Information Acquisition and Reputation Dynamics

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Abstract
We study reputation games where a long-lived player with a possible commitment type faces a sequence of short-lived players who must pay to observe the long-lived player’s past behavior. In this costly information model we show that equilibrium behavior is cyclical. The long-lived player builds her reputation up only to exploit it; then builds it up again, and so on. We call this behavior reputation renewal and show that for a wide class of reputation games as well as for a host of possible alternatives to the information cost structure, all equilibria display reputation renewal. We provide a method to construct the equilibria, and explicitly construct an essentially unique equilibrium for the case of linear cost functions. We also find that in the reputation renewal equilibria, the short-lived players always randomize how much information they purchase, but play pure strategies once the information is obtained.

1 Introduction

In this paper we use a reputation model to study how costly discovery of past play affects the dynamic incentives and behavior in repeated interactions. In many transactions a long-lived player interacts with many short-lived players over time. For example, firms sell to different customers, borrowers take out loans from different lenders, restaurants

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serve different clients, VC firms provide funding and advice to different entrepreneurs, eBay sellers transact with different buyers.

An important feature of such transactions is that the short-lived players change over time. As a result, the short lived players usually do not have full information about the past play of the long-lived player. They need to acquire such information at a cost. For instance, banks may invest substantial effort to unravel the borrower’s credit history, or a customer may need to seek out previous customers to retrieve their experience with the same service provider.

To understand these dynamic interactions, we analyze a reputation model where the long-lived player who is potentially a commitment type—a type that always plays the same single action—interacts repeatedly with a sequence of short-lived players. At the beginning of each period, a short-lived player strategically and covertly decides whether to acquire information regarding the long-lived player’s past play, and if so, how much he should acquire. The long-lived player’s reputation is captured by her opponents’ current beliefs about her type. If they believe she is more likely to be a commitment type—has a better reputation—they will tend to take a more trusting action. But she has an incentive to abuse such trust if she is not the commitment type.

Our main result is that in the presence of costly information acquisition the equilibrium behavior of the patient long-lived player has a (stochastically) cyclical nature. The long-lived player builds up her reputation and then, when she is well trusted, turns to exploit her reputation. She then builds it up again, and so on. The intuition is that costly acquisition of information limits the short-lived players’ learning and makes it easy for the long-lived player to manipulate the learning process of her partners. This behavior pattern is substantially different in character from that of standard reputation models with freely available information.

While our main interest is in the behavior of the long-lived player, information acquisition causes a moral hazard problem for the short-lived players. A short-lived player has an incentive to gather less information, free-riding on the costly information acquisition of other short-lived players. In particular when the cost of information is strictly increasing in the amount acquired, we show that the short-lived players always randomize how much information they purchase, but play pure strategies once the information is obtained.

Reputation games where information is not costly have been extensively studied since the seminal work of Kreps and Wilson (1982a) and Milgrom and Roberts (1982). With infinite horizon the multiplicity of equilibria is generated by trigger strategies and coordination.\(^1\) Fudenberg and Levine (1989) show that reputation has payoff implications.

\(^1\)Note that trigger strategies motivated a different notion of reputation in repeated games with complete information; see, e.g., Shapiro (1982), Barro and Gordon (1983), Stokey (1989), and Chari and
thereby circumventing the equilibria multiplicity problem—there may be many kinds of behavior but very tight payoff limits.

In an imperfect monitoring model with very general assumptions, Cripps, Mailath and Samuelson (2004) show that reputation is a short-run phenomenon depleted over time. The reasoning is that the long-lived player can only benefit from her private information by strategically utilizing, and hence eventually revealing, her type. Hence, the equilibrium behavior converges to an equilibrium of the game with complete information.

In contrast, under perfect monitoring, our costly information acquisition model leads to a very distinct pattern of behavior. It displays the building of reputation and reputation is constantly being renewed and depleted. Furthermore, under imperfect monitoring, the play never converges to an equilibrium of the game with complete information, if the noise is small.

In our model the nature of types does not change over time. There are models that allow the long-lived player’s types to vary exogenously over time and such changes are unobservable to the short-lived players; see, e.g., Holmström (1982), Cole, Dow and English (1995), and Phelan (2005). The equilibrium dynamics in these papers are driven by the exogenous stochastic process that governs types and leads to cyclical behavior. However, reputation is not built by behavior, but reinstated via an exogenous shock. Tadelis (1999, 2002) and Mailath and Samuelson (2001) study markets for reputation where the long-lived players can sell and acquire their identities covertly. In a parallel paper, Liu and Skrzypacz (2006) analyze reputation games where the monitoring is limited to an exogenously bounded number of previous observations. It is shown that cyclical behavior can emerge. There are also models that study the mechanisms that enforce certain outcomes; see, e.g., Hörner (2002) considers market competition and Ekmekci (2005) considers rating systems.

We use the following example to provide a more detailed description of our results.

\[
\begin{array}{c|cc}
 & h & l \\
\hline
H & 2,3 & 0,2 \\
L & 4,0 & 1,1 \\
\end{array}
\]

**Figure 1:** The product-choice game

Consider the stage game\(^2\) in Figure 1. The row player is long-lived and the column player is short-lived. The stage game has \((L, l)\) as the unique Nash equilibrium outcome. In the repeated game, the long-lived player meets a different short-lived player in each

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period. The short-lived players assign a small prior probability to the long-lived player being committed to the Stackelberg action $H$ and the rest of the probability to her being of a normal type with the payoff described in Figure 1. We can think of the long-lived player as a firm that chooses the quality of the products (High or Low) and the short-lived players as customers who decide whether to buy a high quantity or a low quantity.

In Section 2 we introduce the generic model of such product-choice games with costly information acquisition. A short-lived player, before taking an action, can pay $C(n)$ to learn the long-lived player’s play in the last $n$ periods. Before purchasing any information the short-lived players are assumed to be perfectly symmetric. We elaborate on the definition of an equilibrium under this assumption in Section 3. In equilibrium the acquisition of more information can be identified with choosing a finer information partition of the same state space.

Theorem 1 in Section 4 shows that the infinite repetition of the static Nash equilibrium $(L, l)$ would have been the unique outcome had there been complete information in the class of games we considered. In this equilibrium, the short-lived players have no incentive to acquire information. Therefore, the value of information and the enforceability of the Stackelberg outcome in our model come solely from the incomplete information. The short-lived players acquire information because they want to learn the long-lived player’s type instead of “communication” or “coordination.” Learning is a central feature of the model.

Theorem 2 in Section 4 resolves the technical difficulties of costly information acquisition. Since the short-lived players decide how much information to obtain and the information consists of finite histories induced by the long-lived player’s play, players’ strategies induce a random graph over the state space. The size of the graph increases exponentially with the length of observed histories. Theorem 2 shows that equilibrium strategies can be expressed essentially through a simple index of the history that a short-lived player observes, which equals the number of $H$’s since the most recent $L$. This index grows linearly in the length of observed histories. The index ties together the reputation building of the long-lived player and the learning of the short-lived players: the long-lived player builds up her reputation as she accumulates enough $H$’s; the short-lived players learn her type when they find a single $L$.

Section 5 characterizes the equilibrium dynamics, which is at the heart of the paper. Theorem 3 shows that if the long-lived player’s discount factor exceeds a level determined by the stage game payoff parameters, for any equilibrium there is an integer $n^*$ such that the histories with indices greater than $n^*$ are off the equilibrium path and the histories on the equilibrium path can be expressed through the set of indices $\{0, 1, \ldots, n^*\}$. The subset $\{0, 1, \ldots, n^* - 1\}$ forms the reputation building phase. In this phase, the long-lived player
plays strictly mixed strategies. If an $H$ is the realized action, the index increases by 1; otherwise the index decreases down to 0. The long-lived player builds up her reputation as the index grows. When the index reaches $n^*$, the long-lived player plays the pure strategy $L$. The reputation is fully depleted in this stage and the reputation building process restarts with index 0.

Theorem 4 in Section 5 studies the behavior of short-lived players under the assumption that the cost function $C(n)$ is strictly increasing. Their behavior exhibits three generic features. (1) Limited monitoring. In an equilibrium in which the long-lived player depletes her reputation completely when the index is $n^*$, the maximal amount of information that the short-lived players acquire is exactly $n^*$, even though they could find out her type for sure by acquiring $n^* + 1$ bits of information. (2) Random monitoring. The short-lived players acquire $i$ bits of information with positive probability for each $i \in \{0, 1, ..., n^*\}$. This feature results from the free-riding problem. (3) Simple behavioral strategies. The short-lived players play pure strategy $l$ whenever they see an $L$ and they play $h$ whenever they see straight $H$’s.

Section 6 studies the imperfect monitoring case. We assume that if an action $H$ is played, $L$ happens to be the realized signal with probability $\varepsilon > 0$, and vice versa. With imperfect monitoring, the neat expression of equilibria no longer holds. For example, the number of $L$’s observed by a short-lived player will affect his belief. However, we can show that if $\varepsilon$ is small, the play never converges. Our interpretation is that reputation is built and exploited possibly via mixed strategies.

In Section 7, we demonstrate the method to construct equilibria under general cost functions. In particular, we explicitly construct an essentially unique equilibrium for linear cost functions. Omitted proofs are given in the Appendix.

2 The Model

2.1 The Underlying Game

$\begin{array}{c|cc}
H & h & l \\
\hline
H & u_{HH}, v_{HH} & u_{HL}, v_{HL} \\
L & u_{LH}, v_{LH} & u_{LL}, v_{LL} \\
\end{array}$

Figure 2: The generic product-choice game

A long-lived player 1 (the row player) meets a sequence of short-lived players 2 (the column player), with a new player 2 entering in every period. The underlying stage game is a $2 \times 2$ simultaneous-move game with the following features:
Assumption 1: \((L, l)\) is the unique Nash equilibrium, \((H, h)\) is the unique pure strategy Stackelberg outcome, and best responses are strict. All entries are normalized to be nonnegative.

By this assumption, player 1 has a strictly dominant strategy \(L\) in the stage game. This game captures the tension between the Stackelberg outcome and the Nash equilibrium outcome.

Assumption 2: \(u_{Lh} - u_{Hh} > u_{LI} - u_{HI}\).

This assumption says that player 1 is more tempted to “cheat” when player 2 “trusts” her more. As we will see in Theorem 1, this guarantees that the value of information comes only from the incomplete information. This assumption is intuitive when we think of \(h\) as a larger product quantity or a higher investment level.\(^3\)

Let \(u_H \equiv (u_{HH}, u_{HI})\) denote the payoff vector of player 1 given action \(H\). \(u_L, v_h, v_l\) are defined analogously.

### 2.2 Incomplete Information/Reputation

There are two types of player 1: a normal type who has stage game strategies and payoffs given above, and a commitment type who always plays the Stackelberg action \(H\).\(^4\) Player 2 assigns a small prior probability \(\mu_0 > 0\) to player 1 being the commitment type and probability \(1 - \mu_0\) to player 1 being the normal type. This information structure is common knowledge.

To focus on interesting cases, assume \(\mu_0 < \bar{\mu} = \frac{v_{LI} - v_{Lh}}{v_{LI} - v_{Lh} + v_{Hh} - v_{HI}}\). \(\bar{\mu}\) is chosen so that player 2 is indifferent between \(h\) and \(l\) if the stage game is played only once.

### 2.3 Information Acquisition

Player 1 observes the full history of outcomes. Player 2, upon entering the game, observes neither the previous outcomes nor the time when the game starts. That is, the short-lived players are symmetric ex ante. But player 2 can pay a cost \(C(n)\) to learn player 1’s actions in the previous \(n\) periods, \(n \in \mathcal{N} = \{0, 1, \ldots\}\).\(^5\) In this case, we say that player 2 buys \(n\) bits of information. The acquisition decision is not observable to player 1.

\(^3\)These two assumptions imply the following orders: \(u_{Lh} > u_{Hh} > u_{LI} > u_{HI}, v_{LI} > v_{Lh}\) and \(v_{Hh} > v_{HI}\).

\(^4\)The commitment type can be interpreted as having different payoffs such that playing \(H\) every period is a dominant strategy. See Kreps and Wilson (1982a).

\(^5\)In an early version of the paper we consider the possibility that \(\mathcal{N}\) is any subset of non-negative integers. For example, \(\mathcal{N}\) consists of all even integers. However, the main result will not change qualitatively.
We make the following assumptions on the cost function \( C : \{0, 1, \ldots\} \rightarrow R_+ \).

Assumption 3: \( C(n) \) is weakly increasing.

By this assumption, the cost function can have flat segments. This feature captures the case of “wholesale of information,” i.e., player 2 obtains a bundle at a fixed cost.

Assumption 4: \( C(0) = 0 \) and \( \lim_{n \to \infty} C(n) > \max\{v_{HH}, v_{HL}\} \).

This assumption implies that it is too costly for player 2 to have all the information. \( C(0) = 0 \) is just a normalization. A simple example that satisfies both assumptions is the linear cost: \( C(n) = cn \) for some constant \( c > 0 \).

2.4 Repeated Game Strategies and Payoffs

In each period, a short-lived player 2 makes the information acquisition decision according to a rule \( p \in \Delta(\mathcal{N}) \): he will acquire \( n \) bits of information with probability \( p(n) \). After he learns the information, player 2 adopts a behavioral strategy that is a function of his observations to \( \Delta\{h, l\} \). Player 1’s behavioral strategy in each period is a function of the past histories to \( \Delta\{H, L\} \). Player 1 maximizes the sum of her expected discounted payoffs, with a discount factor \( \delta \). Player 2 maximizes his expected payoff in the stage in which he plays.

3 Equilibrium Concept

The equilibrium notion we define below requires the strategies to be mutual best responses given beliefs, and the beliefs to be consistent with the strategies.

3.1 State Space

For any cost function \( C \), there exists an integer \( N_C \) such that a rational player 2 never buys more than \( N_C \) bits of information by Assumption 4.\(^6\)

Fix an integer \( N \geq N_C \). Let us write \( S = \{H, L\}^N \) as the set of player 1’s feasible plays in the last \( N \) periods. For a state \( s = (s_N, s_{N-1}, \ldots, s_1) \in S \), \( s_1 \) is the most recent outcome and \( s_N \) the oldest.

After acquiring \( n \) bits of information, \( n \leq N \), player 2’s information is represented by a partition \( \mathcal{P}_n \) on \( S \). The partition cell containing \( s \), \( \mathcal{P}_n(s) \), consists of all the states with

\(^6\)We can set \( N_C = \min\{n : C(n) > \max\{v_{HH}, v_{HL}\}\} \). Imposing individual rationality in this stage makes the equilibrium analysis much more convenient, though we miss some off-equilibrium-path behavior.
the same most recent \( n \) entries as \( s \). In particular, \( \mathcal{P}_0(s) = S \) for each \( s \in S \). \( \mathcal{P}_n \) becomes finer and finer as \( n \) increases, implying that more information is more informative.

We can therefore write player 2’s strategy after acquiring \( n \) bits of information, \( \sigma^n \), as a \( \mathcal{P}_n \)-measurable function from \( S \) to \( \Delta\{h, l\} \). Let us write \( \sigma = (\sigma^n)_{0 \leq n \leq N} \). Consequently, player 1 has a best response that depends only on \( S \), \( \pi : S \to \Delta\{H, L\} \). We focus on the stationary strategies of player 1.

We do not specify the strategies for the initial periods. For example, the first short-lived player will observe an empty history if he acquires information. However, the play of the initial players does not affect the long-run dynamics. See also Remark 1 below.

### 3.2 Consistent Beliefs

Since the short-lived players in general have no knowledge about when the game starts, we require that a short-lived player’s beliefs agree with the “frequency” induced by the play of the game. Technically, this is equivalent to saying that the beliefs are updated according to the Bayes’ rule as if the short-lived players held an improper uniform prior over the calendar time. A similar idea was proposed by Rosenthal (1979) for random matching games. We provide the equilibrium concept based on such belief consistence as is appropriate for our model.

For a state \( s = (s_N, s_{N-1}, ..., s_1) \in S \), we define \( t = s \land H \) as \( t_1 = H \) and \( t_n = s_{n-1} \) for \( 2 \leq n \leq N \) to represent a history \( t = (t_N, t_{N-1}, ..., t_1) \) that is derived by appending \( H \) to history \( s \) and dropping the oldest outcome. \( s \land L \) is defined analogously. When \( s \) is the current state, either \( s \land H \) or \( s \land L \) will be the next state. For \( N = 2 \), all possible transitions induce a directed graph as depicted in Figure 3.

![Figure 3: Transition on \( S = \{H, L\}^2 \)](image)

From player 2’s point of view, nature chooses one of the two types of player 1 and hence one of the two stochastic processes on \( S \):

(1) The commitment type plays \( H \) constantly and the process is degenerate in the state \( H^N := HH...H \). Let \( \phi \) be the invariant distribution of this process. \( \phi(H^N) = 1 \).
The normal type’s strategy $\pi$ induces a Markov process. Let $\lambda \in \Delta(S)$ be an invariant distribution of the Markov process. Note that we have suppressed the dependence of $\lambda$ on $\pi$.

After acquiring $n$ bits of information when the true state is $s$, player 2 learns the partition $\mathcal{P}_n(s)$, and he updates his belief about the first process/the commitment type according to the Bayes’ formula

$$\frac{\mu_0 \phi(\mathcal{P}_n(s))}{\mu_0 \phi(\mathcal{P}_n(s)) + (1 - \mu_0) \lambda(\mathcal{P}_n(s))}$$

Consistent with the formulation, player 2 holds the prior belief $\mu_0$ when he does not acquire information and he assigns probability 0 to the commitment type if he sees an $L$.

Conditional on player 1 being the normal type and player 2 acquiring $n$ bits of information in state $s$, player 2 holds belief $\lambda(\cdot | \mathcal{P}_n(s))$ over $\mathcal{P}_n(s)$. $\lambda(\cdot | \mathcal{P}_n)$ is a version of the conditional probability measure of $\lambda$ given $\mathcal{P}_n$. We impose the following natural property:

**Consistency**: $\lambda(\cdot | \mathcal{P}_{n+1}(s))$ is a conditional probability of $\lambda(\cdot | \mathcal{P}_n(s))$ for each $s \in S$.

We call this belief system $\{\lambda(\cdot | \mathcal{P}_n)\}_{0 \leq n \leq N}$ consistent and denote it simply (with some abuse of notation) as $\lambda$.

**Remark 1** In the formulation above, using the invariant (stationary) distribution instead of the long-run average probability avoids specification of play in initial periods and then focuses on the steady-state dynamics. In fact, the first several short-lived players will know when the game started if they acquire enough information. These players can use the usual Bayes’ rule to update their beliefs. As we show in Section 5.1, any equilibrium determines a unique ergodic set on the state space so that the initial play does not affect the long-run dynamics.

### 3.3 Best Responses

**Convention**: For a function $f : S \rightarrow \Delta\{x, y\}$, we write $f_s$ as $f(s)$ and treat it as the row vector $(f(s)(x), f(s)(y))$.

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7 More precisely, this is a Markov process on a directed graph.

8 This property is vacuous if $\lambda(s) > 0$ for each $s$. Consistency imposes restrictions only on states that are off the equilibrium path. Consider $S = \{H, L\}^3$ and suppose player 1 never plays $L$ in equilibrium. If after seeing a single $L$ (off the equilibrium path) player 2 assigns a positive probability to $HLL$, then consistency would require him to assign a positive probability to $HLL$ were he to observe $LL$. That is, $LL$ is stronger evidence of $HLL$ than is a single $L$. 

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3.3.1 Optimality of Player 2’s Behavioral Strategies

Let $b^n(s)(H)$ be player 2’s belief, upon obtaining $n$ bits of information in state $s$, that $H$ is going to be played. $b^n(s)(H)$ is induced according to the Bayes’ formula:

$$b^n(s)(H) = \frac{1}{\mu_0\phi(P_n(s)) + (1 - \mu_0)\lambda(P_n(s))}\left(\mu_0\phi(P_n(s)) + (1 - \mu_0)\sum_{s'\in P_n(s)}\lambda(s')\pi(s')(H)\right)$$

Player 2’s expected payoff, conditional on his acquiring $n$ bits of information in state $s$, is

$$V^n_s(\pi, \sigma, \lambda) = \sigma^n_s(h)v_h \cdot b^n_s + \sigma^n_s(l)v_l \cdot b^n_s.$$ (2)

On the RHS of (2), $v_h \cdot b^n_s = v_{Hh}b^n_s(H) + v_{Lh}b^n_s(L)$ is player 2’s expected payoff when he plays $h$ conditional on his acquiring $n$ bits of information in state $s$.

Player 2’s strategy $\sigma$ is optimal if for each $s \in S$ and $n \leq N$, $V^n_s(\pi, \sigma, \lambda) \geq V^n_s(\pi, \sigma', \lambda)$ for all feasible $\sigma'$.

3.3.2 Optimality of Player 2’s Information Acquisition Rule

Let $V^n(\pi, \sigma, \lambda)$ be player 2’s expected payoff of acquiring $n$ bits of information. Then

$$V^n(\pi, \sigma, \lambda) = \sum_s \Lambda(s)V^n_s(\pi, \sigma, \lambda) - C(n),$$ (3)

where $\Lambda(s) = \mu_0\phi(s) + (1 - \mu_0)\lambda(s)$.

Player 2’s information acquisition rule $p$ is optimal if $p(n) > 0$ implies $V^n(\pi, \sigma, \lambda) \geq V^m(\pi, \sigma, \lambda)$ for all $m$, $0 \leq m \leq N$.

Player 2’s expected payoff by taking strategy $(\sigma, p)$ is $\sum p(n)V^n(\pi, \sigma, \lambda)$.

3.3.3 Optimality of Player 1’s Strategies

Let $\sigma_s := \sum_n p(n)\sigma^n_s$. From player 1’s point of view, $\sigma_s$ is the “expected strategy” taken by player 2 in state $s$ since she is uninformed of his information acquisition.

Let $U_s(\pi, \sigma, p, \lambda)$ be the expected payoff that player 1 obtains in state $s$ when $(\pi, \sigma, p)$ is played and player 2 holds belief $\lambda$. $U_s(\pi, \sigma, p, \lambda)$ can be calculated recursively as follows:

$$U_s(\pi, \sigma, p, \lambda) = \pi_s(H)[(1 - \delta)\sigma_s \cdot u_H + \delta U_{s\cap H}(\pi, \sigma, p, \lambda)] + \pi_s(L)[(1 - \delta)\sigma_s \cdot u_L + \delta U_{s\cap L}(\pi, \sigma, p, \lambda)]$$

Player 1’s strategy $\pi$ is optimal if in every state $s \in S$, $U_s(\pi, \sigma, p, \lambda) \geq U_s(\pi', \sigma, p, \lambda)$ for all $\pi'$. Equivalently, this optimality condition can be stated in terms of the one-step deviation principle as follows: $\pi$ is optimal if in every state $s \in S$, $\pi_s(\tilde{a}_1) > 0$ implies $\tilde{a}_1$
maximizes

\[(1 - \delta)\sigma_u \cdot u_{a_1} + \delta U_{s\land a_1}(\pi, \sigma, p, \lambda)\]

### 3.4 Equilibria and Existence

**Definition 1** \((S, \pi, \sigma, p, \lambda)\) is an equilibrium under the cost function \(C\) if (1) \(S = \{H, L\}^N\), \(N \geq N_C\); (2) \(\pi, \sigma, \) and \(p\) are optimal, and (3) \(\lambda\) is consistent.

Note that an equilibrium is well defined when \(\mu_0 = 0\). From the definition, there seems to be some flexibility on the dimensionality of the state space. However, the choice of \(N\) and \(N_C\) is not essential and the “redundant” dimensionality can be reduced in such a way that the strategies of both players are “aggregated” and the equilibrium payoffs are preserved. The intuition is straightforward. For example, from player 2’s point of view, the additional \(N - N_C\) bits of information simply serve as a randomizing device whose outcomes are observed only by player 1.

**Proposition 1** An equilibrium exists.

The proof uses the usual fixed-point arguments. See, for example, the proof of Theorem 1 in Rosenthal (1979). The idea is as follows. We first prove, for a fixed information acquisition rule \(p\), the existence of a triple \((\pi, \sigma, \lambda)_p\) such that \(\pi\) and \(\sigma\) are mutual best responses with a belief system \(\lambda\). We then choose \(p\) to maximize player 2’s payoff. To get consistent beliefs, we only need to make player 1 tremble with completely mixed strategies and consider the limit of the induced beliefs. As usual, the best-response correspondences are upper semi-continuous.

### 4 Preliminary Results

This section presents two results. Theorem 1 shows that the infinite repetition of the static Nash equilibrium would be the unique outcome were the game to have complete information, \(\mu_0 = 0\). This result guarantees that the value of information comes solely from the incomplete information when \(\mu_0 > 0\). In other words, the short-lived players’ motivation to acquire information is not for “communication” or “coordination” but to learn the long-lived player’s type.

Theorem 2 shows that learning and reputation building can be expressed in a simple form: players are concerned about the position of the most recent non-commitment action in the past play.
4.1 Games with Complete Information

**Theorem 1** In the complete information game \((\mu_0 = 0)\), the infinite repetition of static Nash equilibrium \((L, l)\) is the unique equilibrium outcome. Consequently, player 2 never purchases information if \(C(1) > 0\).

The formal proof is given in the Appendix. Here is the main intuition. Were \((H, h)\) to be supported in an equilibrium, player 1 would contemplate playing \(L\) for one period. The benefit from such a deviation is \((u_{Lh} - u_{Hh})\), while \((u_{LI} - u_{HI})\) measures the loss: instead of possibly bearing a loss of \((u_{Hh} - u_{LI})\) in each subsequent period as in the unlimited information case, player 1’s severest loss from such a deviation is much smaller. This happens for two reasons. First, player 1 can certainly come back to \((H, h)\) after a few periods due to player 2’s costly information acquisition. Second, player 2 may not want to punish player 1 by the pure strategy \(l\) even though he sees a deviation since he knows that player 1 has the incentive, and indeed the ability, to come back to \((H, h)\); then, player 2 will take strategies favorable to player 1 and hence compensate for her loss. Under Assumption 2, \((u_{Lh} - u_{Hh}) > (u_{LI} - u_{HI})\), no credible punishment can deter player 1 from deviating, which results in the static Nash equilibrium outcome.

4.2 Simple Representation of Equilibria

In general, the infinite repetition of the static Nash equilibrium will not be an equilibrium outcome when there is incomplete information.\(^9\) Player 1’s strategies induce a random graph over the state space \(S = \{H, L\}^N\) and the size of the graph grows exponentially as \(N\) increases. We show that a simple linear order cuts through the graph and essentially determines all equilibria.

**Definition 2** The clean history index of state \(s\), \(I(s)\), is the number of commitment actions \(H\) since the most recent non-commitment action \(L\) in state \(s\). Formally,

\[
I(s) = \begin{cases} 
N & \text{if} \forall i \in \{1, \ldots, N\}, s_i = H \\
\min \{i : s_i = L\} - 1 & \text{otherwise}.
\end{cases}
\]

For example, \(I(\cdots LHH) = 2, I(\cdots L) = 0,\) and \(I(H \cdots HH) = N\). Note that \(I\) partitions \(S\). Let \(\mathcal{I}(s) = \{s' : I(s') = I(s)\}\) be the partition cell containing \(s\) and \(\mathcal{I} = \{0, 1, \ldots, N\}\). By definition, an additional \(H\) will increase the index from \(i\) to \(\min\{i+1, N\}\) and an \(L\) will reduce the index down to 0. The transition on \(I\) is depicted in Figure 4.

\(^9\)It will be the equilibrium outcome when the cost is so high as to prevent player 2 from purchasing any information.
Theorem 2 Suppose \((S, \pi, \sigma, p, \lambda)\) is an equilibrium. Then there exists another equilibrium \((S, \bar{\pi}, \bar{\sigma}, \bar{p}, \bar{\lambda})\) such that (1) players’ expected payoffs are preserved, (2) players’ strategies depend only on the clean history index: \(\bar{\pi}\) is \(\mathcal{I}\)-measurable and \(\bar{\sigma}^n\) is measurable w.r.t. the join of \(\mathcal{I}\) and \(\mathcal{P}_n\),\(^{10}\) and (3) the information acquisition rule is unchanged: \(\bar{p} = p\).

In the proof of this theorem, we set
\[
\bar{\pi}_s := \sum_{s' \in \mathcal{I}(s)} \lambda(s'|\mathcal{I}(s))\pi_{s'}
\]
That is, the flexibility of player 1’s strategies beyond index \(I(s)\) can be seen as coming from a randomization device. Furthermore, we show that, in any equilibrium, after buying \(n\) bits of information in two states with the same index, player 2 either plays the same strategy or is indifferent between the strategies taken in the two states. Therefore, the new equilibrium strategy profile \((\bar{\pi}, \bar{\sigma})\) is obtained by “projecting” \((\pi, \sigma)\) onto the index set \(\mathcal{I}\).

The intuition behind the theorem is as follows. The index \(I\) ties reputation building and learning together. Player 1 builds up her reputation as she accumulates enough \(H\)’s. Player 2 cares about two things. First of all, he is concerned about the long-lived player’s type. The higher the index, the more difficult it is for him to find out her type. Second, he also cares about the information of future short-lived players, because their beliefs about her type will affect the continuation play and hence the incentives in the current period. For example, assume all short-lived players acquire 3 bits of information. Upon seeing

---

\(^{10}\)The join of two partitions is their finest common coarsening. Consider player 2’s information structure after buying \(n\) bits of information. \(\mathcal{I}(s)\) indicates the position of the most recent \(L\) in \(s\) while \(\mathcal{P}_n(s)\) contains all information about the last \(n\) entries in \(s\). If \(n \leq I(s)\), player 2 will see straight \(H\)’s, and hence \(\mathcal{I}(s) \subset \mathcal{P}_n(s)\). When \(n > I(s)\), player 2 sees an \(L\), and hence \(\mathcal{P}_n(s) \subset \mathcal{I}(s)\). Therefore, the join of \(\mathcal{I}\) and \(\mathcal{P}_n\) is given by

\[
(\mathcal{I} \lor \mathcal{P}_n)(s) = \begin{cases} 
\mathcal{P}_n(s) & \text{if } n \leq I(s) \\
\mathcal{I}(s) & \text{otherwise.}
\end{cases}
\]
HLH or LLH, the current short-lived player knows that the next short-lived player will know player 1’s type but he doesn’t know whether the second short-lived player after him will see an L. Index I summarizes the depth of this knowledge. One implication of Theorem 1 is that player 2, knowing player 1’s type, cannot effectively punish or reward player 1 based on the different realizations of HLH and LLH. Therefore, player 1 can only benefit by cheating those future short-lived players who do not know her type. Thus her strategy is also essentially determined by the depth of player 2’s knowledge summarized by index I.

4.2.1 A Crucial Step

The detailed proof of Theorem 2 is given in the Appendix. Here we present a crucial step. Recall that $\sigma_s = \sum p(n)\sigma_n^s$ is player 2’s “expected strategy” and $U_s$ is player 1’s expected payoff in state $s \in S$.

**Proposition 2** In any equilibrium $(S, \pi, \sigma, p, \lambda)$, if $I(s) = I(s')$, then $\sigma_s = \sigma_{s'}$ and $U_s = U_{s'}$.

This result says that in any equilibrium player 1 expects the same payoff and player 2 expects to play the same strategy in states with the same index. It is the crucial step needed to show that the convex combination of player 1’s strategies as in (4) remains optimal in the new equilibria. The proofs of Proposition 2 and several key lemmata in the Appendix make use of a relationship, termed similarity, which is auxiliary to the clean history index.

**Definition 3** We say $s$ and $s'$ are $m$-similar, denoted by $s \sim_m s'$, if $s_n = s'_n$ for all $n \in \{1, \cdots, m\}$ and $s_k = L$ for some $k \in \{1, \cdots, m\}$. We say $s$ and $s'$ are similar, denoted by $s \sim s'$, if they are $m$-similar for some $m \in \{1, \cdots, N\}$.

Consider an example for $N = 4$. HHLH and LHLH are 3-similar while they have index 1. By definition, similarity of two states requires that an L lies in their shared tail. This L implies that starting from any two different but similar states, short-lived players’ beliefs about player 1’s type evolve in exactly the same way given the same continuation play. Furthermore, if the same action is taken in two $m$-similar states ($m < N$), the succeeding states are $(m+1)$-similar. In contrast, the clean history index of the succeeding states can move in one of two directions depending on the action taken (either increase by 1 or decrease down to 0). It is this feature of similarity that distinguishes it from the clean history index and facilitates the inductive proof.

We summarize the basic properties of similarity below. The straightforward proof is omitted.
Lemma 1

1. \( \sim^m \) and \( \sim \) are equivalence relations.
2. If \( s \sim^n s' \) and \( s_k = L = s'_k \) for \( k \leq m < n \), then \( s \sim^m s' \).
3. \( I(s) = I(s') \) if and only if \( s \sim s' \) or \( s = s' = H^N \).

Proposition 2 follows from part (3) of Lemma 1 and the next result.

Lemma 2

If \( s \sim s' \), then \( \sigma_s = \sigma_{s'} \) and \( U_s = U_{s'} \) in any equilibrium \((S, \pi, \sigma, p, \lambda)\).

**Proof.**

If \( s \sim^N s' \), then \( s = s' \) and hence \( \sigma_s = \sigma_{s'} \) and \( U_s = U_{s'} \). Assume \( U_s = U_{s'} \) and \( \sigma_s = \sigma_{s'} \) if \( s \sim^n s' \), where \( n = N, N - 1, \ldots, m + 1 \), and \( m \leq N - 1 \).

Suppose \( s \sim^m s' \), but \( \sigma_s \neq \sigma_{s'} \). Without loss of generality, let us assume \( 1 \geq \sigma_s(h) > \sigma_{s'}(h) \geq 0 \). Therefore, \( \sum_{n \geq m+1} p(n)\sigma^n(s)(h) > \sum_{n \geq m+1} p(n)\sigma^n(s')(h) \). There exists \( n \geq m + 1 \) such that \( p(n) > 0 \) and \( 1 \geq \sigma^n(s)(h) > \sigma^n(s')(h) \geq 0 \).

Since \( \sigma^n(s)(h) > 0 \) and \( s_j = L \) for some \( j \leq m \), there exists \( t \in S \), such that \( t \sim^n s \) and \( \pi_t(H) > 0 \). Note that \( s_j = L \) is important: player 2 knows the type of player 1 but believes the normal type will play \( H \) with positive probability in state \( t \). Since \( n \geq m + 1 \), \( \sigma_s = \sigma_t \) by the induction hypothesis. Furthermore, \( t \sim^m s \) by part (2) of Lemma 1. Analogously, \( \sigma^n(s') < 1 \) implies that there exists \( t' \in S \), such that \( t' \sim^n s' \), \( \pi_{t'}(L) > 0 \), \( \sigma_{s'} = \sigma_{t'} \) and \( t' \sim^m s' \). By transitivity (part (1) of Lemma 1), \( t \sim^m t' \) and hence \( t \wedge L \sim^{m+1} t' \wedge L \) and \( t \wedge H \sim^{m+1} t' \wedge H \). Therefore, \( U_{t \wedge L} = U_{t' \wedge L} \) and \( U_{t \wedge H} = U_{t' \wedge H} \) by the induction hypothesis.

However, we will show that the properties derived above and summarized as follows are incompatible:

\[
\begin{align*}
\sigma_s(h) &> \sigma_{s'}(h) \\
\sigma_s &= \sigma_t \text{ and } \sigma_{s'} = \sigma_{t'} \\
\pi_t(H) &> 0 \text{ and } \pi_{t'}(L) > 0 \\
U_{t \wedge L} &= U_{t' \wedge L} \text{ and } U_{t \wedge H} = U_{t' \wedge H}
\end{align*}
\]

Denote \( r_s(a_1) \) as player 1’s expected payoff by playing \( a_1 \in \{H, L\} \) in state \( s \in S \).

Then

\[
r_t(L) = (1 - \delta)u_L \cdot \sigma_t + \delta U_{t \wedge L}
\]

\[
= (1 - \delta)u_L \cdot \sigma_s + \delta U_{t \wedge L} \text{ (by (6))}
\]

Analogously,

\[
r_t(H) = (1 - \delta)u_H \cdot \sigma_s + \delta U_{t \wedge H}
\]

\[
r_{t'}(L) = (1 - \delta)u_L \cdot \sigma_{s'} + \delta U_{t' \wedge L}
\]

\[
r_{t'}(H) = (1 - \delta)u_H \cdot \sigma_{s'} + \delta U_{t' \wedge H}
\]
Since $\pi_t(H) > 0$ (by (7)), $r_t(H) \geq r_t(L)$. Therefore,

$$\delta(U_{t\wedge H} - U_{t\wedge L}) \geq (1 - \delta)(u_L - u_H) \cdot \sigma_s$$

(9)

Since $\pi_{t'}(L) > 0$, $r_{t'}(L) \geq r_{t'}(H)$. Therefore,

$$(1 - \delta)(u_L - u_H) \cdot \sigma_{s'} \geq \delta(U_{t'\wedge H} - U_{t'\wedge L})$$

(10)

Since $U_{t\wedge H} - U_{t\wedge L} = U_{t'\wedge H} - U_{t'\wedge L}$ (by (8)), it follows from (9) and (10) that

$$(u_L - u_H) \cdot (\sigma_s - \sigma_{s'}) \leq 0 \iff ((u_{Lh} - u_{Hh}) - (u_{Li} - u_{Hi}))((\sigma_s(h) - \sigma_{s'}(h)) \leq 0$$

But this is impossible because $u_{Lh} - u_{Hh} > u_{Li} - u_{Hi}$ by Assumption 2 and $\sigma_s(h) > \sigma_{s'}(h)$ by (5).

Therefore, $s \prec^m s'$ implies $\sigma_s = \sigma_{s'}$. Then

$$U_s = \max\{(1 - \delta)u_L \cdot \sigma_s + \delta U_{s\wedge L}, (1 - \delta)u_H \cdot \sigma_s + \delta U_{s\wedge H}\}$$

$$= \max\{(1 - \delta)u_L \cdot \sigma_{s'} + \delta U_{s'\wedge L}, (1 - \delta)u_H \cdot \sigma_{s'} + \delta U_{s'\wedge H}\}$$

$$= U_{s'}$$

Thus $s \prec^m s'$ implies $U_s = U_{s'}$.

We conclude that if $s \prec s'$, then $\sigma_s = \sigma_{s'}$ and $U_s = U_{s'}$. □

5 Equilibrium Dynamics

By virtue of Theorem 2, we will write an equilibrium as $(I, \pi, \sigma, p, \lambda)$ with state space $I = \{0, ..., N\}$. With some abuse of notation, $\pi_i, \sigma_i^n, \lambda(i)$ and $U_i$ are defined on $I$. We also write $\sigma_i = \sum p(n)\sigma_n^i$ as player 2’s “expected strategy” in state $i$.

This section is central to the paper. We show that any equilibrium with information acquisition consists of a stochastic cycle on the equilibrium path: reputation keeps rebuilding and unraveling again and again. We also specify player 2’s behavior pattern.

It makes sense to talk about equilibrium dynamics only when the cost is small enough that player 2 is willing to purchase some information. If no information is ever bought in an equilibrium, the repetition of the static Nash equilibrium $(L, l)$ is the only outcome. As the commitment type plays $H$, player 2 is tempted to learn one bit of information to check player 1’s type. The benefit of such a trial is $\mu_0(v_{Hh} - v_{Hi})$ since with probability $\mu_0$ player 1 is the commitment type and player 2 should play $h$ instead of $l$. The trial is
worthwhile if \( C(1) < \mu_0(v_{Hh} - v_{HI}) \). The analysis leads to the following.

**Lemma 3** Not acquiring information is an equilibrium if and only if \( C(1) \geq \mu_0(v_{Hh} - v_{HI}) \).

\[ C(1) < \mu_0(v_{Hh} - v_{HI}) \] is assumed in the rest of the paper.

### 5.1 Renewal of Reputation

**Definition 4** We say that \((I, \pi, \sigma, p, \lambda)\) is a reputation renewal equilibrium if there exists an integer \( n^* \), \( 0 < n^* < N \), such that the set \( \{0, 1, ..., n^*\} \) consists of all states on the equilibrium path, and it can be characterized by the following two phases: (1) the reputation-building phase \( \{0, 1, ..., n^* - 1\} \), in which the reputation-building process \( \{\pi_i\} \) : \( 0 < \pi_i(H) \leq \bar{\mu} \), is defined, and (2) the reputation exploitation phase \( \{n^*\} : \pi_{n^*}(L) = 1 \).

The term “renewal” refers to the fact that the process “starts afresh” in state 0 as illustrated in Figure 5. In the reputation-building process, player 1’s behavioral strategies are strictly mixed, and they do not assign high probabilities to the Stackelberg action \( (\pi_i(H) \leq \bar{\mu}) \). Therefore, player 1 builds her reputation by manipulating the beliefs of the short-lived players. Once she has played \( H \)’s for \( n^* \) consecutive periods, player 1 depletes her reputation completely, and hence states higher than \( n^* \) are off the equilibrium path.

\[ \text{Figure 5: Reputation renewal equilibria, } 0 < \pi_i(H) \leq \bar{\mu}, i \in \{0, 1, \cdots, n^* - 1\} \]

**Theorem 3** If \( \delta > \frac{u_{Lh} - u_{HI}}{u_{Lh} - u_{Hl}} \), every equilibrium \((I, \pi, \sigma, p, \lambda)\) is a reputation renewal equilibrium with \( 0 < n^* < N \). Furthermore, the equilibrium has the property that both \( \sigma_i(h) \) and \( U_i \) are strictly increasing in \( i \), \( i \in \{0, 1, ..., n^*\} \).

On average, player 2 will trust player 1 more when player 1 has a larger clean history index: \( \sigma_i(h) \) is strictly increasing to provide increasing incentives for player 1 to “climb”
toward higher states. However, $\pi_i(H)$ may not be monotonic in $i$ as it depends on the underlying cost structures. Intuitively, player 1 is more willing to mimic the commitment type in a state when the information about that state is less expensive.

**Corollary 1** If $\delta > \frac{u_{lh} - u_{hh}}{u_{lh} - u_{ll}}$, any equilibrium transition induces a unique ergodic set on the state space.

The existence of a unique ergodic set means the play in the initial periods does not affect the long-run dynamics. From a technical point of view, defining equilibrium beliefs in terms of long-run average probabilities or in terms of invariant distributions makes no difference.

Before proving Theorem 3, we study the short-lived players’ behavior.

### 5.2 Short-Lived Players’ Behavior

In this section we study player 2’s information acquisition decision and behavioral strategies in equilibrium $(I, \pi, \sigma, p, \lambda)$. We find three notable features.

1. **Limited monitoring.** Player 1 depletes her reputation definitely whenever she accumulates $n^*$ commitment actions. Therefore, player 2 can learn player 1’s type and avoid being cheated if he acquires $n^* + 1$ bits of information. But for generic cost functions, the maximal amount he acquires is exactly $n^*$.

2. **Random monitoring.** Player 2 always randomizes between buying and not buying information, and he also randomly determines how much to buy.

3. **Simple behavioral strategies.** Upon acquiring information, player 2 will play pure strategy $l$ whenever he sees a non-commitment action and pure strategy $h$ whenever he sees straight $H$’s. If he does not acquire information, player 2 could either play $l$ or $h$, depending on the cost structure.

We consider strictly increasing cost functions. Since a weakly increasing cost function can be approximated by a sequence of strictly increasing functions, the features under strictly increasing cost functions are robust features of reputation games with costly information acquisition.

**Theorem 4** If $\delta > \frac{u_{lh} - u_{hh}}{u_{lh} - u_{ll}}$ and the cost function is strictly increasing, then the following holds in an equilibrium $(I, \pi, \sigma, p, \lambda)$ with player 1 depleting her reputation completely in state $n^*$.

---

11Player 2’s behavior is somewhat flexible on those flat segments of a weakly increasing cost function. For an example, see Section 7.2.
(1) \( n^* \) is the largest amount that player 2 ever purchases in equilibrium for generic cost functions\(^{12}\).

(2) \( p(i) > 0 \) and \( \pi_i(H) < \bar{\mu} \) for each \( i \in \{0, 1, ..., n^*\} \).

(3) Upon acquiring information, player 2 plays pure strategy \( L \) whenever he sees an \( L \) and he plays pure strategy \( h \) whenever he sees straight \( H \)’s.

We provide the main intuition behind the results.

For part (1), the maximal amount of information that player 2 acquires in equilibrium is at least \( n^* \): had player 2 always purchased \( n^* - 1 \) bits or less, player 1 could have completely depleted her reputation in state \( n^* - 1 \) or earlier because player 1, who discounts her payoff, prefers to exploit the benefit sooner rather than later. On the other hand, were player 2 to acquire \( n^* + 1 \) bits or more, player 1 would have an incentive to wait: her payoff is higher if she depletes her reputation when player 2 trusts her more.

Part (2) results from the free-rider problem. For example, if all short-lived players acquire \( n^* \) bit of information for sure, player 1 will be disciplined to play some strategies to build up her reputation. Hence a short-lived player has an incentive to purchase less information to save the cost. On the other hand, if no short-lived players buy information, player 1 will play \( L \) for sure, and hence a short-lived player will have an incentive to buy information to check player 1’s type. This leads to a mixed strategy for information acquisition. The tricky part of the proof is to show that in equilibrium the short-lived players buy \( i \) bits of information with a positive probability for each \( i \in \{0, 1, ..., n^*\} \).

Part (3) is intuitive: had player 2 been indifferent between his two actions upon obtaining the information, he should have played a pure strategy without paying for the information.

5.3 Proof of the Main Theorem: Theorem 3

The proof of Theorem 3 utilizes “rolling induction” extensively. We divide it into several steps.

**Step 1:** We prove that all equilibria with information acquisition are renewal equilibria in a weaker sense. The result states that player 1 will play \( H \) with strictly positive probabilities in the reputation buildup phase. The result is “weaker” because it does not exclude the case that player 1 plays \( H \) with a probability greater than \( \bar{\mu} \). It can also be the case that player 1 plays pure strategy \( H \). This is not sufficient to support our claim that player 1’s reputation is player 2’s belief about her type. The proof is by contradiction.

---

\(^{12}\)The space of (weakly increasing) cost functions is endowed with the topology of pointwise convergence. A strictly increasing cost function is generic if it lies in a relatively open dense subset of the space of strictly increasing cost functions.
If player 1 weakly prefers $H$ in all states, inductively we can show that player 2 plays a constant “expected strategy,” which in turn implies that player 1 will strictly prefer $L$ in all states.

**Step 2:** Using the weaker renewal result in Step 1, we show that player 2’s expected strategies must be weakly increasing on the equilibrium path in order to provide incentives for player 1 to play $H$ with strictly positive probability.

**Step 3:** Using the weak monotonicity of player 2’s expected strategies, we prove the stronger renewal equilibrium result: player 1 actually plays strictly mixed strategies in the reputation buildup phase with $\pi_i(H) \leq \bar{\mu}$. The proof is by contradiction. If player 1 plays pure strategy $H$ in some state on the equilibrium path, then player 2 will respond with $h$ whenever he observes this state; consequently, player 2’s expected strategies in higher states cannot assign larger probabilities to $h$ than the current stage expected strategy does. They must all be equal by the weak monotonicity result in Step 2. Therefore, player 1 has no incentive to play $H$ in this state and $L$ later: she prefers to play $L$ sooner rather than later as she discounts her payoff.

**Step 4:** With the renewal equilibria result in hand, we prove the strict monotonicity of player 2’s expected strategies and player 1’s payoffs.

### 5.3.1 Step 1: Weaker Renewal Equilibria

**Proposition 3** Suppose $\delta > \frac{u_{HL} - u_{HH}}{u_{LL} - u_{LL}}$. For each equilibrium $(I, \pi, \sigma, p, \lambda)$, there exists an $n$, $0 < n < N$, such that (1) $\pi_i(H) > 0$ if $0 \leq i < n$ and $\pi_n(H) = 0$, and (2) states higher than $n$ are off the equilibrium path.

The transition on $I$ is “linear” and hence very easy to work with. In particular, we can categorize all feasible equilibrium transition patterns. If an equilibrium does not take the form in Proposition 3, there could be five other patterns.

1. $i = 0$ is the unique absorbing state: player 1 plays $L$ almost surely.
2. $i = N$ is the unique absorbing state: player 1 plays $H$ almost surely. $\pi_N(H) = 1$.
3. Both $i = 0$ and $i = N$ are absorbing states: player 1 will play $L$ or $H$ depending on the initial conditions. $\pi_N(H) = 1$ in this case.
4. $\{0, 1, ..., n\}$ is an ergodic set with $0 < n < N$, and $i = N$ is an absorbing state. $\pi_N(H) = 1$ in this case.
5. $I = \{0, 1, ..., N\}$ is an ergodic set. In this case, $\pi_i(H) > 0$ if $0 \leq i \leq N - 1$.

Case (1) is impossible. By Lemma 3, when $C(1) < \mu_0(v_{HH} - v_{HL})$, player 2 will strictly prefer to buy 1 bit of information. If player 2 commits to buying 1 bit of information and
$i = 0$ is the absorbing state, then player 1 would end up with $u_{Ll}$. But if she is patient enough, player 1 can “escape the trap” by playing $H$ for 1 period, in which her average payoff is at least $u_{Hl}$ and after which her average payoff is at least $u_{Hh}$. This is because player 1 can convince the short-lived player that she is the commitment type. Therefore, the discount factor $\delta$ that supports this proposed deviation needs to satisfy the following:

$$u_{Hl}(1 - \delta) + \delta u_{Hh} > u_{Ll}$$

This inequality is satisfied when $\delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}}$.

If $C(n) = C(1)$ for some $n > 1$, it is possible that player 2 buys more than 1 bit of information. The analysis is similar. We will see this in Section 7.2. Therefore $i = 0$ cannot be absorbing when $\delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}}$.

We now show that the other four cases are impossible.

Let $n^{(1)} := \max\{n : p(n) > 0\}$. $n^{(1)} > 0$ when $C(1) < \mu_0(v_{Hh} - v_{Hl})$.

Lemma 4 If player 1 weakly prefers to play $H$ in some state $i \in \{n^{(1)}, \ldots, N\}$, then $\sigma_i = \sigma_{n^{(1)}}$ for each $i \in \{0, 1, \ldots, N\}$.

Proof. Step A: We show that if $n^{(1)} \leq i \leq N$, then $\sigma_i = \sigma_{n^{(1)}}$, $U_i = u_H \cdot \sigma_{n^{(1)}}$ and player 1 weakly prefers to play $H$ in state $i$.

By definition of $n^{(1)}$, $\sigma_i = \sigma_{n^{(1)}}$, if $n^{(1)} \leq i \leq N$. We prove the rest of the claim by induction.

Claim 1: Player 1 weakly prefers to play $H$ in state $N$.

Proof of Claim 1. Suppose to the contrary that player 1 strictly prefers to play $L$ in state $N$. Let

$$i^* := \max\{i : \text{player 1 weakly prefers to play } H \text{ in state } i\}$$

$i^*$ is well defined and $n^{(1)} \leq i^* < N$. In state $i > i^*$, player 1 strictly prefers to play $L$ and hence

$$U_i = (1 - \delta)u_L \cdot \sigma_{n^{(1)}} + \delta U_0$$

is constant. Furthermore, in state $i > i^*$,

$$\left(1 - \delta\right)u_L \cdot \sigma_{n^{(1)}} + \delta U_0 > \left(1 - \delta\right)u_H \cdot \sigma_{n^{(1)}} + \delta U_{\min\{i+1,N\}}$$

\text{By Assumption 2, if } \delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}} \text{ then } \delta > \frac{u_{Hh} - u_{Lh}}{u_{Hh} - u_{Ll}}. \text{ Therefore, if } \delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}} \text{ then } \delta > \frac{u_{Lh} - u_{Hh}}{u_{Hh} - u_{Ll}}(u_{Lh} - u_{Hh} - u_{Ll} + u_{Hl}) > 0. \text{ Therefore,}$$

\text{if } \delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}} \text{ then } \delta > \frac{u_{Lh} - u_{Hh}}{u_{Hh} - u_{Ll}}.
In state $i^*$,
\[
(1 - \delta)u_H \cdot \sigma_{n(1)}^i + \delta U_{i^*+1} \geq (1 - \delta)u_L \cdot \sigma_{n(1)}^i + \delta U_0
\]  
(13)

But by summing them up we will have
\[
U_{i^*+1} > U_{\min(i+1,N)},
\]
which contradicts (11).

Therefore, player 1 weakly prefers to play $H$ in state $N$. This completes the proof of Claim 1.

By Claim 1, player 1 weakly prefers $H$ in state $N$ and $U_N = u_H \cdot \sigma_{n(1)}$. Assume player 1 weakly prefers $H$ in state $i$ and $U_i = u_H \cdot \sigma_{n(1)}$ for $i = k + 1, ..., N$, where $k \geq n(1)$. In these states,
\[
\frac{u_H \cdot \sigma_{n(1)}}{U_i} \geq (1 - \delta)u_L \cdot \sigma_{n(1)} + \delta U_0
\]  
(14)

Consider the state $i = k$.

If $\pi_k(H) < 1$,
\[
\frac{u_H \cdot \sigma_{n(1)}}{U_k} \leq (1 - \delta)u_L \cdot \sigma_{n(1)} + \delta U_0
\]  
(15)

By comparing (15) and (14), we have that the equality holds in (15). Therefore, player 1 weakly prefers $H$ in state $k$ and $U_k = u_H \cdot \sigma_{n(1)}$.

The conclusion is immediate if $\pi_k(H) = 1$. The induction of Step A is complete.

**Step B:** We show that if $0 \leq i \leq n(1)$, then $\sigma_i = \sigma_{n(1)}$, player 1 weakly prefers to play $H$ in state $i$, and $U_i = u_H \cdot \sigma_{n(1)}$.

We prove by induction again. When $i = n(1)$, the conclusion follows from Step 1. Suppose the conclusion holds for $i = k+1, ..., n(1)$, $k < n(1)$. Consider $i = k$. The induction hypothesis leads to the following property.

**Claim 2:** $\sigma_k(h) \leq \sigma_{k+1}(h) = \sigma_{n(1)}(h).

**Proof of Claim 2.** To see this, suppose to the contrary that $\sigma_k(h) > \sigma_{n(1)}(h)$.

By playing $L$ or $H$ for one period in state $k$ and then following the equilibrium strategy, player 1 gets the expected payoff
\[
r_k(L) = (1 - \delta)u_L \cdot \sigma_k + \delta U_0
\]
or
\[
r_k(H) = (1 - \delta)u_H \cdot \sigma_k + \delta u_H \cdot \sigma_{n(1)}
\]
respectively. Thus

\[ r_k(L) - r_k(H) = (1 - \delta)(u_L - u_H) \cdot \sigma_k - \delta(u_H \cdot \sigma_{n(1)} - U_0) \]

Analogously, by playing \( L \) or \( H \) for one period in state \( k + 1 \) and then following the equilibrium strategy, player 1 gets

\[ r_{k+1}(L) = (1 - \delta)u_L \cdot \sigma_{n(1)} + \delta U_0 \]

or

\[ r_{k+1}(H) = (1 - \delta)u_H \cdot \sigma_{n(1)} + \delta u_H \cdot \sigma_{n(1)} \]

respectively. Thus

\[ r_{k+1}(L) - r_{k+1}(H) = (1 - \delta)(u_L - u_H) \cdot \sigma_{n(1)} - \delta(u_H \cdot \sigma_{n(1)} - U_0) \]

Let us consider

\[ D := [r_{k+1}(L) - r_{k+1}(H)] - [r_k(L) - r_k(H)] \tag{16} \]

Simple computation shows that

\[ D = (1 - \delta)(u_{Lh} - u_{Hh} - u_{LI} + u_{HI})(\sigma_{n(1)}(h) - \sigma_k(h)) \tag{17} \]

Since \( u_{Lh} - u_{Hh} > u_{LI} - u_{HI} \) and \( \sigma_{n(1)}(h) - \sigma_k(h) < 0 \) (by assumption), \( D < 0 \). Since player 1 weakly prefers to play \( H \) in state \( k + 1 \) by the induction hypothesis, \( r_{k+1}(L) \leq r_{k+1}(H) \). We have two cases to consider.

(i) \( r_{k+1}(L) = r_{k+1}(H) \).

Then \( r_k(L) > r_k(H) \) by (16), i.e. \( \pi_k(L) = 1 \). Thus \( \sigma_k^n(h) = 0 \) for \( n \geq k + 1 \). Therefore,

\[
\sigma_k(h) = \sum_{n \leq k} p(n)\sigma_k^n(h) + \sum_{n \geq k+1} p(n)0
\]

\[
= \sum_{n \leq k} p(n)\sigma_{k+1}^n(h) + \sum_{n \geq k+1} p(n)0
\]

\[
\leq \sum_{n \leq k} p(n)\sigma_{k+1}^n(h) + \sum_{n \geq k+1} p(n)\sigma_{k+1}^n(h)
\]

\[
= \sigma_{k+1}(h)
\]

The second equality follows because player 2 sees straight \( H \)'s after acquiring \( n \) bits of
information in states $k$ and $k+1$, $n \leq k$. But we have derived $\sigma_{n(i)}(h) = \sigma_{k+1}(h) \geq \sigma_k(h)$, which contradicts the assumption.

(ii) $r_{k+1}(L) < r_{k+1}(H)$.

By the induction hypothesis, $\sigma_m = \sigma_{n(i)} = \sigma_{k+1}$ if $m > k$ and hence $r_m = r_{k+1}$. Thus $r_m(L) < r_m(H)$ and $\pi_m(H) = 1$ if $m > k$. Therefore, player 2 will play $h$ for sure whenever he buys $k + 1$ or more bits of information and then sees straight $H$'s. But we then will have a contradiction:

$$\sigma_k(h) = \sum_{n \leq k} p(n)\sigma^n_k(h) + \sum_{n > k} p(n)\sigma^n_k(h)$$

$$\leq \sum_{n \leq k} p(n)\sigma^n_{k+1}(h) + \sum_{n > k} p(n) 1$$

$$= \sum_{n \leq k} p(n)\sigma^n_{k+1}(h) + \sum_{n > k} p(n)\sigma^n_{k+1}(h)$$

$$= \sigma_{k+1}(h)$$

We conclude that $\sigma_k(h) \leq \sigma_{k+1}(h) = \sigma_{n(i)}(h)$. The proof for Claim 2 is complete.

Now let us continue the main induction on $i$. We have two cases to consider when $i = k$.

(a) $\pi_k(H) = 1$.

Then $\sigma^n_k(h) = 1$ for $n > k$ and hence $\sigma_k(h) \geq \sigma_{n(i)}(h)$. It follows from Claim 2 above that $\sigma_k(h) = \sigma_{n(i)}(h)$. Player 1 prefers $H$ in state $k$ since $\pi_k(H) = 1$.

(b) $\pi_k(H) < 1$.

Then $r_k(L) - r_k(H) \geq 0$. If $\sigma_{n(i)}(h) > \sigma_k(h)$, then $D > 0$ by (17). Thus $r_{k+1}(L) - r_{k+1}(H) > 0$, which contradicts the induction hypothesis; player 1 weakly prefers $H$ in state $k + 1$. Therefore, $\sigma_{n(i)} = \sigma_k$ by Claim 2 above and hence $D = 0$ by (17). Hence, $r_k(L) - r_k(H) = r_{k+1}(L) - r_{k+1}(H) \leq 0$: player 1 weakly prefers $H$ at $k$.

To conclude, (a) and (b) imply that $\sigma_{n(i)} = \sigma_k$ and player 1 weakly prefers $H$ in $k$.

Hence

$$U_k = (1 - \delta)u_H \cdot \sigma_{n(i)} + \delta U_{k+1} = u_H \cdot \sigma_{n(i)}$$

The induction is complete. ■

Corollary 2 If $n^{(1)} \leq i \leq N$, then player 1 strictly prefers to play $L$ in state $i$ and hence $\pi_i(H) = 0$.

Proof. By Lemma 4, player 2’s expected behavioral strategies will be constant in all states in $\{0, 1, ..., N\}$ if player 1 weakly prefers to play $H$ in some state $i$, $n^{(1)} \leq i \leq N$. 

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This implies player 1 will strictly prefer to play the stage game dominant strategy $L$ always, a contradiction.

Corollary 2 implies that $\pi_N(H) = 0$. Therefore, cases (2)–(4) are impossible. It also implies that $\pi_{n(1)}(H) = 0$. Since $n^{(1)} < N^C \leq N$, case (5) is impossible. Thus, there exists $n \leq n^{(1)} < N$ such that $\pi_i(H) > 0$ for each $0 \leq i < n$ and $\pi_i(H) = 0$. We complete the proof of Proposition 3.

5.3.2 Step 2: Weak Monotonicity of Player 2’s Expected Strategies

Lemma 5 $\sigma_i(h)$ is non-decreasing in $i, i \in \{0, 1, ..., n\}$, where $n$ is defined in Step 1.

Proof. Step A: $\sigma_{n-1}(h) \leq \sigma_n(h)$ and $U_{n-1} \leq U_n$.

In state $n-1$,

$$\underbrace{(1 - \delta)u_H \cdot \sigma_{n-1} + \delta U_n}_U \geq (1 - \delta)u_L \cdot \sigma_{n-1} + \delta U_0$$

In state $n$,

$$\underbrace{(1 - \delta)u_L \cdot \sigma_n + \delta U_0}_U \geq (1 - \delta)u_H \cdot \sigma_n + \delta U_{n+1}$$

By summing up the two inequalities,

$$\delta(U_n - U_{n+1}) \geq (1 - \delta)(u_L - u_H) \cdot (\sigma_{n-1} - \sigma_n)$$

(18)

Notice that $U_{n+1} \geq (1 - \delta)u_L \cdot \sigma_{n+1} + \delta U_0$ and $U_n = (1 - \delta)u_L \cdot \sigma_n + \delta U_0$. From (18),

$$\delta(u_{Lh} - u_{Ll})(\sigma_n(h) - \sigma_{n+1}(h)) \geq (1 - \delta)(u_{Lh} - u_{Hh} - u_{Ll} + u_{Hl})(\sigma_{n-1}(h) - \sigma_n(h))$$

(19)

We claim that $\sigma_n(h) \leq \sigma_{n+1}(h)$. The argument is as follows. If $n < n^{(1)}$, then $\pi_n(L) = 1$ and hence $\sigma_n^k(h) = 0 \leq \sigma_{n+1}^k(h)$ when $k > n$. If $n = n^{(1)}$, then $\sigma_n(h) = \sigma_{n+1}(h)$. Therefore, $\sigma_{n-1}(h) \leq \sigma_n(h)$ from (19).

Since $\pi_n(H) = 0$, $\sigma_n^k(h) = 0 \leq \sigma_{n+m}^k(h)$ for any $k > n$ and $m > 0$. Therefore $\sigma_{n+m}(h) \geq \sigma_n(h) \geq \sigma_{n-1}(h)$ for any $m > 0$. In state $n$, player 1’s payoff is at least the payoff from playing $H$ always. That is,

$$U_n \geq (1 - \delta)(u_H \cdot \sigma_n + \delta u_H \cdot \sigma_{\min\{n+1,N\}} + \delta^2 u_H \cdot \sigma_{\min\{n+2,N\}} + \cdots)$$

$$\geq u_H \cdot \sigma_n$$

$$\geq u_H \cdot \sigma_{n-1}$$

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Therefore, \( U_n \geq (1 - \delta)u_H \cdot \sigma_{n-1} + \delta U_n = U_{n-1} \).

**Step B:** Suppose \( \sigma_i(h) \) is not weakly increasing in \( i \). Then the following is well defined: \( i^* = \max\{ i : \sigma_{i-1}(h) > \sigma_i(h) \} \). \( \sigma_{i^*}(h) > \sigma_i(h) \) by the definition of \( i^* \) and \( i^* < n \) by Step A. Since \( \pi_i(H) > 0 \) for each \( i \in \{0, 1, \ldots, n-1\} \),

\[
U_H \cdot \sigma_{i^*} - 1 + \delta u_H \cdot \sigma_{i^*} + \cdots + \delta^{n-i^*}u_H \cdot \sigma_{n-2} + \delta^{n-i^*}U_{n-1} - \delta - 1 \geq U_L \cdot \sigma_{i-1} + \delta U_0. \tag{20}
\]

**Claim 3:** \( 0 < \pi_{i^*}(H) \leq \bar{\mu} \).

**Proof of Claim 3.** \( \pi_{i^*}(H) > 0 \) by Proposition 3. We suppose to the contrary that \( \pi_{i^*}(H) > \bar{\mu} \). Therefore, \( \sigma^{k^*}(h) = 1 \) for each \( k > i^* \). By the definition of \( i^* \), \( \sigma_i(h) \geq \sigma_{i^*}(h) \) for each \( i > i^* \). On the other hand, for \( i > i^* \),

\[
\sigma_i(h) = \sum_{k \leq i^*} p(k)\sigma_i^k(h) + \sum_{k > i^*} p(k)\sigma_i^k(h) = \sum_{k \leq i^*} p(k)\sigma_i^k(h) + \sum_{k > i^*} p(k) \leq \sum_{k \leq i^*} p(k)\sigma_i^k(h) + \sum_{k > i^*} p(k)1 = \sum_{k \leq i} p(k)\sigma_i^k(h) + \sum_{k > i^*} p(k)\sigma_{i^*}(h) = \sigma_{i^*}(h)
\]

The second equality follows from the fact that \( \sigma_i^k(h) = \sigma_{i^*}^k(h) \) when \( i > i^* \geq k \). Therefore, if \( \pi_{i^*}(H) > \mu \), then

\[
\sigma_i(h) = \sigma_{i^*}(h) \text{ for each } i > i^* \tag{21}
\]

In particular, \( \sigma_n(h) = \sigma_{i^*}(h) \). This is impossible when \( n < n^{(1)} \), since \( \sigma_{i^*}^k(h) = 1 \) for \( k > i^* \) but \( \sigma_n^{n^{(1)}}(h) = 0 \). Let us consider the case \( n = n^{(1)} \).

In state \( n - 1 \),

\[
(1 - \delta)u_H \cdot \sigma_{n-1} + \delta U_n \geq (1 - \delta)u_L \cdot \sigma_{n-1} + \delta U_0
\]

In state \( n = n^{(1)} \), by Corollary 2,

\[
(1 - \delta)u_L \cdot \sigma_n + \delta U_n > (1 - \delta)u_H \cdot \sigma_n + \delta U_{n+1}
\]
Therefore,

\[ 0 = \delta(U_n - U_{n+1}) > (1 - \delta)(u_L - u_H) \cdot (\sigma_{n-1} - \sigma_n) \] (22)

This will lead to \( \sigma_n(h) > \sigma_{n-1}(h) \), contradicting (21).

Therefore, \( 0 < \pi_i(H) \leq \bar{\mu} \). The proof of Claim 3 is complete.

By Claim 3, we have

\[
u_H \cdot \sigma_i + \delta u_H \cdot \sigma_{i+1} + \cdots + \delta^{n-i-1} u_H \cdot \sigma_{n-1} + \delta^{n-i} \frac{U_n}{1 - \delta} = u_L \cdot \sigma_i + \delta \frac{U_0}{1 - \delta}. \] (23)

From (20)–(23), we have

\[
(u_L - u_H) \cdot (\sigma_{i-1} - \sigma_i) \leq \sum_{i=i^*}^{n-2} \delta^{i-i^*+1} u_H \cdot (\sigma_i - \sigma_{i+1}) + \delta^{n-i} \frac{U_{n-1} - U_n}{1 - \delta}. \] (24)

For \( i \geq i^* \), \( u_H \cdot (\sigma_i - \sigma_{i+1}) = (u_{Hh} - u_{HI}) (\sigma_i(h) - \sigma_{i+1}(h)) \leq 0 \) since \( u_{Hh} > u_{HI} \) and \( \sigma_i(h) \leq \sigma_{i+1}(h) \). \( U_{n-1} \leq U_n \) by Step A. Thus the RHS of (24) is non-positive. Therefore,

\[
(u_L - u_H) \cdot (\sigma_{i-1} - \sigma_i) = (\sigma_{i-1}(h) - \sigma_{i}(h))(u_{Lh} - u_{Hh} - u_{LL} + u_{HI}) \leq 0.
\]

It follows from \( u_{Lh} - u_{Hh} > u_{LL} - u_{HI} \) that \( \sigma_{i-1}(h) \leq \sigma_{i}(h) \), contradicting the definition of \( i^* \). Therefore, \( \sigma_i(h) \) is non-decreasing in \( i \) when \( i \in \{0, 1, \ldots, n\} \). This completes the proof of monotonicity.

5.3.3 Step 3: Reputation Renewal Equilibria

**Lemma 6** If \( i \in \{0, 1, \ldots, n-1\} \), then \( 0 < \pi_i(H) \leq \bar{\mu} \).

**Proof.** Suppose to the contrary that \( \pi_i(H) > \bar{\mu} \) for some \( i < n \). Let \( i^* := \min\{i : \pi_i(H) > \bar{\mu}\} \). \( i^* < n \) by definition. Therefore, \( \sigma_{i^*}(h) = 1 \) for \( m > i^* \). It follows from Lemma 5 that \( \sigma_{i^*} = \sigma_{i^*+1} = \cdots = \sigma_n \). The argument from (21) to (22) then leads to the required contradiction. ■

It follows immediately from Lemma 6 that \( n^* = n \), where \( n \) is determined in Proposition 3.

**Corollary 3** An equilibrium is renewal if and only if there exists an \( n^* > 0 \) such that \( \pi_i(H) > 0, 0 \leq i \leq n^* - 1, \) and \( \pi_{n^*}(L) = 1 \).
5.3.4 Step 4: Strict Monotonicity

**Lemma 7** \( \sigma_i(h) \) and \( U_i \) are both strictly increasing in \( i, i \in \{0, 1, ..., n^*\} \).

**Proof.** We have shown that \( \sigma_i(h) \) is weakly increasing on \( \{0, 1, ..., n^*\} \). Suppose there is a \( j \in \{0, 1, ..., n^* - 1\} \) such that \( \sigma_j(h) = \sigma_{j+1}(h) \). By Lemma 6,

\[
U_j = (1 - \delta)u_L \cdot \sigma_j + \delta U_0 = U_{j+1}
\]

We claim that \( U_j > u_H \cdot \sigma_j \). To see this, notice that (1) \( \sigma_j(h) \leq \sigma_{j+k}(h) \) for each \( k \) (if \( j + k \leq n^* \), the result follows from monotonicity; otherwise, \( \sigma_j(h) \leq \sigma_{n^*}(h) \leq \sigma_{n^*+k} \) for each \( k \)), (2) in state \( j \), player 1 can achieve at least the payoff by playing \( H \) forever, and (3) in a state greater than or equal to \( n^{(1)} \), player 1 strictly prefers to play \( L \). Thus,

\[
U_j > (1 - \delta)(u_H \cdot \sigma_j + \delta u_H \cdot \sigma_{\min(j+1,N)} + \delta^2 u_H \cdot \sigma_{\min(j+2,N)} + \cdots)
\]

\[
\geq u_H \cdot \sigma_j.
\]

Thus, \( U_j > (1 - \delta)u_H \cdot \sigma_j + \delta U_{j+1} \). This contradicts the fact that player 1 is indifferent in state \( j \). Therefore, \( \sigma_i(h) \) is strictly increasing. It follows immediately that \( U_j = (1 - \delta)u_L \cdot \sigma_j + \delta U_0 \) is also strictly increasing. \( \blacksquare \)

The proof of Theorem 3 is complete.

6 Imperfect Monitoring

So far we have assumed that player 1’s actions are recorded in the public history precisely, i.e., perfect monitoring. This section relaxes this assumption by assuming mistakes happen with probability \( \varepsilon \) in each period. That is, when \( H \) is taken in some period, \( L \) happens to be the realized signal with probability \( \varepsilon \) in that period, and vice versa. To avoid cumbersome notation, we assume that players’ payoff functions depend on the actions actually taken instead of signals.

Let \( S = \{H, L\}^N \) be all possible realized signals in the last \( N \) periods with \( N \geq N_C \). We consider strategies that are functions on \( S \). From player 2’s point of view, nature chooses one of the two types of player 1 and hence one of the two stochastic processes on \( S \):

(1) The commitment type plays \( H \) constantly and each period \( H \) is the realized signal with probability \( 1 - \varepsilon \). Therefore, the induced process on \( S \) is a process with a state-independent transition. Let \( \phi^\varepsilon \) be its invariant distribution. If \( s \) contains \( n \) \( H \)’s, then \( \phi^\varepsilon(s) = (1 - \varepsilon)^n \varepsilon^{N-n} \). Note that as \( \varepsilon \to 0 \), \( \phi^\varepsilon(s) \to \phi(s) \) where \( \phi(H^N) = 1 \).
(2) The normal type’s strategy $\pi^\varepsilon$ and the noise induce a Markov process on $S$. For example, the transition probability from $H^N$ to $H^{N-1}L$ is $\pi^\varepsilon_{HN}(L)(1-\varepsilon) + \pi^\varepsilon_{HN}(H)\varepsilon$. Let $\lambda^\varepsilon \in \Delta(S)$ be an invariant distribution of the Markov process. If $\pi^\varepsilon_{HN}$ converges to $\pi_{HN}$ as $\varepsilon$ approaches 0, then $\lambda^\varepsilon$ converges pointwise to an invariant distribution induced by $\pi_{HN}$.

The belief-updating rule, optimality of strategies, and the equilibrium notion can be defined analogously as before. Note that since $\lambda^\varepsilon(s) > 0$ for each state $s \in S$, the belief system $\{\lambda^\varepsilon(|P_n)\}_{0 \leq n \leq N}$ is consistent. An equilibrium $(S, \pi^\varepsilon, \sigma^\varepsilon, p^\varepsilon, \lambda^\varepsilon)$ can be defined in the same way as the perfect monitoring case.

As a benchmark, we first consider $\mu_0 = 0$.

**Lemma 8** Suppose $\mu_0 = 0$, i.e., there is no uncertainty about types. Then, the infinite repetition of $(L,l)$ is the unique equilibrium. Furthermore, player 2 will not buy any information if $C(1) > 0$.

With imperfect monitoring when $\mu_0 > 0$, we do not have the neat equilibrium representation in terms of the clean history index. For example, consider two histories $\cdots HLH$ and $\cdots LLH$ with the same index 1. After observing 3 bits of information, player 2 will assign a higher probability to the commitment type conditional on the former history as it contains two $H$’s. As a result, an equilibrium will not have the stationary structure as that in Theorem 3. However, the equilibrium play never converges to the equilibrium of the case $\mu_0 = 0$ for small $\varepsilon$. Reputation is built and depleted by mixed strategies.

**Lemma 9** Suppose $C(1) < \mu_0(v_{HH} - v_{HL})$ and $\delta > \frac{u_{HL} - u_{HH}}{u_{HL} - u_{LL}}$. Then, there exist $\bar{\varepsilon} > 0$, such that if $\varepsilon < \bar{\varepsilon}$, reputation is repeatedly built and depleted by mixed strategies in all equilibria.

We only need to show that player 1 cannot always play $H$ or always play $L$. The argument is tricky. For example, we cannot conclude from player 1’s always playing $H$ in an absorbing state that player 2 needs not buy information. In fact, it is feasible that $H$ and $L$ are played in two different absorbing states. In this case, player 2 has an incentive to buy information to check which absorbing state is the true state. The proof is based on Theorem 3, Corollary 2 and the upper semi-continuity of best-response correspondences.

### 7 Construction of Equilibrium

The results established in previous sections enable us to construct equilibria for a given cost structure. We first describe the general procedure of equilibrium construction and then apply it in two special cases.
7.1 General Procedure

By Theorem 3, essentially any equilibrium takes the form of that in Figure 6.

\textbf{Figure 6:} The unique equilibrium pattern: $0 < \pi_i(H) \leq \bar{\mu}$, $0 \leq i \leq n^* - 1$, $\sigma_i$ is strictly increasing, $0 \leq i \leq n^*$

We need to determine $\pi, \sigma, p$ and $n^*$.

\textbf{Step 1:} Let us first look at the “expected strategies” of player 2, $\sigma_i$, $i = 0, 1, ..., n^*$. Since $0 < \pi_i(H) \leq \bar{\mu}$, $0 \leq i \leq n^* - 1$, we can derive the indifference conditions of player 1 to determine $\sigma_i$. In state $i \in \{0, 1, ..., n^* - 1\}$,

$$(1 - \delta) u_L \cdot \sigma_i + \delta U_0 = (1 - \delta) u_H \cdot \sigma_i + \delta U_{i+1},$$

where

$$U_0 = u_L \cdot \sigma_0$$

$$U_i = (1 - \delta) u_L \cdot \sigma_i + \delta u_L \cdot \sigma_0$$

Therefore, for $i \in \{0, 1, ..., n^* - 1\}$,

$$(1 - \delta) u_L \cdot \sigma_i + \delta u_L \cdot \sigma_0 = (1 - \delta) u_H \cdot \sigma_i + \delta (1 - \delta) u_L \cdot \sigma_{i+1} + \delta^2 u_L \cdot \sigma_0,$$

which can be simplified further as

$$(u_L - u_H) \cdot \sigma_i + \delta u_L \cdot \sigma_0 - \delta u_L \cdot \sigma_{i+1} = 0 \quad (25)$$

From (25), we have $n^*$ equations with $n^* + 1$ unknowns ($\sigma_0, \sigma_1, ..., \sigma_{n^*}$). Note that since $\sigma_i(h)$ is strictly increasing by Theorem 3, we have three cases to consider:

1) The initial condition $\sigma_0(h) = 0$ and (25) give a unique solution.
2) The initial condition $\sigma_{n^*}(h) = 1$ and (25) give a unique solution.
3) $0 < \sigma_0(h) < \sigma_{n^*}(h) < 1$ and (25) give a continuum of solutions.

We record a small lemma for later reference.
Lemma 10 It is impossible that both $\sigma_0(h) = 0$ and $\sigma_{n^*}(h) = 1$ if $\delta > \frac{u_{LH} - u_{HH}}{u_{Lh} - u_{HL}}$.

Proof. If $\sigma_0(h) = 0$ and $\sigma_{n^*}(h) = 1$, then consider $i = n^* - 1$ in (25), where we have

$$(u_L - u_H) \cdot \sigma_i + \delta u_{LI} - \delta u_{Lh} = 0$$

Solving this equation, we have $\sigma_i(h) = \frac{\delta u_{Lh} - \delta u_{LH} - u_{LI} + u_{LH}}{u_{Lh} - u_{HL} - u_{LI} + u_{HL}}$. We will have $\sigma_i(h) > 1$ if $\delta > \frac{u_{Lh} - u_{HH}}{u_{Lh} - u_{HL}}$.

Step 2: Once we have $\sigma_i, i = 0, 1, ..., n^*$, we can use equations $\sigma_i = \sum p(n)\sigma_i^n$, $i = 0, 1, ..., n^*$, to determine $p$. Note that by Theorem 4, $\sigma_i^n$ is a pure strategy when $n > 0$: $\sigma_i^n(h) = 1$ if $i \geq n$ and $\sigma_i^n(h) = 0$ otherwise. Player 2's play when he does not buy information will be determined by the initial condition above. We call the procedure of deriving $\sigma_i^n$ and $p$ from $\sigma_i$ “decomposition,” which we will analyze in detail. Figure 7 depicts player 2’s expected strategies with the initial condition $\sigma_{n^*}(h) = 1$.

![Figure 7: Player 2's expected strategies with the initial condition $\sigma_{n^*}(h) = 1$](image)

We decompose player 2’s expected strategies in Figure 7 into simple strategies as in Figure 8.

![Figure 8: Player 2's simple strategies when he plays $h$ if he does not acquire information.](image)
As depicted in Figure 8, player 2 first chooses how much information to acquire, \( p(n) > 0 \) for each \( n = 0, 1, ..., n^* \). Player 2 plays \( h \) if he does not buy information \( (p(0) > 0) \) or if he buys information but sees straight \( H \)’s; he plays \( l \) if he observes an \( L \).

For example, if he acquires 1 bit of information, player 1 can separate state 0 from other higher states. His information partition is represented by circles in (green) dashed line in Figure 8. \( p(n), n = 0, 1, ..., n^* \), is chosen so that \( \sum p(n)\sigma_i^n(h) \) in each state \( i \) equals the expected strategy \( \sigma_i \) in Figure 7. For instance, in state 0 (depicted by the rectangle in (red) dashed line in Figure 8), we need \( \sum p(n)\sigma_0^n(h) = \sigma_0(h) \). Therefore, \( p(0) = \sigma_0(h) \).

When the initial condition is \( \sigma_0(h) = 0 \), we have analogous decomposition as in Figure 9: player 2 plays \( l \) if he does not acquire information.

![Figure 9: Player 2’s simple strategies when he plays \( l \) if he does not acquire information.](image)

The decomposition of the expected strategies into simple strategies is feasible because of the monotonicity property in Theorem 3. Furthermore, the decomposition is uniquely determined if \( \sigma_0(h) = 0 \) or \( \sigma_{n^*}(h) = 1 \) as shown in the next result. \( \sigma_0 \) being mixed strategies is a non-generic case where there could be many ways of decomposition, depending on how player 2 plays if he does not acquire information.

**Proposition 4** If \( (\sigma_0, \sigma_1, ..., \sigma_{n^*}) \) is player 2’s expected strategy profile in an equilibrium, where either \( \sigma_0(h) = 0 \) or \( \sigma_{n^*}(h) = 1 \), then the information acquisition rule \( p \) associated with the “simple strategies” is given by the following: \( p(n) = \sigma_n(h) - \sigma_{n-1}(h) \) for \( 1 \leq n \leq n^* \) and \( p(0) = 1 + \sigma_0(h) - \sigma_{n^*}(h) \).

**Proof.** If \( \sigma_{n^*}(h) = 1 \), we need \( \sigma_n(h) = \sum_{k=0}^{n} p(k), 0 \leq n \leq n^* \). If \( \sigma_0(h) = 0 \), we need \( \sigma_n(h) = \sum_{k=1}^{n} p(k), 1 \leq n \leq n^* \). Simple computation yields the required result. ■

**Step 3:** By Theorem 4, \( p(n) > 0, n = 0, 1, ..., n^* \). That is, player 2 randomizes how much information to buy. We can derive \( \pi_i, i = 0, 1, ..., n^* - 1 \), from player 2’s indifference conditions. We do not specify the equations here to avoid the cumbersome notation. Detailed analysis is provided for the case of linear cost functions in the Appendix.
Step 4: The value of $n^*$ is determined by two incentive compatibility conditions of player 2: he does not strictly prefer to buy more information, neither does he want to buy less information.

The four steps provide a procedure to construct equilibria for general cost structures. We apply this procedure to two specific cost structures below.

7.2 Benchmark

Let us consider the case where $C(i) = 0$ if $i \leq n$ and $C(n + 1) > \min\{v_H, v_L\}$. That is, player 2 can obtain $n$ bits of information for free and the cost of more information is prohibitively high. We consider equilibria where player 2 buys exactly $n$ bits of information. Under this condition, the game here is strategically equivalent to that in Liu and Skrzypacz (2006). This is the simplest case: $n^* = n$ and $p(n) = 1$. We only need to specify $\pi_i$ and $\sigma_i$, $i = 0, 1, \ldots, n$. The generically unique equilibrium (in terms of the expected strategies) is characterized by the following result:

**Proposition 5** The equilibria for $\delta > \frac{u_L - u_H}{u_H - u_L}$ are as follows.

1. $n > \frac{\log \mu_0}{\log \bar{\mu}} - 1$. The equilibrium is depicted in Figure 10. $\sigma_i$ is the solution to (25) with initial condition $\sigma_n(h) = 1$.

2. $n < \frac{\log \mu_0}{\log \bar{\mu}} - 1$. The equilibrium is depicted in Figure 11. $\sigma_i$ is the solution to (25) with initial condition $\sigma_0(h) = 0$. $\pi_0(H) = \frac{\mu_0(1 - \bar{\mu})}{\bar{\mu}^n - \mu_0}$.

3. (the non-generic case) If $\frac{\log \mu_0}{\log \bar{\mu}} - 1$ is an integer and $n = \frac{\log \mu_0}{\log \bar{\mu}} - 1$, then there is a continuum of equilibria between (1) and (2).

![Figure 10: n > \frac{\log \mu_0}{\log \bar{\mu}} - 1; \bar{\pi} = (\bar{\mu}, 1 - \bar{\mu}), 0 < \sigma_i(h) < 1, i \in \{0, 1, \ldots, n - 1\}.](image-url)

The proof is given in the Appendix. Here are some comments. First, player 2 is willing to trust player 1 to some extent even when he knows player 1 is the normal type, though such “collusion” exploits future short-lived players. Second, when $n < \frac{\log \mu_0}{\log \bar{\mu}} - 1$, player 1’s payoff is close to the static Nash equilibrium payoff $u_{LI}$ (player 1 is indifferent in state 0 and player 2 plays $l$ there. The payoffs in all stages converge to the same value as
Therefore, the $n$ that maximizes player 1’s payoff is $\frac{\log \mu_0}{\log \mu} - 1$. Third, there are many other equilibria in which player 2 acquires information randomly: they may buy only $n - 1$ bits of information even though the additional information is free, because it is player 2’s “expected strategy” that disciplines player 1. Additional equilibria can be obtained through “decomposition” procedures. In particular, the “simple strategy” equilibrium as in Figures 8 and 9 is an equilibrium robust to the perturbation of cost functions.

### 7.3 Linear Cost

We consider a linear cost function $C(n) = cn$, $0 < c < \mu_0(v_{Hh} - v_{Hl})$. By following the general procedure, we construct the equilibria for all generic $c$. The proof and the details of the construction are given in the Appendix.

**Proposition 6** If $C(n) = cn$, the equilibrium is unique for generic $c \in (0, \mu_0(v_{Hh} - v_{Hl}))$ when $\delta > \frac{\mu_0(v_{Hh} - v_{Hl})}{v_{Lh} - v_{Ll}}$. Furthermore, there exist a strictly decreasing sequence $\{c_n\}_{n=1}^\infty$, and a cutoff $c^* > 0$, within $(0, \mu_0(v_{Hh} - v_{Hl}))$, such that

1. If $c \in (c_n, c_{n+1})$ and $c < c^*$, then $n^* = n$ and player 2 takes pure strategy $h$ if he decides not to acquire information.

2. If $c \in (c_n, c_{n+1})$ and $c > c^*$, then $n^* = n$ and player 2 takes pure strategy $l$ if he decides not to acquire information.

This proposition implies that $n^*(c)$ is a decreasing step function as in Figure 12.

### 8 Conclusion

In this paper, we study the implications of costly acquisition of information regarding the past play in a reputation model. Under general assumptions on information cost structures, we show that all equilibria display renewal of reputation: the long-lived player
Figure 12: The relationship between \( n^* \) and \( c \).

builds her reputation up only to exploit it; then builds it up again, and so on. We also find that the short-lived players play simple strategies in our model. These equilibrium patterns lead to an easy method for equilibrium construction. Further results regarding the payoff bounds and the relaxation of Assumption 2 can be found at the author’s webpage: www.stanford.edu/~qiliu.

.1 Proof of Theorem 1

Definition 5 We say \( s \) and \( s' \) are weakly \( m \)-similar, denoted by \( s \simeq^m s' \), if \( s_n = s'_n \) for all \( n \in \{1, \ldots, m\} \).

The following lemma is immediate from the definition.

Lemma 11 (1) \( \simeq^m \) and \( \simeq \) are equivalence relations. (2) \( s \simeq^0 s' \) for any \( s, s' \in S \). (3) \( s \simeq^m s' \) implies \( s \simeq^n s' \).

Lemma 12 If \( \mu_0 = 0 \), then \( \sigma_s \) is constant in any equilibrium.

Proof. The proof basically replicates the proof of Lemma 2 in the main text. By Lemma 11, we only need to show that \( \sigma_s = \sigma_{s'} \) whenever \( s \simeq^0 s' \).

\[ \sigma_s = \sigma_{s'} \quad \text{and} \quad U_s = U_{s'} \quad \text{if} \quad s \simeq^N s' . \]

Assume \( U_s = U_{s'} \) and \( \sigma_s = \sigma_{s'} \) if \( s \simeq^n s' \), where \( n = N, N - 1, \ldots, m + 1 \), and \( m \leq N - 1 \).

Suppose \( s \simeq^m s' \), but \( \sigma_s \neq \sigma_{s'} \). Without loss of generality, let us assume \( 1 \geq \sigma_s(h) > \sigma_{s'}(h) \geq 0 \). Therefore, \( \sum_{n \geq m+1} p(n)\sigma^n(s)(h) > \sum_{n \geq m+1} p(n)\sigma^n(s')(h) \). There exists \( \bar{n} \geq m + 1 \) such that \( p(\bar{n}) > 0 \) and \( 1 \geq \sigma^{\bar{n}}(s)(h) > \sigma^{\bar{n}}(s')(h) \geq 0 \).

Since \( \sigma^{\bar{n}}(s)(h) > 0 \), there exists \( t \in S \), such that \( t \simeq^{\bar{n}} s \) and \( \pi_1(H) > 0 \). Analogously, \( \sigma^{\bar{n}}(s') < 1 \) implies that there exists \( t' \in S \), such that \( t' \simeq^{\bar{n}} s' \), \( \pi_{t'}(L) > 0 \), \( \sigma_{s'} = \sigma_{t'} \).

By part (3) of Lemma 11, \( t \simeq^m s \simeq^m s' \simeq^m t' \). By transitivity (part (1) of Lemma 11),
$t \simeq^m t'$ and hence $t \land L \simeq^{m+1} t' \land L$ and $t \land H \simeq^{m+1} t' \land H$. Therefore, $U_{t \land L} = U_{t \land L}$ and $U_{t \land H} = U_{t \land H}$ by the induction hypothesis.

However, we will show that the properties derived above and summarized as follows are incompatible:

\begin{align*}
\sigma_s(h) &> \sigma_{s'}(h) \\
\sigma_s & = \sigma_t \text{ and } \sigma_{s'} = \sigma_{t'} \\
\pi_t(H) & > 0 \text{ and } \pi_{t'}(L) > 0 \\
U_{t \land L} & = U_{t \land L} \text{ and } U_{t \land H} = U_{t \land H}
\end{align*}

(26) (27) (28) (29)

Note that

$$r_t(L) = (1 - \delta)u_L \cdot \sigma_t + \delta U_{t \land L}$$
$$r_t(H), r_{t'}(L) \text{ and } r_{t'}(H) \text{ are defined analogously.}$$

Since $\pi_t(H) > 0$ (by (28)), $r_t(H) \geq r_t(L)$. Therefore,

$$\delta(U_{t \land H} - U_{t \land L}) \geq (1 - \delta)(u_L - u_H) \cdot \sigma_s$$

(30)

Since $\pi_{t'}(L) > 0$, $r_{t'}(L) \geq r_{t'}(H)$. Therefore,

$$(1 - \delta)(u_L - u_H) \cdot \sigma_{s'} \geq \delta(U_{t \land H} - U_{t \land L})$$

(31)

Since $U_{t \land H} - U_{t \land L} = U_{t \land H} - U_{t \land L}$ (by (29)), we have from (30) and (31) that

$$(u_L - u_H) \cdot (\sigma_s - \sigma_{s'}) \leq 0 \iff ((u_{Lh} - u_{Hh}) - (u_{Ll} - u_{Hl}))(\sigma_s(h) - \sigma_{s'}(h)) \leq 0$$

But this is impossible because $(u_{Lh} - u_{Hh}) > (u_{Ll} - u_{Hl})$ and $\sigma_s(h) > \sigma_{s'}(h)$ (by (26)).

Therefore, $s \simeq^m s'$ implies $\sigma_s = \sigma_{s'}$. The induction shows that $\sigma_s = \sigma_{s'}$ whenever $s \simeq^0 s'$. ■

It follows from Lemma 12 that player 1 will always play $L$ and hence player 2 will respond with $l$. The proof of Theorem 1 is complete.

\section*{2 Proof of Theorem 2}

The following property of consistent belief systems is intuitive.
Lemma 13 Suppose $0 \leq n, n + m \leq N$, $m > 0$, and $t, s, s' \in S$. The following holds:

1. if $\lambda(s|\mathcal{P}_n(t)) > 0$, then $\lambda(s|\mathcal{P}_{n+m}(s)) > 0$; 
2. if in addition $\lambda(s'|\mathcal{P}_{n+m}(s)) > 0$, then $\lambda(s'|\mathcal{P}_n(t)) > 0$.

Suppose player 2 assigns a positive probability to state $s$ when he observes $(s_n, s_{n-1}, ..., s_1)$. Then part (1) of Lemma 13 implies that he will also assign a positive probability to $s$ when he observes $(s_{n+m}, ..., s_n, ..., s_1)$. Intuitively, the extra information $(s_{n+m}, ..., s_{n+1})$ should reinforce his belief that $s$ is the true state. Part (2) of Lemma 13 implies that if player 2 assigns probability 0 to $s'$ (but assigns a positive probability to $s$) upon observing $(s_n, s_{n-1}, ..., s_1)$, then he will continue to assign probability 0 to $s'$ after observing the extra information $(s_{n+m}, ..., s_{n+1})$.

Proof. (1) Since $\lambda(s|\mathcal{P}_n(t)) > 0$ and $\lambda(\cdot|\mathcal{P}_n(t))$ is confined on $\mathcal{P}_n(t)$, $s \in \mathcal{P}_n(t)$. Therefore, $\mathcal{P}_n(s) = \mathcal{P}_n(t)$ and $\lambda(s|\mathcal{P}_n(s)) > 0$. Since $\mathcal{P}_n$ is an increasing sequence of fields, $s \in \mathcal{P}_{n+1}(s) \subset \mathcal{P}_n(s)$. Since $\lambda(\cdot|\mathcal{P}_{n+1}(s))$ is a version of conditional probability of $\lambda(\cdot|\mathcal{P}_n(s))$,

$$\lambda(s|\mathcal{P}_{n+1}(s)) = \frac{\lambda(s|\mathcal{P}_n(s))}{\lambda(\mathcal{P}_{n+1}(s)|\mathcal{P}_n(s))} > 0.$$ 

Therefore, $\lambda(s|\mathcal{P}_{n+1}(s)) > 0$. Inductively, we have $\lambda(s|\mathcal{P}_{n+m}(s)) > 0$.

(2) Since $\lambda(s'|\mathcal{P}_{n+m}(s)) > 0$, $s' \in \mathcal{P}_{n+m}(s)$, and hence $s' \in \mathcal{P}_n(t) = \mathcal{P}_n(s)$. Suppose to the contrary $\lambda(s'|\mathcal{P}_n(t)) = \lambda(s'|\mathcal{P}_n(s)) = 0$; then

$$\lambda(s'|\mathcal{P}_{n+1}(s)) = \frac{\lambda(s'|\mathcal{P}_n(s))}{\lambda(\mathcal{P}_{n+1}(s)|\mathcal{P}_n(s))} = 0.$$ 

Inductively, we will have $\lambda(s'|\mathcal{P}_{n+m}(s)) = 0$. This is a contradiction.

By Lemma 2, the short-lived player’s “expected strategies” are constant on the class of similar states. The following lemma examines $\sigma^n$, a short-lived player’s behavioral strategy taken after acquiring $n$ bits of information. It roughly says that over the class of similar states, $\sigma^n$ is constant except possibly in states in which the long-lived player takes a strategy that makes him indifferent. Therefore, a short-lived player is indifferent even though he may take different strategies in similar states in an equilibrium. Since player 1 only cares about player 2’s “expected strategies,” this flexibility can be seen as resulting from player 2’s privately observed randomizing device.

Lemma 14 If $s \prec^k s'$, $k \geq \hat{n}$, and $\lambda(s|\mathcal{P}_n(t)) > 0$ for some $t \in S$, and $\hat{n} \geq 0$, then at least one of the following holds: (1) $\pi(s)(H) = \bar{\mu}$; (2) $\sigma^n(s) = \sigma^n(s') \forall n$ such that $p(n) > 0$.

Proof. If $s \prec^N s'$, then $\sigma^n(s) = \sigma^n(s')$. Suppose we have proved that for $k = N, ..., m+1 \geq \hat{n}$, $\lambda(s|\mathcal{P}_n(t)) > 0$ and $s \prec^k s'$ imply either (1) $\pi(s)(H) = \bar{\mu}$ or (2) $\sigma^n(s) = \sigma^n(s')$, $\forall n$, $p(n) > 0$. 

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Consider \( s \prec^m s' \), \( m \geq \hat{n} \). Let us suppose \( \pi(s)(H) \neq \bar{\mu} \). Let 
\[
\tilde{n} = \max\{n : \sigma^n(s)(h) > \sigma^n(s')(h), p(n) > 0\}
\]
\( \tilde{n} \geq m + 1 \) by definition. Since \( \sum_{n \geq m+1} p(n)\sigma^n(s) = \sum_{n \geq m+1} p(n)\sigma^n(s') \) by Lemma 2, there exists \( \tilde{n}, \hat{n} \geq m + 1 \), such that \( 0 \leq \sigma^n(s'(h)) < \sigma^n(s'(h)) \leq 1 \). By definition, \( \hat{n}, \tilde{n} \geq \hat{n} \).

\( \sigma^n(s)(h) > 0 \) and it is a best response to \( \sum_{r \prec \hat{n}} \lambda(r|\mathcal{P}_{\hat{n}}(s))\pi(r) \). Therefore,
\[
\sum_{r \prec \hat{n}} \lambda(r|\mathcal{P}_{\hat{n}}(s))\pi(r)(H) \geq \bar{\mu}.
\]

\( \sigma^n(s)(h) < 1 \) and it is a best response to \( \sum_{r \sim \tilde{n}} \lambda(r|\mathcal{P}_{\tilde{n}}(s))\pi(r) \). Therefore,
\[
\sum_{r \sim \tilde{n}} \lambda(r|\mathcal{P}_{\tilde{n}}(s))\pi(r)(H) \leq \bar{\mu}.
\]

Since \( \lambda(s|\mathcal{P}_{\hat{n}}(t)) > 0 \) and \( \tilde{n}, \hat{n} \geq \hat{n} \), it follows from Lemma 13 that \( \lambda(s|\mathcal{P}_{\hat{n}}(s)) > 0 \) and \( \lambda(s|\mathcal{P}_{\tilde{n}}(s)) > 0 \).

Define 
\[
r^* = \arg \max_r \{\pi(r)(H) : r \prec \hat{n} \text{ and } \lambda(r|\mathcal{P}_{\hat{n}}(s)) > 0\}
\]

Since \( \sum_{r \prec \hat{n}} \lambda(r|\mathcal{P}_{\hat{n}}(s))\pi(r)(H) \geq \bar{\mu}, \pi(r^*)(H) \geq \bar{\mu} \). By definition, \( \lambda(r^*|\mathcal{P}_{\hat{n}}(s)) > 0 \) and \( r^* \prec \hat{n} \), which together with \( \lambda(s|\mathcal{P}_{\hat{n}}(t)) > 0 \) imply \( \lambda(r^*|\mathcal{P}_{\hat{n}})(s)) > 0 \) (by Lemma 13).

Since \( \pi(r^*)(H) \geq \bar{\mu} \), we have two cases to consider.

Case I: \( \pi(r^*)(H) = \bar{\mu} \). Then \( \pi(r)(H) = \bar{\mu} \) for all \( r \prec \hat{n} \) with \( \lambda(r|\mathcal{P}_{\hat{n}}(s)) > 0 \) since \( \sum_{r \prec \hat{n}} \lambda(r|\mathcal{P}_{\hat{n}}(s))\pi(r)(H) \geq \bar{\mu} \). In particular, \( \pi(s)(H) = \bar{\mu} \), contradicting the hypothesis.

Case II: \( \pi(r^*)(H) > \bar{\mu} \). Then by the induction hypothesis (recall that \( \lambda(r^*|\mathcal{P}_{\hat{n}}(s)) > 0 \), \( r^* \prec \hat{n} \) and \( \hat{n} > m + 1 \), \( \sigma^n(s) = \sigma^n(r^*) \) if \( n > \hat{n} \) and \( p(n) > 0 \). In particular, \( \sigma^N(s)(h) = \sigma^N(r^*)(h) = 1 \). Now suppose \( r \prec \tilde{n} \) and \( \lambda(r|\mathcal{P}_{\tilde{n}}(s)) > 0 \) (these together with \( \lambda(s|\mathcal{P}_{\tilde{n}}(t)) \) imply \( \lambda(r|\mathcal{P}_{\tilde{n}}(s)) > 0 \)) by Lemma 13). By the induction hypothesis, \( \pi(r)(H) = \bar{\mu} \) or \( \sigma^N(r)(h) = \sigma^N(s)(h) = 1 \). The latter implies \( \pi(r)(H) \geq \bar{\mu} \). Therefore, \( \pi(r)(H) \geq \bar{\mu} \).

But recalling that \( \sum_{r \sim \tilde{n}} \lambda(r|\mathcal{P}_{\tilde{n}}(s))\pi(r)(H) \leq \bar{\mu}, \) we then have \( \pi(r)(H) = \bar{\mu} \) if \( r \prec \tilde{n} \) and \( \lambda(r|\mathcal{P}_{\tilde{n}}(s)) > 0 \). In particular, \( \pi(s)(H) = \bar{\mu} \). This is a contradiction. 

**Corollary 4** At least one of the following holds in equilibrium \( (S, \pi, \sigma, p, \lambda) : (1) \) for each \( n \) such that \( p(n) > 0 \), \( \sigma^n \) is constant over \( I^{-1}(i) \); (2) if \( \lambda(s|I^{-1}(i)) > 0 \), then \( \pi(s)(H) = \bar{\mu} \).

**Proof.** If \( \lambda(s|I^{-1}(i)) > 0 \) and \( \pi(s)(H) \neq \bar{\mu} \), then, by Lemma 14, \( \sigma^n \) is constant over \( I^{-1}(i) \) for each \( n \geq i + 1 \). \( \sigma^n \) is constant over \( I^{-1}(i) \) if \( n \leq i \) because player 2 sees straight \( H \)’s. If \( \sigma^n \) is not constant over \( I^{-1}(i) \), then \( \lambda(s|I^{-1}(i)) > 0 \) implies that \( \pi(s)(H) = \bar{\mu} \).
We now study a property for a “grouping” Markov process. Consider a discrete time Markov process \((X, \kappa)\) with (countable) state space \(X\) and a transition probability \(\kappa : X \times 2^X \to [0, 1]\). Let \(\lambda\) be an invariant distribution of \((X, \kappa)\). Let \(g : X \to Y\) be a surjective measurable function. Therefore, \(\{g^{-1}(y) : y \in Y\}\) is a partition of \(X\). We interpret \(g\) as an information coarsening device: when the true state of the process is \(x\), an outside observer learns the coarser information \(g(x)\). Suppose \((x_1, x_2, \ldots)\) is a sample path of the underlying process; then \((g(x_1), g(x_2), \ldots)\) is the observation. Clearly, the process on \(Y\) is not a Markov process for general \(g\), but the invariant distribution \((X, \kappa)\) indeed induces a probability measure on \(Y\) by \(g^{-1}\): We ask the following question: Can we define a Markov transition probability on \(Y\), such that the Markov process \((Y, \kappa)\) has \(g^{-1}\) as an invariant distribution? \((Y, \kappa)\), if it exists, will to some extent capture the steady-state dynamics of \(g(x)\).

For any \(y \in Y\) and \(x \in g^{-1}(y)\), define \(\alpha_g(x) = \frac{\lambda(x)}{\lambda(g^{-1}(y))}\) if \(\lambda(g^{-1}(y)) \neq 0\) and let \(\alpha_y\) be an arbitrary probability over \(g^{-1}(y)\) if \(\lambda(g^{-1}(y)) = 0\). For any \(y, y' \in Y\), we define a transition probability \(\nu(y, y') = \sum_x \alpha_y(x) \kappa(x, g^{-1}(y'))\). \((Y, \nu)\) is a Markov process.

**Lemma 15** \((Y, \nu)\) has an invariant distribution \(\lambda g^{-1}\).

**Proof.** Since \(\lambda\) is an invariant distribution of \((X, \mu)\), \(\sum_{x \in X} \lambda(x) \kappa(x, x') = \lambda(x')\) for any \(x\) and \(x'\) in \(X\). Therefore,

\[
\sum_{y \in Y} \lambda g^{-1}(y) \nu(y, y') = \sum_{y \in Y} \lambda(g^{-1}(y)) \sum_{x \in X} \alpha_y(x) \kappa(x, g^{-1}(y')) = \sum_{y \in Y} \sum_{x \in g^{-1}(y)} \lambda(x) \kappa(x, g^{-1}(y')) = \sum_{x \in X} \lambda(x) \kappa(x, g^{-1}(y')) = \lambda g^{-1}(y')
\]

On the other hand, \(\lambda g^{-1}(Y) = \lambda(X) = 1\). Therefore, \(\lambda g^{-1}\) defines an invariant distribution of \((Y, \nu)\). \(\blacksquare\)

We are now in a position to prove Theorem 2.

**Proof of Theorem 2.** **Step 1:** Constructing the candidate equilibrium \((S, \bar{\pi}, \bar{\sigma}, \bar{p}, \bar{\lambda})\).

For each \(s \in I^{-1}(i)\), define

\[
\bar{\pi}_s : = \sum_{s' \in \mathcal{I}(s)} \lambda(s'|\mathcal{I}(s)) \pi_{s'}
\]

\[
\bar{p} : = p
\]
\( \tilde{\pi}_s \) is \( \mathcal{I} \)-measurable. By Lemma 15, \( \tilde{\pi} \) induces a stationary distribution \( \tilde{\lambda} \) such that 
\[ \tilde{\lambda}(I^{-1}(i)) = \lambda(I^{-1}(i)), \quad \tilde{\lambda}(\cdot|\mathcal{P}_n) \] and \( \tilde{b}^n \) are defined analogously to \( \lambda(\cdot|\mathcal{P}_n) \) and \( b^n \). In particular, \( \tilde{\lambda} \) is consistent. Let us construct player 2’s strategy \( \tilde{\sigma} \).

Consider an arbitrary function \( f : \mathcal{I} \to \mathbb{S} \) such that \( f(i) \in I^{-1}(i) \). For each \( i \in \mathcal{I} \), \( f(i) \) fixes an element in \( I^{-1}(i) \). By definition, if \( s \in I^{-1}(i) \), then \( \mathcal{P}_{i+1}(f(i)) = I(s) \).

If \( p(n) > 0 \), define \( \tilde{\sigma}^n(s) := \sigma^n(f(i)) \). If \( p(n) = 0 \), let \( \tilde{\sigma}^n \) be any strategy that is measurable with respect to the join of \( \mathcal{I} \) and \( \mathcal{P}_n \) and is a best response to \( \tilde{b}^n(s) \). Write \( \tilde{\sigma}_s = \sum_n \tilde{p}(n) \tilde{\sigma}^n_s \). By definition, \( \tilde{\sigma}_s = \sigma_s \). \( \tilde{\sigma}^n \)’s and \( \tilde{\sigma} \) are also \( \mathcal{I} \)-measurable.

**Step 2:** Deriving player 1’s continuation payoff under \((S, \tilde{\pi}, \tilde{\sigma}, \tilde{\rho}, \tilde{\lambda})\).

Under \((S, \pi, \sigma, p, \lambda)\), for each \( s \in \mathcal{S} \),
\[
U_s = \pi_s(H)[(1 - \delta)\sigma_s \cdot u_H + \delta U_{s\wedge H}] + \pi_s(L)[(1 - \delta)\sigma_s \cdot u_L + \delta U_{s\wedge L}] 
\]
(32)

By Lemma 2, under \((S, \pi, \sigma, p, \lambda)\), \( \sigma_s, U_s, U_{s\wedge H} \) and \( U_{s\wedge L} \) are \( \mathcal{I} \)-measurable. Taking the convex combination of (32) within \( \mathcal{I}(s) \) with coefficients \( \lambda(\cdot|\mathcal{I}(s)) \), we have
\[
U_s = \tilde{\pi}_s(H)[(1 - \delta)\tilde{\sigma}_s \cdot u_H + \delta U_{s\wedge H}] + \tilde{\pi}_s(L)[(1 - \delta)\tilde{\sigma}_s \cdot u_L + \delta U_{s\wedge L}] 
\]  
(33)

By definition, \( \tilde{\sigma}_s = \sigma_s \). \( \sigma_s \)’s and \( \tilde{\sigma} \) are also \( \mathcal{I} \)-measurable by Lemma 2.

**Step 3:** Optimality of player 1’s strategy under \((S, \tilde{\pi}, \tilde{\sigma}, \tilde{\rho}, \tilde{\lambda})\).

For each \( s \in \mathcal{S} \), \( \sigma_s(H) \) is a maximizer of
\[
(1 - \delta)\sigma_s \cdot u_H + \delta U_{s\wedge H} + (1 - \delta)\sigma_s \cdot u_L + \delta U_{s\wedge L} 
\]
(34)

The RHS of (35) is \( \mathcal{I} \)-measurable by Steps 1 and 2. Therefore, it follows from the convexity of the set of maximizers in this problem that \( \tilde{\pi}_s = \sum_{s' \in \mathcal{I}(s)} \lambda(s'|\mathcal{I}(s)) \pi_{s'} \) is a maximizer of the RHS of (35).

**Step 4:** Optimality of player 2’s strategy, \( \sigma \), under \((S, \tilde{\pi}, \tilde{\sigma}, \tilde{\rho}, \tilde{\lambda})\).
If \( p(n) = 0 \), then \( \sigma^n \) is optimal by definition. We now consider \( p(n) > 0 \).

**Step 4a: \( n \leq i \).**

Let us first consider the optimality condition in equilibrium \((S, \pi, \sigma, p, \lambda)\). Suppose the true state is \( f(i) \in \mathcal{I}^{-1}(i) \). Since \( n \leq i \), the most recent \( n \) elements in \( f(i) \) are \( H \ldots H \); i.e., player 2 sees straight \( H \)'s after buying \( n \) bits of information and hence,

\[
P_n(f(i)) = \{ s : s_n s_{n-1} \ldots s_1 = H \ldots H \} = \cup_{i \geq n} \mathcal{I}^{-1}(i)
\]

Thus

\[
\lambda(P_n(f_i)) = \sum_{i \geq n} \lambda(I^{-1}(i)) = \tilde{\lambda}(P_n(f(i)))
\]

Recall that

\[
b_n^i(H) = \frac{\mu_0}{\mu_0 + (1 - \mu_0) \lambda(P_n(f(i)))} + \frac{(1 - \mu_0)}{\mu_0 + (1 - \mu_0) \lambda(P_n(f(i)))} \sum_{s \in P_n(f(i))} \lambda(s) \pi_s(H)
\]

\[
= \frac{1}{\mu_0 + (1 - \mu_0) \lambda(P_n(f(i)))} (\mu_0 + (1 - \mu_0) \sum_{s \in P_n(f(i))} \lambda(s) \pi_s(H))
\]

Notice that

\[
\sum_{s \in P_n(f(i))} \lambda(s) \pi(s) = \sum_{i \geq n} \sum_{I(s)=i} \lambda(s) \pi_s
\]

\[
= \sum_{i \geq n} \lambda(I^{-1}(i)) \sum_{I(s)=i} \alpha_i(s) \pi_s
\]

\[
= \sum_{i \geq n} \tilde{\lambda}(I^{-1}(i)) \tilde{\pi}_s
\]

\[
= \sum_{i \geq n} \tilde{\lambda}(I^{-1}(i)) \tilde{\pi}_s
\]

\[
= \sum_{s \in P_n(f(i))} \tilde{\lambda}(s) \tilde{\pi}_s
\]

We therefore have \( b^n(f(i)) = \tilde{b}^n(f(i)) \).

\( \sigma^n(f(i))(h) = \sigma^n(f(i))(h) \) maximizes

\[
x b^n(f(i)) \cdot v_h + (1-x) \tilde{b}^n(f(i)) \cdot v_l = x b^n(f(i)) \cdot v_h + (1-x) \tilde{b}^n(f(i)) \cdot v_l
\]

where the argument \( x \in [0, 1] \). \( \bar{\sigma}^n \) is optimal under \((S, \tilde{\pi}, \bar{\sigma}, \bar{p}, \tilde{\lambda})\) when \( n \leq i \).

**Step 4b: \( n > i \).**
For an \( s \in I^{-1}(i) \), the last \( n \) bits of information in \( s \) take the form \( \underbrace{LH...H}_n \) with an \( L \) inside. Therefore, player 2 knows player 1 is the normal type upon buying \( n \) bits of information.

Let us first consider the optimality condition in equilibrium \((S, \pi, \sigma, p, \lambda)\). By Corollary 4, we have either (1) \( \sigma^n \) is constant on \( I^{-1}(i) \), or (2) \( \pi(s)(H) = \bar{\mu} \) if \( \lambda(s|P_{i+1}(f(i))) > 0 \). Notice that \( \lambda(s|P_{i+1}(f(i))) > 0 \) implies \( P_{i+1}(s) \subset P_{i+1}(f(i)) \) and \( \lambda(s|P_n(s)) > 0 \) (by part (1) of Lemma 13). Hence \( P_n(s) \subset P_{i+1}(f(i)) \) and \( \lambda(s|P_n(s)) \) is updated from \( \lambda(|P_{i+1}(s)) = \lambda(|P_{i+1}(f(i))) \). By part (2) of Lemma 13, if \( \lambda(s'|P_n(s)) > 0 \), then \( \lambda(s'|P_{i+1}(s)) > 0 \). Therefore, in both cases (1) and (2), \( \sigma^n(f(i))(h) \) maximizes

\[
\sum_{s'} \lambda(s'|P_n(s)) \pi_{s'} \cdot (xv_h + (1 - x)v_l)
\]

for each \( s \in I^{-1}(i) \) such that \( \lambda(s|P_{i+1}(f(i))) > 0 \). Therefore, for each \( s \in I^{-1}(i) \), \( \sigma^n(f(i))(h) \) maximizes

\[
\lambda(s|P_{i+1}(f(i))) \times (37)
\]

Hence, \( \sigma^n(f(i))(h) \) maximizes

\[
\sum_{s \in I^{-1}(i)} \lambda(s|P_{i+1}(f(i))) \times (37)
\]

\[
= \sum_{s \in I^{-1}(i)} \sum_{s' \in P_n(s)} \lambda(s|P_{i+1}(f(i))) \lambda(s'|P_n(s)) \pi_{s'} \cdot (xv_h + (1 - x)v_l)
\]

\[
= \sum_{\{P_n(s): s \in I^{-1}(i)\}} \sum_{s' \in P_n(s)} \lambda(P_n(s)|P_{i+1}(f(i))) \lambda(s'|P_n(s)) \pi_{s'} \cdot (xv_h + (1 - x)v_l)
\]

\[
= \sum_{s' \in I^{-1}(i)} \lambda(s'|P_{i+1}(f(i))) \pi_{s'} \cdot (xv_h + (1 - x)v_l)
\]

By definition, the RHS of (38) is exactly \( \bar{\pi}_s \cdot (xv_h + (1 - x)v_l) \) for each \( s \in I^{-1}(i) \). Therefore, \( \bar{\sigma}^n(s) = \sigma^n(f(i)) \) is optimal under \((S, \bar{\pi}, \bar{\sigma}, \bar{p}, \bar{\lambda})\) when \( n > i \).

**Step 5:** Derivation of player 2’s payoff.

**Step 5a:** When \( n \leq i \), the last \( n \) bits of information in any \( s \in I^{-1}(i) \) are \( \underbrace{HH...H}_n \). Therefore, player 2’s payoff upon buying \( n \) bits of information under equilibria \((S, \pi, \sigma, p, \lambda)\), \( V_s^n \), is constant on \( I^{-1}(i) \). Similarly, player 2’s payoff \( V_s^n \) from \((S, \bar{\pi}, \bar{\sigma}, \bar{p}, \bar{\lambda})\) is constant on \( I^{-1}(i) \). It follows from (36) that \( V_s^n = \bar{V}_s^n \) when \( n \leq I(s) \).
Step 5b: When \( n > i \), player 2 will see an \( L \) upon buying \( n \) bits of information and hence \( P_{i+1}(f(i)) = I^{-1}(i) \). It follows from Corollary 4 that one of the following holds on \( I^{-1}(i) \): (1) \( \sigma^n \) is constant; (2) \( \pi(s)(H) = \bar{\mu} \) if \( \lambda(s) > 0 \). The same argument as in (38) in Step 4b shows that, under equilibrium \((S, \pi, \sigma, p, \lambda)\), player 2’s payoff conditional on buying \( n \) bits of information and on the true state lying in \( I^{-1}(i) \) is

\[
\begin{align*}
\sum_s \lambda(s|P_{i+1}(f(i)))V^n_s &= \sum_s \lambda(s|P_{i+1}(f(i))) \sum_{s'} \lambda(s'|P_n(s))\pi_s' \cdot (\sigma^n(f(i))(h)v_h + \sigma^n(f(i))(l)v_l) \\
&= \sum_{s' \in I^{-1}(i)} \lambda(s'|P_{i+1}(f(i)))\pi_s' \cdot (\sigma^n(f(i))(h)v_h + \sigma^n(f(i))(l)v_l) \\
&= \bar{\pi}_{f(i)} \cdot (\sigma^n(f(i))(h)v_h + \sigma^n(f(i))(l)v_l) \\
&= \sum_s \bar{\lambda}(s|P_{i+1}(f(i)))\bar{V}^n_s
\end{align*}
\]

The last equality follows from the fact that \( \bar{V}^n_s \) is constant on \( P_{i+1}(f(i)) \). This shows that \( V^n = \bar{V}^n \).

Step 5a shows that conditional on \( n \leq i \), player 2’s expected payoffs are the same under \((S, \pi, \sigma, p, \lambda)\) and \((S, \bar{\pi}, \bar{\sigma}, \bar{p}, \bar{\lambda})\). Step 5b further shows that they are the same conditional on \( n > i \). Furthermore, \( \bar{\rho} = p \) by construction. We therefore have that \( V = \bar{V} \).

**Step 6: Optimality of information acquisition rule \( \bar{\rho} \).**

Let us consider those \( n \) such that \( \bar{\rho}(n) = p(n) = 0 \). When \( n \leq i \), \( \bar{V}^n_s = V^n_s \) for each \( s \in I^{-1}(i) \) by Step 5a (where we didn’t use the fact that \( p(n) > 0 \)). When \( n > i \), player 2 knows that player 1 is the normal type; therefore player 2’s strategy \( \bar{\sigma}^n_s \) is a best response to \( \bar{\pi}_s = \sum_{s' \in I(s)} \lambda(s'|P_{i+1}(s))\pi_{s'} \) by definition. Thus \( \bar{V}^n_s = V^{i+1}_s \). That is, player 2 achieves the payoff from buying \( i+1 \) bits of information in equilibrium \((S, \pi, \sigma, p, \lambda)\) even though he buys \( n \) bits of information. However, in equilibrium \((S, \pi, \sigma, p, \lambda)\), \( \sigma^n \) achieves a higher payoff since for each \( s \in I^{-1}(i), \sigma^n_s \) is a best response to finer information, \( \sum_{s' \in I(s)} \lambda(s'|P_n(s))\pi_{s'} \). Therefore, \( V^n \geq \bar{V}^n \) when \( \bar{\rho}(n) = p(n) = 0 \). Step 5 shows that \( V^n = \bar{V}^n \) when \( \bar{\rho}(n) = p(n) > 0 \). By the optimality of \( p \) in \((S, \pi, \sigma, p, \lambda)\), \( \bar{\rho} \) is optimal in \((S, \bar{\pi}, \bar{\sigma}, \bar{p}, \bar{\lambda})\).

**.3 Proof of Theorem 4**

We first prove parts (2) and (3) in Theorem 4, since they do not depend on generic cost functions.

Let \( n^{(1)} \) and \( n^{(2)} \) be the largest and the second largest amount that is ever bought in equilibrium, respectively.

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Lemma 16 If the cost function is strictly increasing and \( \delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}} \), then \( n^* = n^{(2)} \) or \( n^* = n^{(1)} \).

Proof. Suppose \( n^{(2)} < n^* < n^{(1)} \). Since \( \pi_{n^*}(L) = 1 \) and \( n^* < n^{(1)} \), we therefore have \( \sigma^*_{n^{(1)}}(h) = 0 \). Note that \( \sigma^*_{n^{(2)}} = \sigma^*_{n^{(1)}} \) by \( n^{(2)} < n^* < n^{(1)} \) and \( \sigma^*_{n^{(i)}-1}(h) < \sigma^*_{n^{(i)}-1}(h) \) (Lemma 7), and hence we must have that \( \sigma^*_{n^{(2)}-1}(h) < 0 \). This is impossible. Also note that \( n^* < n^{(2)} \) is impossible because \( \pi_{n^*}(H) = 0 \), then states \( \{n^{(2)}, n^{(2)} + 1, \ldots, N\} \) are off the equilibrium path (Proposition 3), and hence player 2 needs only to buy at most \( n^{(2)} \) bits of information. This contradicts the definition of \( n^{(1)} \). \( \blacksquare \)

Lemma 17 If \( n^* = n^{(2)} \), then \( n^{(2)} = n^{(1)} - 1 \).

Proof. If \( n^* = n^{(2)} < n^{(1)} - 1 \), then states \( n^{(2)} + 1, \ldots, n^{(1)} \) are off the equilibrium path. Player 2 will be better off buying \( n^* = n^{(2)} + 1 < n^{(1)} \) bits of information. \( \blacksquare \)

Part (2) and (3) of Theorem 4 are immediate implications of Lemma 16 and the following.

Proposition 7 If the cost function is strictly increasing and \( \delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}} \), then \( p(i) > 0 \) and \( \pi_i(H) < \bar{\mu} \) for each \( i \in \{0, 1, \ldots, n^{(1)}\} \). Upon acquiring information, player 2 plays pure strategy \( l \) whenever he sees an \( L \) and he plays pure strategy \( h \) whenever he sees \( H \)’s.

Proof. \( p(n^{(1)}) > 0 \) by definition and \( \pi(n^{(1)})(H) = 0 \) by Theorem 3. Since \( \pi_i(h) \) is strictly increasing in \( i \), we have \( \sigma^{(1)}_n(h) > \sigma^{(1)}_{n+1}(h) \). That is, \( \sum p(k)\sigma^{(1)}_n(h) > \sum p(k)\sigma^{(1)}_{n+1}(h) \).

Since \( \sigma^{(1)}_n(h) = \sigma^{(1)}_{n+1}(h) \) when \( k \leq n^{(1)} - 1 \), we have \( \sigma^{(1)}_n(h) > \sigma^{(1)}_{n+1}(h) \).

Claim 1: \( \sigma^{(1)}_{n^{(1)}}(h) = 1 \) and \( \sigma^{(1)}_{n^{(1)}-1}(h) = 0 \).

To see this, suppose to the contrary that one of them is in \( (0, 1) \); i.e., player 2 plays a strictly mixed strategy in that state upon observing \( n^{(1)} \) bits of information. Suppose \( a_2 \) is played with positive probability in both states \( n^{(1)} \) and \( n^{(1)} - 1 \), \( a_2 \in \{h, l\} \). Then, we can construct a profitable deviation \( (\tilde{p}, \tilde{\sigma}) \) for player 2 as follows. Set \( \tilde{p}(n^{(1)} - 1) := p(n^{(1)} - 1) + p(n^{(1)}) \) and \( \tilde{p}(k) = p(k) \) when \( k < n^{(1)} - 1 \). That is, player 2 buys \( n^{(1)} - 1 \) instead of \( n^{(1)} \) bits of information. Set \( \tilde{\sigma}^{(1)}_{n^{(1)}}(a_2) = \tilde{\sigma}^{(1)}_{n^{(1)}-1}(a_2) = 1 \) and \( \tilde{\sigma}^{(1)}_{i+1}(h) := \sigma^{(1)}_{i+1}(h) \) if \( i < n^{(1)} - 1 \). Let \( \tilde{\sigma}^k := \sigma^k \) if \( k \neq n^{(1)} \), \( n^{(1)} - 1 \). \( \tilde{\sigma}^{(1)}_{n^{(1)}} \) and \( \tilde{\sigma}^{(1)}_{n^{(1)}-1} \) are optimal since \( a_2 \) is optimal in both states \( n^{(1)} \) and \( n^{(1)} - 1 \). The optimality of \( \tilde{\sigma}^k \) follows from the optimality of \( \sigma^k \) for \( k \neq n^{(1)} \), \( n^{(1)} - 1 \). By this alternative strategy, player 2 achieve the same payoff but saves \( (C(n^{(1)}) - C(n^{(1)} - 1))p(n^{(1)}) \). We have derived a contradiction. Therefore, \( \sigma^{(1)}_{n^{(1)}}(h) = 1 \) and \( \sigma^{(1)}_{n^{(1)}-1}(h) = 0 \).

Claim 2: \( \pi_{n^{(1)}-1}(H) < \bar{\mu} \).
To see this, suppose to the contrary that $\pi_{n(1)-1}(H) = \bar{\mu}$. Then, player 2 is indifferent after buying $n^{(1)}$ bits of information in state $n^{(1)} - 1$. The identical argument that establishes Claim 1 also applies here. We thus complete the proof of Claim 2.

We now prove the proposition by induction. Suppose for $i = n^{(1)}, n^{(1)} - 1, \ldots, m + 1$, $m > 0$, we have $p(i) > 0$, $\pi_{i-1}(H) < \bar{\mu}$, $\sigma_i^i(h) = 1$ and $\sigma_{i-1}^k(h) = 0$ if $k \geq i$. Consider $i = m$. Suppose $p(m) = 0$. Then

$$\sigma_m(h) - \sigma_{m-1}(h) = \sum_{k \geq m+1} p(k)(\sigma_m^k(h) - \sigma_{m-1}^k(h))$$

$$= -\sum_{k \geq m+1} p(k)\sigma_{m-1}^k(h)$$

$$\leq 0$$

This is impossible because $\sigma_m(h) > \sigma_{m-1}(h)$ by strict monotonicity. Therefore, $p(m) \neq 0$ and

$$\sigma_m(h) - \sigma_{m-1}(h) = \sum_{k \geq m} p(k)(\sigma_m^k(h) - \sigma_{m-1}^k(h))$$

$$= p(m)\sigma_m^m(h) - \sum_{k \geq m} p(k)\sigma_{m-1}^k(h)$$

It must be that $\sigma_m^m(h) = 1$, $\sigma_{m-1}^m(h) = 0$, and $\pi_{m-1}(H) < \bar{\mu}$, by arguments similar to those used in the proof of Claims 1 and 2. Since $\pi_{m-1}(H) < \bar{\mu}$, we have $\sigma_{m-1}^k(h) = 0$ if $k \geq m$. We therefore have that for each $i \in \{1, 2, \ldots, n^{(1)}\}$, $p(i) > 0$, $\pi_{i-1}(H) < \bar{\mu}$, $\sigma_i^k(h) = 0$ if $k \geq i$ and $\sigma_i^i(h) = 1$.

We still need to show that $p(0) > 0$. Suppose to the contrary $p(0) = 0$. Then we will have $\sigma_0(h) = 0$ and $\sigma_{n^{(1)}}(h) = 1$. If $n^* = n^{(2)}$, then $n^{(2)} = n^{(1)} - 1$ by Lemma 17. Player 1 plays pure strategy $L$ in state $n^{(2)}$,

$$(1 - \delta)u_H \cdot \sigma_n(h) + \delta \left( (1 - \delta)u_L \cdot \sigma_{n(1)}(h) + \delta u_L \cdot \sigma_0 \right) \leq (1 - \delta)u_L \cdot \sigma_n(h) + \delta u_L \cdot \sigma_0$$

Substituting $\sigma_0(h) = 0$ and $\sigma_{n^{(1)}}(h) = 1$ and rearranging terms, we have

$$\delta(u_{Lh} - u_{Ll}) \leq (u_L - u_H) \cdot \sigma_n(h)$$

$$< u_{Lh} - u_{Hh}$$

We have an immediate contradiction if $\delta > \frac{u_{Lh} - u_{Hh}}{u_{Lh} - u_{Ll}}$. The same argument will show that $n^* = n^{(1)}$ is impossible. □
Now let us prove Part (1) of Theorem 4.

By Lemma 16, we only need to show that \( n^* = n^{(2)} \neq n^{(1)} \) is the non-generic case. Note that by Proposition 7, \( n^{(1)} = n^{(2)} + 1 \) if \( n^{(1)} \neq n^{(2)} \).

Also by Proposition 7, we have two cases to consider.

**Case I:** Player 2 plays \( l \) with positive probability when he does not acquire information.

If player 2 does not acquire information, we have

\[
V^0(\pi, \sigma) = \mu_0 v_{Hl} + (1 - \mu_0) \sum_{j=0}^{n^{(1)}} \lambda(j)\pi_j \cdot v_l
\]

and if player 2 acquires \( n^{(1)} \) bits of information, we have

\[
V^{n^{(1)}}(\pi, \sigma) = \mu_0 v_{Hh} + (1 - \mu_0) \sum_{j=0}^{n^{(1)}-1} \lambda(j)\pi_j \cdot v_l
+ (1 - \mu_0)\lambda(n^{(1)})\pi_{n^{(1)}} \cdot v_h - C(n^{(1)}).
\]

Since \( V^0(\pi, \sigma) = V^{n^{(1)}}(\pi, \sigma) \),

\[
\mu_0(v_{Hh} - v_{Hl}) - (1 - \mu_0)\lambda(n^{(1)})(v_{Ll} - v_{Lh}) = C(n^{(1)}).
\]

If \( n^* = n^{(2)} \neq n^{(1)} \), \( \lambda(n^{(1)}) = 0 \). Then \( C(n^{(1)}) = \mu_0(v_{Hh} - v_{Hl}) \). This is clearly non-generic.

**Case II** Player 2 plays \( h \) when he does not acquire information.

For \( 0 \leq i \leq n^{(1)} \),

\[
V^i(\pi, \sigma) = \mu_0 v_{Hh} + (1 - \mu_0) \sum_{j=0}^{i-1} \lambda(j)\pi_j \cdot v_l
+ (1 - \mu_0) \sum_{j=i}^{n^{(1)}} \lambda(j)\pi_j \cdot v_h - C(i).
\]

From \( V^0(\pi, \sigma) = \cdots = V^{n^{(1)}}(\pi, \sigma) \), we have \( n^{(1)} \) equations:

\[
(1 - \mu_0)\lambda(i)\pi_i \cdot (v_l - v_h) = C(i + 1) - C(i), \ 0 \leq i \leq n^{(1)} - 1.
\]  

(40)

From the definition of invariant distribution, we have \( n^{(1)} + 1 \) equations:

\[
\lambda(i + 1) = \lambda(i)\pi_i(H) \text{ for } 0 \leq i \leq n^{(1)},
\]

\[
\sum \lambda(i) = 1.
\]  

(41)  

(42)
If $n^* = n^{(2)} \neq n^{(1)}$, then $\pi_{n^{(2)}}(H) = 0$. This together with $\pi_{n^{(1)}}(H) = 0$ gives two initial conditions. Therefore, we have $n^{(1)} + (n^{(1)} + 1) + 2 = 2(n^{(1)} + 1) + 1$ equations, but only $2(n^{(1)} + 1)$ unknowns: $\pi_i$ and $\lambda(i)$ for $0 \leq i \leq n^{(1)}$. This case is non-generic.

.4 Proof of Lemmata 8 and 9

Proof of Lemma 8. Since players’ payoffs depend only on the actions actually taken, the analogous inductive argument that is used to prove Theorem 1 can be used to prove the current lemma.

Proof of Lemma 9. Suppose to the contrary that the result does not hold. Then for any $\varepsilon > 0$, there exist $\varepsilon < \tilde{\varepsilon}$, such that either $H$ or $L$ (or both) is played in some absorbing state in an corresponding equilibrium $(S, \pi^\varepsilon, \sigma^\varepsilon, p^\varepsilon, \lambda^\varepsilon)$. Without loss of generality, suppose $H$ is played in an absorbing state—the state is $H^N$. By upper semi-continuity, as $\varepsilon$ approaches 0, $\{(S, \pi^\varepsilon, \sigma^\varepsilon, p^\varepsilon, \lambda^\varepsilon)\}$ has a limit point $(S, \pi, \sigma, p, \lambda)$ which is an equilibrium of the perfect monitoring game—$H$ is played in state $H^N$ in $(S, \pi, \sigma, p, \lambda)$—contradicting Corollary 2. Analogously, $L$ being played in an absorbing state contradicts Theorem 3.

We now turn to the proof of Proof of Proposition 5 The idea is as follows. As $\sigma_i$ is strictly increasing by Theorem 3, we must have $0 < \sigma_i(h) < 1$ if $1 \leq i \leq n - 1$. Since player 2 observes all past $n$ bits of information, $\pi_i = \bar{\pi}$ if $1 \leq i \leq n - 1$. We are left to determine $\pi_0$. To understand the threshold, note that a large $n$ and a favorable prior $\mu_0$ is crucial for player 1 to build her reputation. To build a reputation (i.e., player 2’s high posterior belief about her commitment type upon seeing straight $H$’s), player 1 needs to make the straight $H$’s a small probability event. When $n$ is small relative to $\mu_0$, player 1 needs to reduce $\pi_0(h)$ from $\bar{\mu}$ which leads to Case (2) in the proposition.

Proof of Proposition 5. As $\sigma_i$ is strictly increasing by Theorem 3, we must have $0 < \sigma_i(h) < 1$ if $1 \leq i \leq n - 1$. Since player 2 observes all past $n$ bits of information, $\pi_i = \bar{\pi}$ if $1 \leq i \leq n - 1$. We are left to determine $\pi_0$. By Lemma 10, the two generic cases involves $\sigma_0(h) = 0$ and $0 < \sigma_0(h) < 1$.

Case I: $\sigma_0(h) = 0$.

$\pi_0(H) \leq \bar{\mu}$. By Lemma 10, $0 < \sigma_n(h) < 1$. That is, player 2 is indifferent between $h$ and $l$ after seeing $n$ $H$’s. Player 2’s belief after seeing $n$ $H$’s is

$$\frac{\mu_0}{\mu_0 + (1 - \mu_0)\lambda(n)} = \bar{\mu}$$

\[\text{[14]}\text{A simple computation by substituting } \pi_i \text{ from (41) into (40) leads to an independent system of linear equations.}\]
The “balanced flow equations” for the invariant distribution are

\[
\lambda(i + 1) = \lambda(i)\pi_i(H) \\
\sum \lambda(i) = 1
\]

We then have \(\pi_0(H) = \frac{\mu_0(1 - \bar{\mu})}{\bar{\mu}^n - \mu_0}\). For \(0 < \pi_0(H) \leq \bar{\mu}\), we must have \(\mu_0 < \bar{\mu}^{n+1}\). That is, 
\[n \leq \frac{\log \mu_0}{\log \bar{\mu}} - 1.\]

Case II: \(\sigma_n(h) = 1\).

By Lemma 10, \(\sigma_0(h) > 0\) and hence \(\pi_0(H) = \bar{\mu}\). Therefore,

\[
\frac{\mu_0}{\mu_0 + (1 - \mu_0)\lambda(n)} \geq \bar{\mu}
\]

Together with the flow equations, we have \(\mu_0 \geq \bar{\mu}^{n+1}\), i.e., \(n \geq \frac{\log \mu_0}{\log \bar{\mu}} - 1\).

We omit the non-generic case \(n = \frac{\log \mu_0}{\log \bar{\mu}} - 1\). It can be analyzed analogously.

.5 Proof of Proposition 6

By the definition of invariant distribution,

\[
\lambda(i + 1) = \lambda(i)\pi_i(H) \text{ for } 0 \leq i \leq n^* \\
\sum \lambda(i) = 1 \\
\pi_{n^*}(H) = 0
\]

As in the proof in Section.3, we have several cases to consider.

Case I: Player 2 plays \(h\) when he doesn’t buy information.

This implies that \(\sigma_{n^*}(h) = 1\). For \(0 \leq i \leq n^*\),

\[
V^i(\pi, \sigma) = \mu_0 v_{Hh} + (1 - \mu_0) \sum_{j=1}^{n^*} \lambda(j)\pi_j \cdot v_h + (1 - \mu_0) \sum_{j=0}^{i-1} \lambda(j)\pi_j \cdot v_l - ic
\]

Since \(V^0(\pi, \sigma) = \cdots = V^{n^*}(\pi, \sigma)\),

\[
(1 - \mu_0)\lambda(i)\pi_i \cdot (v_l - v_h) = c, \quad 0 \leq i \leq n^* - 1
\]
Solving those equations, we have

\[
\begin{align*}
\pi_i(H) &= \frac{(1 - \bar{\mu})\bar{\mu}^{i+1}(c + cn^* + K) - c(1 - \bar{\mu}^{n^*+1})}{(1 - \bar{\mu})\bar{\mu}^i(c + cn^* + K) - c(1 - \bar{\mu}^{n^*+1})}, 1 \leq i \leq n^* - 1 \\
\pi_0(H) &= \frac{(1 - \bar{\mu})\bar{\mu}K + c(\bar{\mu}^{n^*+1} - \bar{\mu}(n^* + 1) + (n^* + 1)\tilde{\mu} - 1)}{(1 - \bar{\mu})K + c(\bar{\mu}^{n^*+1} - \bar{\mu}(n^* + 1) + n^*)} \\
\lambda(i) &= \frac{(1 - \bar{\mu})\bar{\mu}^i(c + cn^* + K) - c(1 - \bar{\mu}^{n^*+1})}{(1 - \bar{\mu}^{n^*+1})K} \\
\lambda(0) &= \frac{c(\bar{\mu}^{n^*+1} + n^* - \bar{\mu}(n^* + 1)) + (1 - \bar{\mu})K}{(1 - \bar{\mu}^{n^*+1})K} \\
K &= (1 - \mu_0)(v_{Hh} - v_{Hl}) \text{ is a constant.}
\end{align*}
\]

We have to check two sets of IC conditions.

(1) \(\pi_i\)'s are well-defined strategies and player 2's behavioral strategies are optimal. By Proposition 8 below, we only need to check one IC condition: \(h\) is optimal if player 2 does not acquire information.

The total (ex ante) probability of \(H\) being played is

\[
\mu_0 + (1 - \mu_0)\sum_{i=0}^{n^*} \lambda(i)\pi_i(H) = \mu_0 + (1 - \mu_0)\sum_{i=1}^{n^*} \lambda(i) = \mu_0 + (1 - \mu_0)(1 - \lambda_0)
\]

For \(h\) to be optimal when player 2 does not acquire information, we need

\[
\mu_0 + (1 - \mu_0)(1 - \lambda_0) \geq \tilde{\mu} \tag{47}
\]

(2) The maximal amount of information that player 2 buys is exactly \(n^*\).
Player 2 does not strictly prefer \(n^* - 1\) to \(n^*\) bits of information if

\[
\lambda(n^*) > 0
\]

Player 2 does not strictly prefer \(n^* + 1\) bits of information if

\[
(1 - \mu_0)\lambda(n^*)(v_{Ll} - v_{Lh}) \leq c
\]

Therefore, for \(n^*\) to be the maximal number of bits acquired in equilibrium,

\[
0 < \lambda(n^*) \leq \frac{c}{(1 - \mu_0)(v_{Ll} - v_{Lh})} \tag{48}
\]

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Substituting $\lambda_{n(1)}$, (47) becomes
\[
\frac{c \leq (1 - \bar{\mu})(\mu_0 - \bar{\mu}^{n^{*}+1})(v_{Hh} - v_{Hl})}{\bar{\mu}^{n^{*}+1} + n^{*} - (n^{*} + 1)\bar{\mu}}
\] (49)

(48) becomes
\[
g(n^{*} + 1) \leq c < g(n^{*}),
\] (50)

where $g(n^{*}) = \frac{(1 - \bar{\mu})\bar{\mu}^{n^{*}+1}}{1 + n^{*}\bar{\mu}^{n^{*}+1} - (n^{*} + 1)\bar{\mu}^{n^{*}}}$ is strictly decreasing in $n^{*}$. Analogously to the proof in Section 3, we can show that $c = g(n)$, $n = 1, 2, \ldots$, is the non-generic case. (49) and (50) characterize the equilibrium in this case.

**Case II: Player 2 plays pure strategy $l$ when he does not buy information.**

This implies that $\sigma_0(h) = 0$. If player 2 does not acquire information, we have
\[
V^0(\pi, \sigma) = \mu_0 v_{HI} + (1 - \mu_0) \sum_{j=0}^{n^{*}} \lambda(j) \pi_j \cdot v_l
\] (51)

and if player 2 acquires $i$ bits of information, $1 \leq i \leq n^{*}$, we have
\[
V^i(\pi, \sigma) = \mu_0 v_{Hh} + (1 - \mu_0) \sum_{j=0}^{n^{*}} \lambda(j) \pi_j \cdot v_h + (1 - \mu_0) \sum_{j=0}^{i-1} \lambda(j) \pi_j \cdot v_l - c
\] (52)

Since $V^0(\pi, \sigma) = \cdots = V^{n^{*}}(\pi, \sigma)$, we have,
\[
(1 - \mu_0)\lambda(i) \pi_i \cdot (v_l - v_h) = c, \quad 1 \leq i \leq n^{*} - 1
\] (53)
\[
\mu_0(v_{Hh} - v_{HI}) - (1 - \mu_0)\lambda(n^{*})(v_{Ll} - v_{Lh}) = n^{*}c
\] (54)

These equations together with the equations for invariant distributions give us $2(n^{*} + 1)$ equations with $2(n^{*} + 1)$ unknowns ($n^{*} + 1$ $\pi_i$'s and $n^{*} + 1$ $\lambda(i)$'s). In particular, by (54),
\[
\lambda(n^{*}) = \frac{\mu_0(v_{Hh} - v_{HI}) - n^{*}c}{(1 - \mu_0)(v_{Ll} - v_{Lh})}
\] (55)

$\sigma_0(l) = 1$ implies that player 1 plays $l$ when he does not buy information ($p(0) > 0$). For the solutions to form an equilibrium, two sets of IC conditions are both necessary and sufficient.

(1) $\pi_i$, $0 \leq i \leq n^{*} - 1$, are well-defined strategies and player 2’s simple behavioral strategies are optimal: he plays $l$ whenever he sees an $L$ and $h$ whenever he sees straight $H$’s; he plays $l$ whenever he does not acquire information.

Proposition 9 below shows that we only need to consider one IC condition involving player 2’s behavioral strategies: $l$ is optimal if player 2 does not acquire information. That
is,
\[ \mu_0 + (1 - \mu_0)(1 - \lambda(0)) \leq \tilde{\mu} \quad \text{(56)} \]

(2) The maximal number of bits of information player 2 buys is exactly \( n^* \).
Player 2 does not strictly prefer \( n^* - 1 \) to \( n^* \) bits of information if
\[ \lambda(n^*) > 0 \]
Player 2 does not strictly prefer \( n^* + 1 \) bits of information if
\[ (1 - \mu_0)\lambda(n^*)(v_{LI} - v_{LH}) \leq c \]

Therefore, for \( n^* \) to be the maximal number of bits acquired in equilibrium,
\[ 0 < \lambda(n^*) \leq \frac{c}{(1 - \mu_0)(v_{LI} - v_{LH})} \quad \text{(57)} \]

Substituting \( \lambda(n^*) \), (56) is
\[ c \geq \frac{(1 - \tilde{\mu})(\mu_0 - \tilde{\mu}n^* + 1)(v_{HH} - v_{HI})}{\tilde{\mu}n^* + 1 + n^* - (n^* + 1)\tilde{\mu}} \quad \text{(58)} \]

(57) is
\[ \frac{\mu_0(v_{HH} - v_{HI})}{n^* + 1} \leq c < \frac{\mu_0(v_{HH} - v_{HI})}{n^*} \quad \text{(59)} \]

(58) and (59) characterize the relation of \( n^* \) and \( c \) for this case. \( c = \frac{\mu_0(v_{HH} - v_{HI})}{n^*} \), \( n = 1, 2, \ldots \), is the non-generic case.

Case III: Player 2 plays strictly mixed strategies when he does not buy information.

This implies that \( 0 < \sigma_0(h) < \sigma_{n^*}(h) < 1 \). Equations (53), (54), and (46) hold. In particular, from (49) and (58) we have
\[ c = \frac{(1 - \tilde{\mu})(\mu_0 - \tilde{\mu}n^* + 1)(v_{HH} - v_{HI})}{\tilde{\mu}n^* + 1 + n^* - (n^* + 1)\tilde{\mu}} \]

From the two expressions of \( \lambda(n^*) \),
\[ \frac{\mu_0(v_{HH} - v_{HI}) - n^*c}{(1 - \mu_0)(v_{LI} - v_{LH})} = \frac{(1 - \tilde{\mu})\tilde{\mu}n^* (c + cn^* + K) - c(1 - \tilde{\mu}n^* + 1)}{(1 - \tilde{\mu}n^* + 1)K} \]

The two equations above give exact values of \( c = c^* \) and \( n^* \). It is noteworthy that when \( c = c^* \),
\[ \frac{(1 - \tilde{\mu})\tilde{\mu}n^* K}{1 + n^*\tilde{\mu}n^* + 1 - n^* + 1)\tilde{\mu}n^*} = \frac{\mu_0(v_{HH} - v_{HI})}{n^*}. \]
This is the non-generic case.

Finally, we define
Proof. The “only if” part is obvious. “If”: Substituting (43) into (46), we obtain
\[
\lambda(i) - \lambda(i + 1))(v_{Ll} - v_{Lh}) = \frac{c}{1 - \mu_0} + \lambda(i + 1)(v_{Hh} - v_{Hi}), \quad 0 \leq i \leq n^* - 1
\]
From this expression, \(\lambda(i + 1) > 0\) implies that \(\lambda(i) > 0, \quad 0 \leq i \leq n^* - 1\). Therefore, \(\lambda(n^*) > 0\) implies that \(\lambda(i) > 0, \quad 0 \leq i \leq n^*, \) and that \(\lambda(i)\) is strictly decreasing in \(i\). Thus, \(0 < \pi_i(H) = \frac{\lambda(i+1)}{\lambda(i)} < 1, \quad 0 \leq i \leq n^* - 1\). Indeed, since \(\pi_i \cdot (v_l - v_h) = 0\) and \((u_{Lh} - u_{Hh}) - (u_{Ll} - u_{Hl}) > 0\), (46) implies that \(0 < \pi_i(H) < \bar{\mu}\). □

From \(0 < \pi_i(H) < \bar{\mu}\), it follows immediately that player 2’s playing \(l\) is optimal whenever he sees an \(L\). We need to show that it is optimal for player 2 to play \(h\) whenever he sees straight \(H\)’s. This requires that

\[
\frac{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k) \pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)} \geq \bar{\mu}, \quad 0 \leq i \leq n \quad (60)
\]

Lemma 19 If \(\frac{x+a}{y+b} > \frac{a}{b}\), with \(y, b > 0\), then \(\frac{x}{y} > \frac{x+a}{y+b}\). □

Proof. Let \(r = \frac{x+a}{y+b}\). Therefore, \(ry + rb = x + a\) and \(rb > a\). Thus, \(ry < x\). That is, \(\frac{x}{y} > \frac{x+a}{y+b}\). □

Lemma 20 If \(0 < \pi_i(H) < \bar{\mu}, \quad 0 \leq i \leq n^* - 1\), then (60) holds if and only if \(\mu_0 + (1 - \mu_0)(1 - \lambda(0)) \geq \bar{\mu}\).
**Proof.** The “only if” part holds by considering $i = 0$. To see the “if” part, notice that

$$
\mu_0 + (1 - \mu_0)(1 - \lambda(0)) = \frac{\mu_0 + (1 - \mu_0) \sum_{k=0}^{n^*} \lambda(k) \pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)}
$$

By applying Lemma 19, if \( \frac{\mu_0 + (1 - \mu_0) \sum_{k=0}^{n^*} \lambda(k) \pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)} \geq \bar{\mu} > \frac{(1 - \mu_0) \lambda(0) \pi_0(H)}{(1 - \mu_0) \lambda(0)} \), then

$$
\frac{\mu_0 + (1 - \mu_0) \sum_{k=2}^{n^*} \lambda(k) \pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)} > \frac{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k) \pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)} \geq \bar{\mu}
$$

The proof is completed by induction. Suppose we have shown

$$
\frac{\mu_0 + (1 - \mu_0) \sum_{k=i}^{n^*} \lambda(k) \pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=i}^{n^*} \lambda(k)} \geq \bar{\mu}, \ i \leq n^* - 1
$$

Then, since \( \bar{\mu} > \frac{(1 - \mu_0) \lambda(i) \pi_i(H)}{(1 - \mu_0) \lambda(i)} \), we have

$$
\frac{\mu_0 + (1 - \mu_0) \sum_{k=i+1}^{n^*} \lambda(k) \pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=i+1}^{n^*} \lambda(k)} \geq \bar{\mu}
$$

by applying Lemma 19. ■

**Simplification of IC Conditions in Case II**

**Proposition 9** The first set of IC conditions is met if and only if \( \mu_0 + (1 - \mu_0)(1 - \lambda(0)) \leq \bar{\mu} \). That is, \( l \) is optimal if player 2 does not acquire information.

This result is immediate from the following lemmata.

**Lemma 21** The solutions \( \pi_i, \ 1 \leq i \leq n^* - 1 \), are well-defined strategies if and only if \( \lambda(n^*) \geq 0 \).

**Proof.** The “only if” part is straightforward. We now show the “if” part. Substituting (43) into (53), we have

$$
(\lambda(i) - \lambda(i+1))(v_{Li} - v_{LH}) = \frac{c}{1 - \mu_0} + \lambda(i+1)(v_{Hi} - v_{HH}), \ 1 \leq i \leq n^* - 1.
$$

From this expression, \( \lambda(i+1) > 0 \) implies that \( \lambda(i) > 0, \ 1 \leq i \leq n^* - 1 \). Therefore, \( \lambda(n^*) > 0 \) implies that \( \lambda(i) > 0, \ 1 \leq i \leq n^* \), and that \( \lambda(i) \) is strictly decreasing in \( i \).
0 < \pi_i(H) = \frac{\lambda(i+1)}{\lambda(i)} < 1, 1 \leq i \leq n^* - 1. Indeed, since \bar{\pi} \cdot (v_l - v_h) = 0 and (u_{Lh} - u_{Hh}) - (u_{Li} - u_{Hi}) > 0, (53) implies that 0 < \pi_i(H) < \bar{\mu}, 1 \leq i \leq n^* - 1. ■

Note that (53) does not necessarily hold for i = 0. We therefore need to show that \lambda(0) > 0 and 0 < \pi_0(H) < \bar{\mu} separately.

**Lemma 23**

\[ \mu_0 + (1 - \mu_0)(1 - \lambda(0)) \leq \bar{\mu} \]

**Proof.** By the indifference condition for the proposed equilibrium, \( V^0(\pi, \sigma) = V^1(\pi, \sigma) \). That is,

\[
\mu_0 v_{Hi} + (1 - \mu_0) \sum_{k=0}^{n^*} \lambda(k)\pi_k \cdot v_l \n
= \mu_0 v_{Hh} + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k \cdot v_h + (1 - \mu_0)\lambda(0)\pi_0 \cdot v_l - c.
\]

Suppose \( \frac{\mu_0 + (1 - \mu_0)(1 - \lambda(0))}{\mu_0 + (1 - \mu_0)(1 - \lambda(0))} < \bar{\mu} \); i.e., \( \mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k(H) < \mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k) \). Then \( l \) is optimal for player 2 if he buys one bit of information and sees an \( H \). Thus,

\[
\mu_0 v_{Hh} + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k \cdot v_h + (1 - \mu_0)\lambda(0)\pi_0 \cdot v_l - c
\]

\[
< \mu_0 v_{Hi} + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k \cdot v_l + (1 - \mu_0)\lambda(0)\pi_0 \cdot v_l - c
\]

We then have

\[
\mu_0 v_{Hi} + (1 - \mu_0) \sum_{k=0}^{n^*} \lambda(k)\pi_k \cdot v_l
\]

\[
< \mu_0 v_{Hi} + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k \cdot v_l + (1 - \mu_0)\lambda(0)\pi_0 \cdot v_l - c
\]

Therefore, \( c < 0 \). A contradiction. ■

**Lemma 23** If \( \mu_0 + (1 - \mu_0)(1 - \lambda(0)) \leq \bar{\mu} \), then \( \pi_0(H) < \bar{\mu} \).

**Proof.** Suppose to the contrary that \( \lambda(0) \leq 0 \). Then, \( \mu_0 + (1 - \mu_0)(1 - \lambda(0)) \geq 1 \). Therefore, \( \bar{\mu} \geq \mu_0 + (1 - \mu_0)(1 - \lambda(0)) \geq 1 \), which is a contradiction. Given \( \bar{\mu} \leq \frac{\mu_0 + (1 - \mu_0)(1 - \lambda(0))}{\mu_0 + (1 - \mu_0)(1 - \lambda(0))} \)

(Lemma 22), we have

\[ \frac{1 - \bar{\mu}}{1 - \mu_0} \mu_0 + (1 - \mu) \geq \frac{\lambda(1)}{1 - \mu_0} + (1 - \mu)\lambda(0) > \lambda(1) + (1 - \bar{\mu})\lambda(0) \quad (61) \]

Suppose to the contrary that \( \pi_0(H) = \frac{\lambda(1)}{\lambda(0)} \geq \bar{\mu} \). Then, \( \lambda(1) + (1 - \bar{\mu})\lambda(0) \geq \lambda(0) \). It follows from (61) that \( \frac{1 - \bar{\mu}}{1 - \mu_0} \mu_0 + (1 - \mu) > \lambda(0) \). That is,

\[ \mu_0 + (1 - \mu_0)(1 - \lambda(0)) > \bar{\mu} \]

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We have a contradiction. ■

From Lemmata 21 and 23, \(0 < \pi_i(H) < \bar{\mu}, i \in \{0, 2, ..., n^* - 1\}\). It follows immediately that player 2 playing \(l\) is optimal whenever he sees an \(L\). We also need to show that player 2 playing \(h\) is optimal whenever he sees straight \(H\)'s. This requires that for \(0 \leq i \leq n^*\),

\[
\frac{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)} \geq \bar{\mu}
\]  

(62)

**Lemma 24** (62) holds.

**Proof.** Let us show that if \(\frac{\mu_0 + (1 - \mu_0)(1 - \lambda(0) - \lambda(1))}{\mu_0 + (1 - \mu_0)(1 - \lambda(0))} \geq \bar{\mu}\), then (62) holds. The lemma will then follow from Lemma 22. Since

\[
\frac{\mu_0 + (1 - \mu_0)(1 - \lambda(0) - \lambda(1))}{\mu_0 + (1 - \mu_0)(1 - \lambda(0))} = \frac{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)}
\]

and \(\bar{\mu} > \frac{(1 - \mu_0)\pi_1(H)}{\lambda_1}\), it follows from Lemma 19 that if \(\frac{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)} \geq \bar{\mu}\), then

\[
\frac{\mu_0 + (1 - \mu_0) \sum_{k=2}^{n^*} \lambda(k)\pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=2}^{n^*} \lambda(k)} \geq \frac{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)\pi_k(H)}{\mu_0 + (1 - \mu_0) \sum_{k=1}^{n^*} \lambda(k)} \geq \bar{\mu}
\]

That is, if (62) holds for \(i = 1\), it holds for \(i = 2\). The result follows by induction. ■

**References**


