{K-1}^* \), \( p_K (t) \) is decreasing but not continuous because \( \lim_{t 
arrow t_k^*} p_K (t) > p_K (t_k^*) \), \( \forall k < K-1 \) and \( \hat{p}_K = (M+1-i) / (M+1) \hat{p}_{K-i} < 0 \) where it exists.

Now consider \( t < t_{K-1}^* \). In the “fire sales period”: \( (l-1) \Delta < t < t_{K-1}^* < l \Delta \), the H-buyer’s indifference condition is:
\[
v_H - p_K (t) = e^{-\lambda (t_{K-1}^* - t)} U_{t_{K-1}^*}^K + \sum_{k=1}^{K-1} \lambda^k e^{-\lambda (l \Delta - t)} \sum_{i=1}^{k} \frac{(t_{K-1}^* - t)^i}{i!} \frac{(l \Delta - t_{K-1}^*)^{k-i}}{(k-i)!} U_{l \Delta}^{K-k}.
\]

In the “no fire sales period”: \( (l-1) \Delta < t < l \Delta \), the condition becomes:
\[
v_H - p_K (t) = \sum_{k=0}^{K-1} e^{-\lambda (l \Delta - t)} \frac{\lambda^k (l \Delta - t)^k}{k!} U_{l \Delta}^{K-k}.
\]
(A.6)

The expected continuation values \( U_{l \Delta}^{K} \) and \( U_{t_{K-1}^*}^{K} \), defined in the same fashion as before, are the expected utilities of an H-buyer if her next attention time is in the next period or at \( t_{K-1}^* \), whichever comes first. The analytical expression for \( p_K (t) \) is then obtained using the continuation
values in a recursive way. It is straightforward to show that \( p_K(t) \) is continuous at \( t_{K-1}^* \). In addition, we have

\[
\dot{p}_K(t) = -\lambda (p_{K-1}(t) - p_K(t)) \quad \text{for } t < t_{K-1}^*.
\] (A.7)

**H-buyers’ Best Response: Characterization.**

**Lemma A.2.** For each \( t \in [0, 1) \), \( p_{k+1} < p_k < 0 \) where \( k = \{1, 2, \ldots, K-1\} \).

Proof. First, for each \( t \geq t_k^* \), by equation (15), \( p_{k+1}(t) < p_k(t) \). Second, for \( t < t_k^* \), by equation (A.6), we have

\[
p_{K-1}(t) - p_K(t) = \sum_{k=0}^{K-2} \frac{e^{-\lambda(l\Delta-t)}\lambda^k(l\Delta-t)^k}{k!} [U_{l\Delta}^{K-k} - U_{l\Delta}^{K-1-k}] + \frac{e^{-\lambda(l\Delta-t)}\lambda^{K-1}(l\Delta-t)^{K-1}}{(K-1)!} U_{l\Delta}^1
\]

where \( U_{l\Delta}^k = v_H - p_k(l\Delta) > 0 \) is the H-buyer’s expected payoff at time \( l\Delta \) when \( K(l\Delta) = k \). The strictly inequality comes from the fact that, an H-buyer can always wait for the last minute low price and obtain \( v_H - v_L \) with a positive probability. Moreover, we have

\[
U_{l\Delta}^{K-k} - U_{l\Delta}^{K-1-k} = p_{K-1-k}(l\Delta) - p_{K-k}(l\Delta)
\]

We already know that \( p_1(t) > p_2(t) \) for each \( t \in [0, t_1^*] \), so \( p_2(t) > p_3(t) \) for \( t \in [0, t_2^*] \). A simple induction argument implies our desired result. \( \square \)

By the construction of \( p_K(\cdot) \), we immediately have the following result.

**Corollary 1.** \( \dot{p}_K(t) < 0 \) whenever \( p_K(\cdot) \) is differentiable.

Notice that \( p_K(\cdot) \) is not continuous at \( t_k^* \) for \( k < K-1 \).

**Lemma A.3.** For \( t < t_k^* \), \( \dot{p}_{k+1} - \dot{p}_k < 0 \) where \( k = \{1, 2, \ldots, K-1\} \).

Proof. We solve the closed-form solution of \( p_{k+1} - p_k \) for \( t < t_k^* \). Simple algebra implies that

\[
\dot{p}_{k+1}(t) - \dot{p}_k(t) = \lambda (p_{k+1} - p_k) + \lambda (p_{k-1} - p_k),
\]

which is equivalent to

\[
\frac{d}{dt} [(p_{k+1} - p_k) e^{-\lambda t}] = -\lambda (p_k - p_{k-1}) e^{-\lambda t}.
\]

Recursively, we have

\[
\frac{d^k}{dt^k} [(p_{k+1} - p_k) e^{-\lambda t}] = -(-\lambda)^k \frac{v_H - v_L}{M+1} e^{-\lambda t},
\]

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Hence, when so to the H-buyer’s equilibrium strategy Lemma A.4.
After any seller’s history the following lemma is true.

\[ p_{k+1} - p_k = -\lambda^k \frac{v_H - v_L}{M + 1} e^{-\lambda t} \frac{t^k}{k!} \sum_{i=1}^{k} C_i \frac{t^{k-i}}{(k-i)!} e^{\lambda t} \]

where \( C_i \) is a constant number for each \( i \), and

\[ \dot{p}_{k+1} - \dot{p}_k = -\lambda^k \frac{v_H - v_L}{M + 1} e^{-\lambda t} \frac{t^{k-1}}{(k-1)!} \sum_{i=1}^{k-1} C_i \frac{t^{k-i-1}}{(k-i-1)!} e^{\lambda t} + \lambda (p_{k+1} - p_k) \]

\[ = (p_k - p_{k-1}) + \lambda (p_{k+1} - p_k) + (\lambda + 1) (-\lambda)^{k-1} \frac{v_H - v_L}{M + 1} e^{-\lambda t} \frac{t^{k-1}}{(k-1)!} e^{\lambda t}. \]

Hence, when \( k \in \{2, 4, 6, 8, \ldots\} \), we have \( \dot{p}_{k+1} - \dot{p}_k < 0 \). By the same logic, we have

\[ p_2 - p_1 = \frac{\lambda (v_H - v_L)}{M + 1} e^{-\lambda(1-t)} t + e^{\lambda t} C_1 < 0 \]  
\[ \text{(A.8)} \]

and

\[ \frac{d^{k-1}}{dt^{k-1}} [ (p_{k+1} - p_k) e^{-\lambda t} ] = (-\lambda)^{k-1} \left[ \frac{\lambda (v_H - v_L)}{M + 1} e^{-\lambda t} + C_1 \right] \]

so

\[ p_{k+1} - p_k = (-\lambda)^{k-1} \left[ \frac{\lambda (v_H - v_L)}{M + 1} e^{-\lambda t} \frac{t^{k-1}}{(k-1)!} + C_1 \frac{t^{k-2}}{(k-2)!} \right] e^{\lambda t} + \sum_{i=2}^{k} C_i \frac{t^{k-i}}{(k-i)!} e^{\lambda t} \]

and

\[ \dot{p}_{k+1} - \dot{p}_k = \lambda (p_{k+1} - p_k) + (p_k - p_{k-1}) \]

\[ - (\lambda + 1) (-\lambda)^{k-2} \frac{t^{k-3}}{(k-2)!} \left[ \frac{\lambda (v_H - v_L)}{M + 1} e^{-\lambda t} + C_1 (k-2) \right] e^{\lambda t} \]

where the first two terms are negative by Lemma A.2, and the last term is negative when \( k \in \{3, 5, 7, 9\ldots\} \) by the inequality (A.8), \( \dot{p}_{k+1} - \dot{p}_k < 0 \). In short, \( \dot{p}_{k+1} - \dot{p}_k < 0 \) for any \( k \in \mathbb{N} \).

**The Seller’s Problem: No Accumulation Result.** Similar to the two-unit case, we claim the following lemma is true.

**Lemma A.4.** After any seller’s history \( h_t^S \) with \( K(t) = k \) and \( t < 1 \), the seller’s best response to the H-buyer’s equilibrium strategy \( \sigma_B(t,k) \) specifies an offer either \( \sigma_S(h_t^B) = (p_2(t),1) \) or \( \sigma_S(h_t^B) = (v_L,Q) \) where \( Q \in \{1,2,\ldots,K(t)\} \).
The proof is exactly similar to the proof of lemma A.1, so it is omitted here. As a result, after any history, the seller has no incentive to accumulate buyers, so the remaining problem is to pin down the optimal fire sale times \( t^*_k \) for each \( k = 1, 2, \ldots, K \) provided that the seller believes there is no H-buyer waiting in the market.

**The Seller’s Problem: Optimal Fire Sale Time.** We employ the mathematical induction again to prove the result. Suppose for \( k = 1, 2, \ldots, K - 1 \), both the buyers and the seller follows the equilibrium strategy. Given the H-buyer’s best response, we first prove the following lemma. Given the H-buyer’s best response, we first prove the following lemma.

**Lemma A.5.** For each \( t < t^*_{K-2} \), \( \dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) < 0 \).

**Proof.** We know that \( \dot{\Pi}_2(t) - \dot{\Pi}_1(t) < 0 \). Now at \( t^*_{K-1} \), \( \dot{\Pi}_{K-1}(t^*_{K-1}) - \dot{\Pi}_{K-2}(t^*_{K-1}) = 0 \), and the limit \( \lim_{t \searrow t^*_{K-1}} \left[ \dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) \right] = -\lambda \left[ \dot{p}_{K-1}(t) - \dot{p}_{K-2}(t) + \dot{\Pi}_{K-2}(t) - \dot{\Pi}_{K-3}(t) \right] > 0 \). Hence \( \dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) < 0 \) for \( t \in (t^*_{K-1} - \varepsilon, t^*_{K-1}) \) where \( \varepsilon \) is small but positive. If \( \dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) > 0 \) for some \( t \), by continuity of \( \dot{\Pi}_{K-1}(\cdot) - \dot{\Pi}_{K-2}(\cdot) \), there must be a \( \hat{t} \) s.t. \( \dot{\Pi}_{K-1}(\hat{t}) - \dot{\Pi}_{K-2}(\hat{t}) = 0 \) and \( \dot{\Pi}_{K-1}(t) - \dot{\Pi}_{K-2}(t) < 0 \) for any \( t \in (\hat{t}, t^*_K) \). However, \( \dot{\Pi}_{K-1}(\hat{t}) - \dot{\Pi}_{K-2}(\hat{t}) > 0 \), which is a contradiction! (\( \Box \))

When \( k = K \) and the fact that the seller does not charge unacceptable price after any history, we can formulate the problem faced by the seller as follows.

\[
\Pi_K(t) = \max_{t_{K-1} \in [t, 1]} \int_t^{t_{K-1}} e^{-\lambda(s-t)} \lambda [p_K(s) + \Pi_{K-1}(s)] ds + e^{-\lambda(t_{K-1}-t)} [v_L + \Pi_{K-1}(t_{K-1})], \tag{A.9}
\]

where \( \Pi_k(\cdot) \) is defined recursively for each \( k = 1, 2, \ldots, K \). By the induction hypothesis, we have

\[
\Pi_{K-1}(t_{K-1}) = \begin{cases} (i - 1)v_L + \Pi_{K-2}(t_{K-2}), & t_{K-2} \in [t_{K-2}, t^*_K) \\ (K - 2)v_L, & t_{K-2} = 1 \end{cases}
\]

for \( i = 2, 3, \ldots, K - 1 \). We are going to show that as long as \( t^*_K > 0 \), the optimal fire sale time is strictly greater than \( t^*_{K-2} \).

In equilibrium, the H-buyers’ belief is correct, so the seller’s optimal choice is indeed \( t^*_K \). If \( t^*_K \neq t^*_K \), the equilibrium first-order condition (FOC) must hold at \( t = t^*_K \):

\[
\lambda [p_K(t^*_K) - v_L] + \dot{\Pi}_{K-1}(t^*_K) \leq 0 \tag{A.10}
\]

and when \( t^*_K > 0 \) the weak inequality is replaced by an equality.

**Lemma A.6.** Suppose that \( t^*_K > 0 \). There is a unique \( t^*_K < t^*_K \) s.t.

\[
\lambda [p_K(t^*_K) - v_L] + \dot{\Pi}_{K-1}(t^*_K)
\]

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1. is strictly negative at any \( t_{K-1} \in (t^*_K - 1, 1) \),

2. is strictly positive at any \( t_{K-1} \in [0, t^*_K - 1) \) when \( t^*_K - 1 > 0 \), and

3. is equal to zero at \( t_{K-1} = t^*_K - 1 \).

\[ \text{Proof.} \] The function form of the left-hand side of the FOC, \( \lambda[p_K(t_{K-1}) - v_L] + \hat{\Pi}_{K-1}(t_{K-1}) \), depends on the seller’s continuation value after the fire sale, \( \Pi_{K-1}(t^*_K - 1) \), so we need to consider its value case by case.

First, consider \( t_{K-1} \in [0, t^*_K - 2) \). Let

\[
f^1_K(t) = v_L + \Pi_{K-1}(t) - \Pi_{K-2}(t) - \frac{p_{K-1}(t) - v_L}{M + 1} \text{ for } t \in [0, t^*_K - 2)
\]

Similar to the two-unit case, a simple observation is that \( \lim_{t \rightarrow t^*_K - 2} \Pi_{K-1}(t) - \Pi_{K-2}(t) \rightarrow v_L \) by the induction hypothesis and \( \lim_{t \rightarrow t^*_K - 2} \left[ \mathbb{E}_i | t - \tau(t) | p_{K-1}(t) - v_L \right] > 0 \) and both \( \Pi_{K-1}(t) - \Pi_{K-2}(t) \) and \( p_{K-1}(t) \) are continuous function, so \( f^1_K(t) < 0 \) for \( t \) close to \( t^*_K - 2 \). If \( f^1_K(t) < 0 \) for any \( t \in [0, t^*_K - 2) \), we claim that \( t^*_K - 1 = 0 \). Otherwise, we let \( t^*_K - 1 = \sup\{t| t \leq t^*_K - 2, f^1_K(t) = 0 \text{ and } \exists \varepsilon > 0 \text{ s.t. } f^1_K(t') > 0 \text{ for } t' \in (t - \varepsilon, t). \}

If \( t^*_K - 1 > 0 \), let

\[
f^0_K(t) = p_K(t_{K-1}) - v_L - p_{K-1}(t_{K-1}) + \Pi_{K-1}(t_{K-1}) - \Pi_{K-2}(t_{K-1})
\]

for \( t \in [0, t^*_K - 1) \) where \( p_K(t) \) is defined in equation (A.7). Similar to the two-unit case, we can show that \( f^0_K(t) > 0 \) for \( t < t^*_K - 1 \) and \( \lim_{t \rightarrow t^*_K - 1} f^0_K(t) = 0 \). Notice that, when \( t < t^*_K - 1 \), we have

\[
f^0_K(t) = \dot{p}_K - \dot{p}_{K-1} + \dot{\Pi}_{K-1} - \dot{\Pi}_{K-2}
\]

We claim that \( f^0_K(t) < 0 \) for \( t < t^*_K - 1 \) since \( \dot{p}_K - \dot{p}_{K-1} < 0 \) by Lemma A.3 and \( \dot{\Pi}_{K-1} - \dot{\Pi}_{K-2} < 0 \) by Lemma A.5.

Let

\[
f_K(t) = \begin{cases} f^0_K(t), & t < t^*_K - 1 \\ f^1_K(t), & t \in [t^*_K - 1, t^*_K - 2) \end{cases}
\]

Hence, the FOC becomes \( f^0(t^*_K - 1) = 0 \), which has a unique solution in the interval \( [0, t^*_K - 2) \).

Further more \( f_K(t) \) is strictly positive for \( t < t^*_K - 1 \) and strictly negative for \( t \in (t^*_K - 1, t^*_K - 2) \).

Now consider the case in which \( t_{K-1} \in (t^*_K - 1 - i, t^*_K - 2 - i) \) for \( i = 1, 2, \ldots, K - 2 \). The first derivative of the seller’s objective function is given by

\[
p_K(t_{K-1}) + \Pi_{K-1}(t_{K-1}) - iv_L - \Pi_{K-1-i}(t_{K-1}) - [p_{K-1-i}(t) + \Pi_{K-2-i} - \Pi_{K-1-i}]
\]
for \( t_{K-1} \in (t_{K-1-i}^*, t_{K-2-i}^*) \). By construction of \( p_k \), we have
\[
p_K(t_{K-1}) = \frac{i + 1}{M + 1} v_L + \frac{M - i}{M + 1} p_{K-1-i}(t),
\]
where \( \tilde{t} \) is the H-buyer’s next regular attention time. Let
\[
f'_K(t) = \Pi_{K-1}(t_{K-1}) - i v_L - \Pi_{K-2-i}(t_{K-1}) + [p_K(t_{K-1}) - p_{K-1-i}]
\]
\[
= [\Pi_{K-1}(t_{K-1}) - i v_L - \Pi_{K-2-i}(t_{K-1})]
\]
\[
+ \left\{ \frac{i + 1}{M + 1} v_L + \frac{M - i}{M + 1} p_{K-1-i}(t) - p_{K-1-i}(t) \right\}
\]
\[
= [\Pi_{K-1}(t_{K-1}) - i v_L - \Pi_{K-2-i}(t_{K-1})] + \frac{i + 1}{M + 1} [v_L - p_{K-1-i}(t)].
\]
And by the construction of \( \Pi_k, \Pi_{K-1}(t_{K-1}) - i v_L - \Pi_{K-2-i}(t_{K-1}) = 0 \) for \( t_{K-1} \in (t_{K-1-i}, t_{K-2-i}) \).
So \( f'_K < 0 \). For \( t_{K-1} \in (t_{K-1-i}, t_{K-2-i}) \), let \( f_K(t) = f'_K(t) \), and the FOC is negative in these time interval.

Lemma A.6 implies that there is at most one \( t_{K-1}^* < t_{K-2}^* \) at which the FOC holds. Hence, there is a unique equilibrium fire sale time \( t_{K-1}^* \geq 0 \). When \( t_{K-1}^* > 0 \), it satisfies the equilibrium FOC in an equality.

In short, the equilibrium of the \( K \)-unit game uniquely exists. Similar to the two-unit case, off the path of play, if the fire sale is postponed such that (1) \( K(t) = k \) and (2) \( t_{K-2}^* \) are in the same period, the H-buyer’s reservation price is lower than \( p_k(t) \) and \( p_{k-1}(t) \) when \( K(t) = k \) and \( k - 1 \) respectively. However, the seller’s profit by running the fire sale in such a period is strictly less than that in the auxiliary problem. Hence, it is strictly dominated. Consequently, in the real problem, the seller does not have any incentive to choose a deal time later than \( t_{K-1}^* \) when \( K(t) = k \). Q.E.D.

### A.4 Proofs for the Applications

We first prove the following lemma which is useful to prove Proposition 4.

**Lemma A.7.** For any \( k \in \mathbb{N} \) and \( t \in [0, t_k^*] \), \( v_L \leq \Pi_{k+1}(t) - \Pi_k(t) < \Pi_k(t) - \Pi_{k-1}(t) \)

**Proof.** Fix a \( k \in \mathbb{N} \) and \( t < t_k^* \). Consider the law of motion of \( \Pi_{k+1} \) and \( \Pi_k \): \( \dot{\Pi}_{k+1}(t) = \lambda [\Pi_{k+1}(t) - p_{k+1}(t) - \Pi_k(t)] \) and \( \dot{\Pi}_k(t) = \lambda [\Pi_k(t) - p_k(t) - \Pi_{k-1}(t)] \), hence we have
\[
\lambda [\Pi_{k+1}(t) - \Pi_k(t)] - [\Pi_k(t) - \Pi_{k-1}(t)] = [\dot{\Pi}_{k+1}(t) - \dot{\Pi}_k(t)] + \lambda [p_{k+1}(t) - p_k(t)] \tag{A.11}
\]
Since both term on the right-hand-side of equation (A.11) are strictly negative for \( t < t_k^* \) by Lemma A.3 and Lemma A.5, we have the desired result. Q.E.D.
Proof of Proposition 4

Lemma A.7 implies that the marginal benefit of adding inventory is decreasing. For some \( \bar{c} \geq v_L \), we always find \( c > \bar{c} \) and an associated \( K(c) \) s.t. \( \Pi_{K(c)+1}(t) - \Pi_{K(c)}(t) < c \leq \Pi_{K(c)}(t) - \Pi_{K(c)-1}(t) \) so that \( K(c) \) is the optimal initial inventory size to the seller. The remaining is to show that \( \bar{c} = v_L \) so that for any \( c \geq v_L \) the optimal initial inventory to the seller \( K(c) \) is finite.

Suppose not and \( \bar{c} > v_L \). Then for some \( c \in (v_L, \bar{c}) \), for any \( K \), \( \Pi_{K+1}(0) - \Pi_K(0) > c \), and the seller’s inventory choice problem has no solution.

Notice that for a given \( K \in \mathbb{N} \), the seller’s profit has an upper bound as follows.

\[
\Pi_K(0) - Kc < \sum_{k=0}^{K-1} e^{-\lambda} \frac{\lambda^k}{k!} [kv_H + (K-k)v_L] + \sum_{k=K}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} Kv_H - Kc
\]

\[
= \sum_{k=0}^{K-1} e^{-\lambda} \frac{\lambda^k}{k!} [kv_H + (K-k)v_L] + \sum_{k=K}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} Kv_H - Kv_L + K(v_L - c)
\]

\[
= \sum_{k=0}^{K-1} e^{-\lambda} \frac{\lambda^k}{k!} k(v_H - v_L) + \sum_{k=K}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} K(v_H - v_L) + K(v_L - c)
\]

\[
< (v_H - v_L) \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} k + K(v_L - c)
\]

where \( \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} = \frac{1}{\lambda} < \infty \) is independent from \( K \) but \( K(v_L - c) < 0 \) if \( c > v_L \). More importantly, as \( K \) goes to infinity, \( K(v_L - c) \) goes to negative infinite, so for sufficient big \( K \), the seller’s profit is negative, which is a contradiction! \( Q.E.D. \)

References


