Contests with Identity-Dependent Externalities

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Abstract. In an \( n \)-player contest with externalities, each player has a multidimensional payoff function. This player’s payoff function specifies the payoff she would get for each of the \( n \) possible outcomes: player \( i \) \( (i = 1, \ldots, n) \) gets the prize. We provide conditions for existence and uniqueness of full-participation equilibrium where each player exerts positive effort.

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1. Introduction

Traditionally, the focus of contest theory has been on models that view contests as isolated events.\(^1\) However, most often contests are parts of larger environment. In many such economic environments, contestants care about not only winning the contest, but also the prize allocation in case of losing. This situation is known in the literature as identity-dependent externalities. There are many examples of contests with identity-dependent externalities in sports, rent seeking, lobbying etc. See examples in Funk (1996), Jehiel, Moldovanu, and Stacchetti (1996), Das Varma (2002), Klose and Kovenock (2011).

In this paper we analyze contests with identity-dependent externalities. We suggest a new approach how to analyze contests. Our approach is based on a particular set of determinants. First, we calculate each determinant in this set. Then, we show how to find a full-participation equilibrium based on the calculated determinants.

It is well-known in the contest literature that some modifications to standard contests might lead to multiple equilibria, or even to non-existence of an equilibrium. See, for example, Cohen and Sela (2005), Chowdhury and Sheremeta (2011), Klose and Kovenock (2011). That is why existence and uniqueness of equilibrium deserve serious attention in Cornes and Hartley (2005), Yamazaki (2008). The problem of existence and uniqueness of equilibrium in contests with identity-dependent externalities is difficult to address. Klose and Kovenock (2011) consider all-pay auctions with

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\(^1\)See, for example, Tullock (1980), Nitzan (1994), Congleton, Hillman, and Konrad (2008), Konrad (2009).

1
identity-dependent externalities and demonstrate that strict conditions are required for the existence of equilibrium. We demonstrate that our set of determinants can establish both the existence and uniqueness of full-participation equilibrium. In order to guarantee a unique full-participation equilibrium, all determinants in our set of determinants have to have the same sign. This simple requirement and a standard assumption that all players prefer winning to losing are sufficient.

Our new approach is very general. We demonstrate how to find equilibria in the classic symmetric and asymmetric contests without externalities based on our set of determinants.

Finally, we analyze a particular $n = 3$ player case. It turns out that in this case our approach helps to find a unique equilibrium which might or might not be the full-participation equilibrium.

The rest of the paper is organized as follows. Section 2 presents the Model. Applications of the model are discussed in Section 3. Section 4 analyzes the case of $n = 3$ players. Concluding remarks are given in Section 5.

2. THE MODEL

Consider an $n$-player contest with externalities. We assume that $n$ risk-neutral players spend resources simultaneously in order to win one prize. If player $i$ wins the prize, she gets payoff $V_{ii}$. If player $i$ does not win the prize, her payoff depends on the identity of the winner: she obtains payoff $V_{ij}$ if player $j$ wins. Therefore, player $i$’s payoff function is an $n$-dimensional vector $V_i = (V_{i1}, ..., V_{in})$. We assume that players’ payoff functions are common knowledge.

Formally, player $i$ exerts effort $x_i$ in order to maximize the following expression:

$$\max_{x_i \geq 0} \left( \frac{x_1}{\sum_{j=1}^{n} x_j} V_{i1} + \ldots + \frac{x_i}{\sum_{j=1}^{n} x_j} V_{ii} + \ldots + \frac{x_n}{\sum_{j=1}^{n} x_j} V_{in} \right) - x_i,$$  

(1)

where $\frac{x_l}{\sum_{j=1}^{n} x_j}$ is the probability that player $l$ wins the contest and player $i$ gets payoff $V_{il}$ where $l = 1, ..., n$. The last term in (1) is the cost of effort.²

2.1. Full-Participation Equilibrium. The first order conditions of maximization problem (1) are:

$$- \frac{x_1 V_{i1}}{(\sum_{j=1}^{n} x_j)^2} - \ldots - \frac{\sum_{j \neq i} x_j}{(\sum_{j=1}^{n} x_j)^2} V_{ii} - \ldots - \frac{x_n V_{in}}{(\sum_{j=1}^{n} x_j)^2} = 1.$$  

(2)

²We assume that if $x_1 = ... = x_n = 0$, then $\frac{x_l}{\sum_{j=1}^{n} x_j} \equiv 1/n$ for any $l = 1, ..., n$. 
The second order conditions are
\[
\frac{2x_1 V_{i1}}{\left(\sum_{j=1}^{n} x_j\right)^3} + \ldots + \frac{(-2) \sum_{j \neq i} x_j V_{ii}}{\left(\sum_{j=1}^{n} x_j\right)^3} + \ldots + \frac{2x_n V_{in}}{\left(\sum_{j=1}^{n} x_j\right)^3} = 0
\]
\[
\frac{2}{\left(\sum_{j=1}^{n} x_j\right)^3} \left[ x_1 V_{i1} + \ldots + (-1) \left(\sum_{j \neq i} x_j\right) V_{ii} + \ldots + x_n V_{in}\right] = 0
\]
\[
\frac{-2}{\left(\sum_{j=1}^{n} x_j\right)^3} \left[ x_1 (V_{ii} - V_{i1}) + \ldots + x_n (V_{ii} - V_{in})\right] \leq 0. \quad (3)
\]

Denote
\[
S \equiv \sum_{j=1}^{n} x_j.
\]

Then, (2) becomes
\[
x_1 (V_{ii} - V_{i1}) + \ldots + x_n (V_{ii} - V_{in}) = S^2.
\]

Hence, we get the following system of linear equations:
\[
\begin{pmatrix}
0 & (V_{11} - V_{12}) & \ldots & (V_{11} - V_{1(n-1)}) & (V_{11} - V_{1n}) \\
(V_{22} - V_{21}) & 0 & \ldots & (V_{22} - V_{2(n-1)}) & (V_{22} - V_{2n}) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(V_{nn} - V_{n1}) & \ldots & \ldots & (V_{nn} - V_{n(n-1)}) & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= \begin{pmatrix} S^2 \\
S^2 \\
\vdots \\
S^2
\end{pmatrix}.
\]

By Cramer's rule,
\[
x_i = \frac{\begin{vmatrix}
0 & (V_{11} - V_{12}) & \ldots & S^2 & \ldots & (V_{11} - V_{1(n-1)}) & (V_{11} - V_{1n}) \\
(V_{22} - V_{21}) & 0 & \ldots & S^2 & \ldots & (V_{22} - V_{2(n-1)}) & (V_{22} - V_{2n}) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
(V_{nn} - V_{n1}) & \ldots & \ldots & S^2 & \ldots & (V_{nn} - V_{n(n-1)}) & 0
\end{vmatrix}}{\begin{vmatrix}
0 & (V_{11} - V_{12}) & \ldots & (V_{11} - V_{1(n-1)}) & (V_{11} - V_{1n}) \\
(V_{22} - V_{21}) & 0 & \ldots & (V_{22} - V_{2(n-1)}) & (V_{22} - V_{2n}) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(V_{nn} - V_{n1}) & \ldots & \ldots & (V_{nn} - V_{n(n-1)}) & 0
\end{vmatrix}}.
\]
Contests with Identity-Dependent Externalities

\[
S = S^2 \left[ \frac{|A_i|}{|A|} \right]
\]

where

\[
|A_i| = \begin{vmatrix}
0 & (V_{11} - V_{12}) & \cdots & 1 & \cdots & (V_{11} - V_{1(n-1)}) & (V_{11} - V_{1n}) \\
(V_{22} - V_{21}) & 0 & \cdots & 1 & \cdots & (V_{22} - V_{2(n-1)}) & (V_{22} - V_{2n}) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
(V_{nn} - V_{n1}) & (V_{nn} - V_{n2}) & \cdots & 1 & \cdots & (V_{nn} - V_{n(n-1)}) & 0 \\
\end{vmatrix}
\]

\[
|A| = \begin{vmatrix}
0 & (V_{11} - V_{12}) & \cdots & 1 & \cdots & (V_{11} - V_{1(n-1)}) & (V_{11} - V_{1n}) \\
(V_{22} - V_{21}) & 0 & \cdots & 1 & \cdots & (V_{22} - V_{2(n-1)}) & (V_{22} - V_{2n}) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
(V_{nn} - V_{n1}) & (V_{nn} - V_{n2}) & \cdots & 1 & \cdots & (V_{nn} - V_{n(n-1)}) & 0 \\
\end{vmatrix}
\]

Therefore,

\[
S = x_1 + \ldots + x_n = S^2 \frac{|A_1| + \ldots + |A_n|}{|A|}
\]

and

\[
S = \frac{|A|}{|A_1| + \ldots + |A_n|}
\]

We can find equilibrium spending now.

\[
x_i = S^2 \frac{|A_i|}{|A|} = \frac{|A|}{(|A_1| + \ldots + |A_n|)^2} |A_i|
\]

Since we are looking for a full-participation equilibrium, it has to be \(x_i > 0\) for any \(i\). It means that we need to impose the following conditions:

\(|A_i| |A_i| > 0\) for any \(i\),
or

\[ \text{sign} |A_i| = \text{sign} |A|. \]

In a full-participation equilibrium with \( n \) active players, \( S > 0 \), and therefore

\[ |A| \neq 0. \]

We can state the main result now.

**Theorem 1.** Suppose that

\[ V_{ii} > V_{ij} \text{ for any } j \neq i \text{ and any } i = 1, \ldots, n \]

and

\[ |A| |A_i| > 0 \text{ for any } i = 1, \ldots, n. \]  \hspace{1cm} (8)

Then, there exists a unique full-participation Nash equilibrium in pure strategies, where player \( i \) spends

\[ x_i = \frac{|A|}{(|A_1| + \ldots + |A_n|)^2} |A_i|, \]  \hspace{1cm} (9)

where \( |A_i| \) and \( |A| \) are defined in (4) and (5).

3. Applications

In this section we obtain several classic contest results based on formula (9).

3.1. No externalities. Suppose that there are no externalities: \( V_{jk} = 0 \) for \( j \neq k \).

There are two standard models in this case. Tullock (1980) analyzes an \( n \)-player contest with the same prize value for all contestants. Hillman and Riley (1989) consider an \( n \)-player contest in which players have different prize values.

**Symmetric Contest: Tullock (1980).** Consider symmetric contest among \( n \) risk-neutral players. Players exert effort simultaneously in order to win one main prize. In our notation, it means

\[ V_{jk} = 0 \text{ for } j \neq k \]

and

\[ V_{11} = \ldots = V_{nn} = V > 0. \]
Then,
\[
|A| = \begin{vmatrix}
0 & V & \cdots & V & V \\
V & 0 & \cdots & V & V \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
V & V & \cdots & V & 0
\end{vmatrix}
\]
and
\[
|A_i| = \begin{vmatrix}
0 & V & \cdots & 1 & \cdots & V & V \\
V & 0 & \cdots & 1 & \cdots & V & V \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
V & V & \cdots & 1 & \cdots & V & 0
\end{vmatrix}
\]

We will use the following result.

**Lemma 1.**

\[
W(n) \equiv \begin{vmatrix}
0 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0
\end{vmatrix} = (-1)^{n-1} (n - 1)
\]

and

\[
W_i(n) \equiv \begin{vmatrix}
0 & 1 & \cdots & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & \cdots & 0 & 1 \\
1 & 1 & \cdots & 1 & \cdots & 1 & 0
\end{vmatrix} = (-1)^{n-1}.
\]

**Proof.** See the Appendix.

Note that
\[
|A| = V^n W(n) = (-1)^{n-1} (n - 1) V^n,
\]

and
\[
|A_i| = V^n W_i(n) = (-1)^{n-1} V^{n-1}.
\]
Since condition (8) holds, from Theorem 1 and expressions (10) and (11), we get Tullock’s equilibrium prediction:

\[ x_i = \frac{|A|}{(|A_1| + \ldots + |A_n|)^2} |A_i| = \frac{(n-1)}{n^2} V \text{ for any } i = 1, \ldots, n. \]

**Asymmetric Contest: Hillman and Riley (1989).** Consider asymmetric contest among \( n \) risk-neutral players. Players exert effort simultaneously in order to win one main prize. Players’ prize values differ.

Define the harmonic mean of the \( k \) largest valuations as

\[ \hat{v}_k \equiv \frac{k}{\sum_{j=1}^{k} \frac{1}{v_{jj}}}, \text{ for } k = 1, \ldots, n. \]  \hspace{1cm} (12)

Hillman and Riley (1989) derive the following condition for the full-participation equilibrium:

\[ V_{nn} > \frac{n-2}{n-1} \hat{v}_{n-1}. \]  \hspace{1cm} (13)

Stein (2002) makes one more step and describes equilibrium strategies:

\[ x_i = \frac{(n-1)}{n} \hat{v}_n - \left( \frac{(n-1)}{n} \hat{v}_n \right)^2 \frac{1}{V_{ii}}. \]  \hspace{1cm} (14)

In our notation, asymmetric contest means

\[ V_{jk} = 0 \text{ for } j \neq k \]

and

\[ V_{11} \geq \ldots \geq V_{nn} > 0. \]

Then,

\[ |A| = \begin{vmatrix} 0 & V_{11} & \ldots & V_{11} & V_{11} \\ V_{22} & 0 & \ldots & V_{22} & V_{22} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ V_{nn} & V_{nn} & \ldots & V_{nn} & 0 \end{vmatrix} = (V_{11} \times \ldots \times V_{nn}) W(n) = (-1)^{n-1} (n-1) (V_{11} \times \ldots \times V_{nn}), \]

and

\[ |A_i| = \begin{vmatrix} 0 & V_{11} & \ldots & 1 & \ldots & V_{11} & V_{11} \\ V_{22} & 0 & \ldots & 1 & \ldots & V_{22} & V_{22} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ V_{nn} & V_{nn} & \ldots & 1 & \ldots & V_{nn} & 0 \end{vmatrix}. \]

We will use the following result.
Lemma 2.

\[ T_i(v_1, \ldots, v_n) \equiv \begin{vmatrix} 0 & 1 & \cdots & v_1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & v_2 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & v_n & \cdots & 1 & 0 \end{vmatrix} = (-1)^n \left( (n - 2) v_i - \sum_{j \neq i} v_j \right). \]

**Proof.** See the Appendix.

Note that

\[ |A_i| = (V_{11} \times \ldots \times V_{nn}) T_i \left( \frac{1}{V_{11}}, \ldots, \frac{1}{V_{nn}} \right) = (-1)^n \left( V_{11} \times \ldots \times V_{nn} \right) \left( (n - 2) \frac{1}{V_{ii}} - \sum_{j \neq i} \frac{1}{V_{jj}} \right). \]

Now, we can apply Theorem 1. Condition (8) takes care of full participation and (9) describes equilibrium spending:

\[ x_i = \frac{|A|}{(|A_1| + \ldots + |A_n|)^2} |A_i| = \]

\[ \frac{(-1)(n-1)}{\left(\sum_{j=1}^{n} \frac{1}{V_{jj}}\right)^2} \left( \frac{n - 2}{V_{ii}} - \frac{n - 1}{\hat{v}_i} \right) = \]

\[ \frac{(n - 1)}{\left(\sum_{j=1}^{n} \frac{1}{V_{jj}}\right)} \left( \frac{(n - 1)}{\left(\sum_{j=1}^{n} \frac{1}{V_{jj}}\right)} \right)^2 \frac{1}{V_{ii}}, \]

which is Stein’s equilibrium spending (14). Condition (8)

\[ 0 < |A| \ |A_i| = \]

\[ = (-1)^{n-1} (n - 1) (V_{11} \times \ldots \times V_{nn}) \times (-1)^n (V_{11} \times \ldots \times V_{nn}) \left( (n - 2) \frac{1}{V_{ii}} - \sum_{j \neq i} \frac{1}{V_{jj}} \right) \]

\[ = (-1)(n - 1) (V_{11} \times \ldots \times V_{nn})^2 \left( (n - 2) \frac{1}{V_{ii}} - \sum_{j \neq i} \frac{1}{V_{jj}} \right). \]
is equivalent to
\[
(n - 2) \frac{1}{V_{ii}} - \sum_{j \neq i} \frac{1}{V_{jj}} < 0,
\]
or
\[
\frac{(n - 2)}{\sum_{j \neq i} \frac{1}{V_{jj}}} < V_{ii},
\]
which is Hillman and Riley’s full-participation equilibrium condition (13).

4. \( n = 3 \) Players

Klose and Kovenock (2011) illustrate non-existence of an equilibrium in all-pay auctions with Identity-Dependent Externalities in the case of \( n = 3 \). In this section, we show that a contest with Identity-Dependent Externalities always has a unique pure-strategy equilibrium.

If \( n = 3 \), then
\[
|A| = \begin{vmatrix}
0 & (V_{11} - V_{12}) & (V_{11} - V_{13}) \\
(V_{22} - V_{21}) & 0 & (V_{22} - V_{23}) \\
(V_{33} - V_{31}) & (V_{33} - V_{32}) & 0
\end{vmatrix}.
\]

Define \( |A(-i)| \) as a determinant \( |A| \) without column \( i \) and row \( i \). For example,
\[
|A(-1)| = \begin{vmatrix}
0 & (V_{22} - V_{23}) \\
(V_{33} - V_{32}) & 0
\end{vmatrix}.
\]

Now, we can present the main result of this section.

**Theorem 2.** Suppose that \( n = 3 \),

\[ V_{ii} > V_{ij} \text{ for any } j \neq i \text{ and any } i = 1, 2, 3. \]

Then, there exists a unique Nash equilibrium in pure strategies. If

\[ |A_i| > 0 \text{ for any } i = 1, 2, 3, \]

then in the equilibrium player \( i \) spends

\[ x_i = \frac{|A|}{(|A_1| + |A_2| + |A_3|)^2 |A_i|}. \]

If

\[ |A_i| \leq 0 \text{ for some } i = 1, 2, 3, \]
then in the equilibrium player $i$ spends zero, and the other players spend

$$x_j = \frac{|A(-i)|}{(|A_j(-i)| + |A_k(-i)|)^2} |A_j(-i)|,$$

where $i \neq j, k$.

**Proof.** Here is the intuition for the proof.

First, note that $|A| > 0$.

Second, if $|A_i| > 0$ for any $i = 1, 2, 3$, then the statement follows from Theorem 1.

Third, note that at most one player $i$ can have $|A_i| \leq 0$. This player spends zero in the equilibrium. The other two players participate actively in the contest.

The following examples illustrate the Theorem.

**Example 1.** Suppose that

$$|A| = \begin{vmatrix} 0 & 1 & 3 \\ 2 & 0 & 4 \\ 5 & 5 & 0 \end{vmatrix} = 15 + 20 = 35$$

and

$$|A_1| \equiv \begin{vmatrix} 1 & 1 & 3 \\ 1 & 0 & 4 \\ 1 & 5 & 0 \end{vmatrix} = 4 + 15 - 20 = -1,$$

$$|A_2| \equiv \begin{vmatrix} 0 & 1 & 3 \\ 2 & 1 & 4 \\ 5 & 1 & 0 \end{vmatrix} = 20 + 6 - 15 = 11,$$

$$|A_3| \equiv \begin{vmatrix} 0 & 1 & 1 \\ 2 & 0 & 1 \\ 5 & 5 & 1 \end{vmatrix} = 5 + 10 - 2 = 13.$$

Then, player 1 spends zero in equilibrium. Consider players 2 and 3.

$$|A(-1)| \equiv \begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} = -20$$

and

$$|A_2(-1)| \equiv \begin{vmatrix} 1 & 4 \\ 1 & 0 \end{vmatrix} = -4$$
Contests with Identity-Dependent Externalities

\[ |A_3(-1)| \equiv \begin{vmatrix} 0 & 1 \\ 5 & 1 \end{vmatrix} = -5. \]

In the equilibrium

\[ x_1 = 0, \]
\[ x_2 = \frac{|A(-1)|}{(|A_2(-1)| + |A_3(-1)|)^2} |A_2(-1)| = \frac{-20}{(-4-5)^2} (-4) = \frac{80}{81}, \]
\[ x_3 = \frac{|A(-1)|}{(|A_2(-1)| + |A_3(-1)|)^2} |A_3(-1)| = \frac{100}{81}. \]

**Example 2.** Suppose that

\[ |A| \equiv \begin{vmatrix} 0 & 3 & 1 \\ 3 & 0 & 1 \\ 2 & 2 & 0 \end{vmatrix} = 6 + 6 = 12 \]

and

\[ |A_1| \equiv \begin{vmatrix} 1 & 3 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 2 + 3 - 2 = 3, \]
\[ |A_2| \equiv \begin{vmatrix} 0 & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = 2 + 3 - 2 = 3, \]
\[ |A_3| \equiv \begin{vmatrix} 0 & 3 & 1 \\ 3 & 0 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 6 + 6 - 9 = 3. \]

Therefore,

\[ x_1 = \frac{|A|}{(|A_1| + |A_2| + |A_3|)^2} |A_1| = \frac{12}{(3 + 3 + 3)^2} 3 = \frac{4}{9}, \]
\[ x_2 = \frac{|A|}{(|A_1| + |A_2| + |A_3|)^2} |A_2| = \frac{12}{(3 + 3 + 3)^2} 3 = \frac{4}{9}, \]
\[ x_3 = \frac{|A|}{(|A_1| + |A_2| + |A_3|)^2} |A_3| = \frac{12}{(3 + 3 + 3)^2} 3 = \frac{4}{9}. \]
5. Conclusion

We consider contests with identity-dependent externalities. We suggest a new approach to analyze contests based on a particular set of determinants. Our approach allows to provide conditions for existence and uniqueness of the full-participation equilibrium for any number of players. It turns out that standard classic equilibrium results for symmetric and asymmetric contests are particular cases in our approach. In the case of \( n = 3 \) players we characterize a unique equilibrium which might be with or without full participation.

6. Appendix A

Proof of Lemma 1.

We want to prove that

\[
W(n) \equiv \begin{pmatrix}
0 & 1 & \cdots & 1 & 1 \\
1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix} = (-1)^{n-1}(n-1),
\]

for \( n \geq 2 \).

First, we multiply the second row of \( W(n) \) by \((-1)\) and add it to the first row:

\[
W(n) = \begin{pmatrix}
-1 & 1 & \cdots & 0 & 0 \\
1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & 0
\end{pmatrix}.
\]

Then, we multiply the third row by \((-1)\) and add it to the second row and so on:

\[
W(n) = \begin{pmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 0
\end{pmatrix}.
\]
Since only one element in the last column is not equal to zero, we get

\[
W(n) = (-1)^{2^{n-1}}
\begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1
\end{bmatrix}_{(n-1) \times (n-1)}.
\]

We multiply the row before last by \((-1)\) and add it to the last row:

\[
W(n) = (-1)^{2(n-1)-1}
\begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 2
\end{bmatrix}_{(n-2) \times (n-2)}.
\]

Since only one element in the last column is not equal to zero, we get

\[
W(n) = (-1) \times (-1)^{2(n-1)-1}
\begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & 2
\end{bmatrix}_{(n-2) \times (n-2)}.
\]

We multiply the row before last by \((-2)\) and add it to the last row:

\[
W(n) = (-1)^{2(n-2)}
\begin{bmatrix}
-1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
1 & 1 & 1 & \cdots & 1 & 3 & 0
\end{bmatrix}_{(n-2) \times (n-2)}.
\]
We can repeat the same steps and obtain

\[ W(n) = (-1)^{(n-2)} \begin{vmatrix} -1 & 1 \\ n-1 & 0 \end{vmatrix} = (-1)^{(n-1)}(n-1). \]

Similar argument leads to the second statement of Lemma 1:

\[ W_i(n) \equiv \begin{vmatrix} 0 & 1 & \cdots & 1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & \cdots & 1 & 0 \\ 1 & 1 & \cdots & 1 & \cdots & 1 & 0 \end{vmatrix} = (-1)^{n-1}. \]

\[ \square \]

**Proof of Lemma 2.**

We want to prove that

\[ T_i(v_1, \ldots, v_n) \equiv \begin{vmatrix} 0 & 1 & \cdots & v_1 & \cdots & 1 & 1 \\ 1 & 0 & \cdots & v_2 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & v_n & \cdots & 1 & 0 \end{vmatrix} = (-1)^n \left( (n-2) v_i - \sum_{j \neq i} v_j \right). \]

for \( n \geq 2 \).

First, we multiply the second row of \( T_i(v_1, \ldots, v_n) \) by \((-1)\) and add it to the first row:

\[ T_i(v_1, \ldots, v_n) = \begin{vmatrix} v_1 - v_2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ v_2 & 0 & 1 & \cdots & 1 & 1 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_n & 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix}. \]

Then, we multiply the third row by \((-1)\) and add it to the second row and so on:

\[ T_i(v_1, \ldots, v_n) = \begin{vmatrix} v_1 - v_2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ v_2 - v_3 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ v_n & 1 & 1 & \cdots & 1 & 1 & 0 \end{vmatrix} = \]
Contests with Identity-Dependent Externalities

\[
\begin{bmatrix}
  v_1 - v_2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\
  v_2 - v_3 & -1 & 1 & 0 & \cdots & 0 & 0 & 0 \\
  \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
  v_{n-1} - v_n & 0 & 0 & 0 & \cdots & 0 & -1 & 1 \\
  v_n & 1 & 1 & 1 & \cdots & 1 & 1 & 0
\end{bmatrix}
\]

Since only one element in the last column is not equal to zero, we get

\[
T_i(v_1, \ldots, v_n) = (-1)^{2n-1}
\]

We multiply the row before last by \((-1)\) and add it to the last row:

\[
T_i(v_1, \ldots, v_n) = (-1)
\]

Since only one element in the last column is not equal to zero, we get

\[
T_i(v_1, \ldots, v_n) = (-1) \times (-1)^{2(n-1)-1}
\]

We multiply the row before last by \((-2)\) and add it to the last row:

\[
T_i(v_1, \ldots, v_n) = (-1)^2
\]
We can repeat the same steps and obtain

\[
T_i(v_1, \ldots, v_n) = (-1)^2 \begin{vmatrix}
    v_1 - v_2 & 1 & 0 & \cdots & 0 & 0 & 0 \\
    v_2 - v_3 & -1 & 1 & \cdots & 0 & 0 & 0 \\
    \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
    v_{n-3} - v_{n-2} & 0 & 0 & \cdots & 0 & -1 & 1 \\
    v_n + v_{n-1} + v_{n-2} - 2v_{n-3} & 1 & 1 & \cdots & 1 & 3 & 0
\end{vmatrix}
\]

\[
(-1)^{n-2} \begin{vmatrix}
    v_1 - v_2 & 1 & 0 \\
    v_2 - v_3 & -1 & 1 \\
    v_n + \cdots + v_3 - (n-3)v_2 & (n-2) & 0
\end{vmatrix} = \]

\[
(-1)^{n-1} \begin{vmatrix}
    v_1 - v_2 & \frac{1}{(n-2)} \\
    v_n + \cdots + v_3 - (n-3)v_2 & (n-2)
\end{vmatrix} = (-1)^{n-1} \left( (n-2) v_1 - (v_2 + \cdots + v_n) \right).
\]

References


