Dynamic Indeterminacy and Welfare in Credit Economies

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Abstract

We characterize the set of dynamic equilibria of a pure credit economy with random matching and limited commitment. For standard trading mechanisms there are a continuum of steady states, a continuum of credit cycle equilibria of any periodicity, a subset of which yield a higher welfare than the ones singled out in the literature, and a continuum of sunspot equilibria. The set of equilibria expands as agents become more patient, trading opportunities are more frequent, and borrowers have more bargaining power. We characterize the constrained-efficient allocations under both pairwise and centralized meetings, and we establish conditions under which the second welfare theorem of Alvarez and Jermann (2000) fails to apply to our economy, i.e., constrained-efficient allocations cannot be implemented with "not-too-tight" solvency constraints.

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1 Introduction

The inability of individuals to commit to honor their future obligations is a key friction of decentralized economies. The lack of commitment jeopardizes the Arrow-Debreu apparatus based on the exchange of securities backed by promises to deliver goods at different dates and in different states. In Kiyotaki and Moore’s (2001, p.9) words,

"Surprisingly, in Arrow-Debreu the time dimension is treated on a par with the type dimension. Trust is ignored. Implicitly, it is assumed either that all economic agents are entirely trustworthy, or that the auctioneer can wield a stick that is so big no-one dare renege on a promise. (...) Factoring in a lack of trust—placing a limitation on the degree of commitment—is of primary importance."

Economies where anonymous individuals are subject to the temptation to renege on their promises have been the primary focus of monetary theory. In pure monetary economies all trades are *quid pro quo* and are mediated with money—credit is (incentive-) infeasible. The opposite limiting case—the pure credit (or cashless) economy—occurs when trades are conducted with credit only and money does not circulate.\(^1\) A rationale for omitting money is that technological advances in record-keeping technologies facilitate the use of credit and reduce the usefulness of currency. Such advances, however, do not make agents entirely trustworthy. In this paper we study a cashless economy—its full set of equilibria and their normative properties—taking seriously the limited commitment friction.

There are two recent contributions, one normative and one positive, that shed some light on non-monetary economies with limited commitment. On the normative side, Alvarez and Jermann (2000), AJ thereafter, establish a second welfare theorem for a pure exchange, one-good economy where agents are subject to endowment shocks and have limited commitment—a special case of the environment in Kehoe and Levine (1993), KL thereafter. They prove that constrained-efficient allocations can be implemented by competitive trades subject to "not-too-tight" endogeneous solvency constraints. These constraints specify that in every period agents can issue the maximum amount of debt that is incentive-compatible with no default, thereby allowing as much risk sharing as possible. From a positive perspective Gu et al. (2013b), G2MW thereafter, study a pure credit economy subject to the same "not-too-tight" solvency constraints and show the possibility

\(^1\) The notion of pure credit economy dates back to Wicksell’s (1936) description of pure credit systems.

"We intend therefor, as a basis for the following discussion, to imagine a state of affairs in which money does not actually circulate at all, neither in the form of coin . . . nor in the form of notes, but where all domestic payments are effected by means of the Giro system and bookkeeping transfers. A thorough analysis of this purely imaginary case seems to me to be worth while, for it provides a precise antithesis to the equally imaginary case of a pure cash system, in which credit plays no part whatever." 

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of endogenous cycles and other exotic dynamics. The conditions for such cycles, however, are much more stringent than the ones in pure monetary economies. For instance, they find that some economies are prone to cycles under monetary exchange (e.g., Lagos and Wright, 2003) but exhibit no such dynamics under credit.2

The objective of this paper is to revisit these two key insights—the implementation of constrained-efficient allocations with endogenous solvency constraints and the existence of endogenous credit cycles—in the context of a pure credit economy with limited commitment. Our main contributions are twofold. On the positive side, we give a complete characterization of the equilibrium set of a pure credit economy. We impose no restriction on equilibria beyond and above the standard requirements of sequential rationality and belief consistency. On the normative side, we characterize the constrained-efficient allocations under two notions of core (for pairwise meetings and for large meetings).

The pure credit economy we consider features random matching—in pairwise meetings or in large groups—and incorporates intertemporal gains from trade that can be exploited with simple one-period debt contracts. In the absence of public record keeping, the environment corresponds to the New-Monetarist framework of Lagos and Wright (2005) so that one can easily compare allocations in credit and monetary economies. In the presence of a public record keeping technology the environment is mathematically equivalent to the one in G2MW.3 Moreover, the environment is readily amenable to game-theoretic analysis and mechanism design, which makes it suitable for both positive and normative investigations.

We start with a simple mechanism where the borrower in each bilateral match sets the terms of the loan contract unilaterally, which allows us to analyze the economy as a standard infinitely-repeated game with imperfect monitoring. If we impose the AJ "not-too-tight" solvency constraints—which amounts to restricting strategies and beliefs such that any form of default is punished with permanent autarky—then there is a unique active steady-state equilibrium and no equilibria with endogenous cycles. When we look for all perfect Bayesian equilibria, we find a continuum of steady-state equilibria, a continuum of periodic equilibria of any periodicity, and much more. Each equilibrium can be reduced to a sequence of debt limits, where the debt limit in a period specifies the amount that agents can be trusted to repay. Moreover, there is a wide variety of outcomes: in some credit cycle equilibria debt limits are binding in all periods, in other equilibria they are never binding, or they bind periodically. These results are robust to the choice of the mechanism to determine the terms of the loan contract—Nash or proportional bargaining, or even Walrasian

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2Both Lagos and Wright (2003) and Rocheteau and Wright (2013) find that monetary economies can generate endogenous cycles under monotone trading mechanisms, such as buyers-take-all or proportional bargaining solutions. Under the same trading mechanisms, Gu et al. (2013b) do not find any cycle.

3As we discuss later in details, there are differences regarding the timing of production that are inconsequential.
pricing if agents meet in large groups.

The large multiplicity of credit equilibria captures the basic notion that trust is a self-fulfilling phenomenon. To that extent, and following Mailath and Samuelson (2004, p.9), "we consider multiple equilibria a virtue." But this multiplicity does not imply that everything goes. Fundamentals, including preferences and market structure, do matter for the feasibility of some outcomes. We show that the set of credit-cycle equilibria expands as trading frictions are reduced, agents are more patient, and borrowers have more bargaining power.

We also show that for a given trading mechanism the set of outcomes of a pure monetary economy (with fiat money but no record keeping) is a strict subset of the outcomes of a pure credit economy (with record keeping but no fiat money). So the exotic dynamics of monetary economies (e.g., cycles, chaos...) also exist in credit environments. The reverse is not true. There are outcomes of the pure credit economy that cannot be sustained as outcomes of the pure monetary economy. For instance, there are equilibria where credit and output shut down periodically. A simple backward-induction logic rules such outcomes out in monetary economies.

In order to understand why the equilibrium set for credit economies is so vastly larger than the one found in G2MW (and related papers) it is worth recalling that the AJ "not-too-tight" solvency constraints were meant to provide a way to decentralize constrained-efficient allocations in an economy with limited commitment. Such constraints are not warranted for positive analysis. We avoid arbitrary restrictions on the set of equilibrium outcomes by following the repeated-game literature and by working with simple strategies that punish both default and excessive lending (i.e., lending in excess to the amount that is deemed trustworthy along the equilibrium path). Any failure to repay a loan that is no greater than the debt limit in the current period triggers permanent autarky for the borrower. However, if a lender agrees on a loan larger than the debt limit, provided that the borrower repays at least the debt limit, then the borrower remains trustworthy to future lenders. We will show that such simple strategies implement the full set of outcomes of perfect Bayesian equilibria (subject to mild restrictions).

In terms of normative analysis we determine the constrained-efficient allocations under different assumptions regarding the physical environment: agents meet in pairs or in large groups—an assumption about feasible communications among agents—and we vary an agent’s temptation to renege in his debt—an assumption about preferences. We show that if agents are matched bilaterally then the constrained-efficient allocation corresponds to the highest steady state when buyers/borrowers make take-it-or-leave-it offers.

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4 This result should not come as a surprise since Kocherlakota (1998) showed in different environments that the set of implementable outcomes of monetary economies is a subset of the implementable outcomes of pure credit economies. In contrast to Kocherlakota we take the trading mechanism as given and we do not restrict outcomes to stationary ones.
Next, we consider large meetings, as in AJ, and impose a core requirement in spot markets, which is consistent with assuming price taking behavior. We isolate a parameter that affects the temptation to renege on one's debt and hence the trade-off between efficiency and incentives. When the temptation to renege is high, the AJ Second Welfare Theorem applies to our credit economy and the constrained-efficient allocation corresponds to the highest steady state. When the temptation to renege is low, constrained-efficient allocations are non-stationary and incentive constraints to ensure repayment are slack. This failure of the AL Second Welfare Theorem is due to a "pecuniary" externality according to which an increase in debt limits allows for higher contemporaneous trade, which raises prices and lowers gains from trade for borrowers, thereby tightening borrowing constraints in earlier periods. These results are robust if we restrict our attention to 2-period credit cycles. There exists a continuum of credit cycles that yield a higher social welfare than those with "not-too-tight" debt constraints. Along these optimal cycles the borrower's repayment constraint is slack periodically.

1.1 Related literature

Our investigation of the equilibrium set of a pure credit economy extends the analysis of G2MW. We adopt a New-Monetarist environment similar to the one in Lagos and Wright (2005) and Rocheteau and Wright (2005). The equilibrium notion in G2MW imposes the "not-too-tight" solvency constraints of AJ. We show that such constraints have normative foundations that do not apply to our environment. Instead, we present our model as a repeated game with imperfect monitoring.\(^5\) Differently from the repeated game literature we study both stationary and non-stationary equilibria, including endogenous cycles and sunspot equilibria, we consider various trading mechanisms, including ultimatum games, axiomatic bargaining solutions (Kalai and Nash), Walrasian pricing, and we conduct a normative analysis to determine constrained-efficient allocations.

Our paper is also part of the literature on limited commitment in macroeconomics. Seminal contributions on risk sharing in endowment economies where agents lack commitment include Kehoe and Levine (1993), Kocherlakota (1996), and Alvarez and Jermann (2000).\(^6\) Kocherlakota (1996) adopts a mechanism design approach while we take both a positive approach focusing on mechanisms commonly used in the literature, e.g., Nash and proportional bargaining, and a mechanism design approach where the mechanism is chosen to maximize society’s welfare, subject to individual rationality constraints and a static core requirement.

KL (Section 7) conjectured that punishments based on partial exclusion might allow the implementa-

\(^5\)Repeated games where agents are matched bilaterally and at random and change trading partners over time are studied in Kandori (1992) and Ellison (1994). A thorough review of the literature is provided by Mailath and Samuelson (2004).

\(^6\)While our economy is a production economy similar to the one studied in monetary theory it can be easily reinterpreted as an endowment economy along the lines of Rocheteau, Rupert, and Wright (2008).
We provide a pure-credit economy where this conjecture is verified, but we also show that the extent of exclusion has to vary over time to obtain a constrained-efficient allocation. Our normative results are also related to the Second Welfare Theorem in AJ according to which constrained-efficient allocations can be implemented with "not-too-tight" solvency constraints. We will provide a necessary and sufficient condition under which this theorem applies to our environment. This condition will be expressed in terms of one variable that parametrizes the incentive to default. When the AJ Second Welfare Theorem fails to apply the constrained-efficient allocation is non-stationary and it is such that borrowers' participation constraints are slack over an infinite number of periods.

Sanches and Williamson (2010) were the first to introduce limited commitment to study the coexistence of money and pairwise credit in the Lagos-Wright environment. They focus on steady-state equilibria and assume that the punishment for not repaying a loan is permanent autarky. The equilibrium set of pure monetary economies has been characterized by Lagos and Wright (2003) and Rocheteau and Wright (2013), among many others, in the context of random matching economies. They find that monetary economies can generate endogenous cycles under monotone trading mechanisms, such as buyers-take-all or proportional bargaining solutions. In contrast to G2MW we will show that such outcomes are also outcomes of pure credit economies.

Our comparison of allocations in pure credit and pure monetary economies is consistent with Kocherlakota (1998). While Kocherlakota adopts an implementation approach focusing on stationary allocations, we compare stationary and non-stationary outcomes in both environments under a given trading mechanism. Hellwig and Lorenzoni (2009) study an environment similar to Alvarez and Jermann (2000) and show that the set of equilibrium allocations with self-enforcing private debt is equivalent to the allocations that are sustained with money. Similarly, Berentsen and Waller (2011) show the equivalence between allocations in an economy with outside money (government bonds) and economy with inside money (private bonds) in the context of the Lagos-Wright model.

2 Description of the game

We study a pure credit economy described as follows. Time is discrete and starts with period 0, preferences are additively separable over dates and stages, and there is a continuum of agents of measure two divided...
evenly into a subset of buyers, $\mathcal{B}$, and a subset of sellers, $\mathcal{S}$.

Each date has two stages: first, pairwise meetings, and then a centralized meeting where settlement takes place. The first stage will be referred to as DM (decentralized market) while the second stage will be referred to as CM (centralized market). There is a single, perishable good at each stage. The CM good will be taken as the numéraire. The labels “buyer” and “seller” refer to agents’ roles in the DM: only the sellers can produce the DM good (and hence will be lenders) and only the buyers desire DM goods (and hence will be buyers). In the DM a fraction $\alpha \in (0, 1]$ of buyers meet with sellers in pairs.

Agents maximize expected discounted utility with discount factor $\beta = 1/(1 + r) \in (0, 1)$. The DM utility of a seller who produces $y \in \mathbb{R}_+$ is $-v(y)$, while that of a buyer who consumes $y$ is $u(y)$, where $v(0) = u(0) = 0$, $v$ and $u$ are strictly increasing and differentiable with $v$ convex and $u$ strictly concave, and $u'(0) = +\infty > v'(0) = 0$. Moreover, there exists $\bar{y} > 0$ such that $v(\bar{y}) = u(\bar{y})$. We denote $y^* = \arg\max [u(y) - v(y)] > 0$ the quantity that maximizes a match surplus. The utility of consuming $z \in \mathbb{R}$ units of the numéraire good is $z$, where $z < 0$ is interpreted as production.

With no loss in generality we restrict our attention to intra-period loans issued in the DM and repaid in the subsequent CM. The terms of the loan contracts are determined according to a simple protocol where buyers make take-it-or-leave-it offers to sellers. We describe alternative mechanisms later in the paper. Agents cannot commit to future actions. Therefore, repayment of loans in the CM has to be self-enforcing.

There exists a technology allowing loan contracts in the DM and repayments in the CM to be publicly recorded. The entry in the public record for each loan is a triple, $(\ell, x, i)$, composed of the size of the loan in terms of the numéraire good, $\ell \in \mathbb{R}_+$, the amount repaid by the buyer, $x \in \mathbb{R}_+$, and the identity of the buyer, $i \in \mathcal{B}$. If no credit takes place in a pairwise meeting, or if $i$ was unmatched, the entry to the public record is simply $(0, 0, i)$. The record is updated at the end of each period $t$ as follows:

$$
\rho^{i,t+1} = \rho^{i,t} \circ (\ell_t, x_t, i),
$$

where $\rho^{i,0} = (\ell_0, x_0, i)$. The list of records for all buyers, $\rho^i = \langle \rho^{i,t} : i \in \mathcal{B} \rangle$, is public information to all agents. Note that agents have private information about their trading histories that are not recorded; in

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1. The assumption of ex-ante heterogeneity among agents is borrowed from Rocheteau and Wright (2005). It is not crucial for our results but it will allow us to separate clearly borrowers (buyers) from lenders (sellers).

2. Kehoe and Levine (1993), Kocherlakota (1996), and AJ consider pure exchange economies. One could reinterpret our economy as an endowment economy as follows. Suppose that sellers receive an endowment $\bar{y}$ in the DM and $\bar{z}$ in the CM. Buyers have no endowment in the DM but an endowment $\bar{z}$ in the CM. The DM utility of the seller is $w(c)$ where $w$ is a concave function with $w'(\bar{y}) = 0$. Hence, the opportunity cost to the seller of giving up $y$ units of consumption is $v(y) = w(\bar{y}) - w(\bar{y} - y)$. It can be checked that $v$ is convex with $v(0) = 0$ and $v'(y) = -w'(\bar{y} - y)$.

3. Under linear payoffs in the CM one-period debt contracts are optimal, i.e., agents have no incentives to smooth the repayment of debt across multiple periods. Moreover, this assumption will facilitate the comparison with pure monetary economies of the type studied in Lagos and Wright (2005).

4. We could make alternative assumptions regarding what is recorded in a match. For instance, the technology could also record the output level, $y$, together with the promises made by the buyer. Not surprisingly, this would expand the set of
particular, if $\rho^i \equiv (0, 0, i)$, then agents other than $i$ do not know whether $i$ was matched but his offer got rejected (in that case, the offer made is not observed either) or was unmatched. However, as discussed later, this private information plays no role in our construction of equilibria.

3 Equilibria

For each buyer $i \in B$, a strategy, $s^i$, consists of two functions $s^i_t = (s^i_{t,1}, s^i_{t,2})$ at each period $t$, conditional on being matched: $s^i_{t,1}$ maps his private trading history, $h^i_t$, and public records of other buyers, $\rho^{-i,t}$, to an offer to the seller, $(y_t, \ell_t)$; $s^i_{t,2}$ maps $((h^i_t, \rho^{-i,t}), (y_t, \ell_t))$, together with the seller’s response, to his CM repayment to the seller, $x_t$.\footnote{Note that the buyer’s private history $h^i_t$ also contains information about his public records, $\rho^i_t$.} For each seller $j \in S$, a strategy, $s^j$, consists of one function at each period $t$, conditional on being matched with buyer $i$: $s^j_{t,1}$ maps the buyer identity $i$, his private trading history, $h^j_t$, public records, $(\rho^{i,t}, \rho^{-i,t})$, and buyer offer, $(y_t, \ell_t)$, to his response, yes or no. We restrict our attention to perfect Bayesian equilibria satisfying the following conditions:\footnote{In most monetary models, conditions (A1) and (A2) are implicitly assumed, while an analogous condition to (A3) is implied by the use of money, by taking the number $d_t$ as the date-$t$ price of equilibrium amount money, a point we establish formally later. Although most monetary models are not explicit about this, agents’ trading histories in those models are their private information and, in principle, beliefs about those are necessary to define a PBE.}

(A1) Belief-freeness. In any DM meeting, the strategies only depend on the buyer’s public trading history and his current offers and the seller’s response in the current match, but not on the private histories of previous actions (nor the public records of other buyers).

(A2) Symmetry. All buyers adopt the same strategy, $s^b$, and all sellers adopt the same strategy, $s^s$. Moreover, the buyer’s offer strategy, $s^b_{t,1}$, is constant over all public trading histories of the buyer that are consistent with equilibrium behavior, in particular, equilibrium offers at date-$t$ are independent of matching histories.

(A3) Threshold rule for repayments. For each buyer $i$ and each date $t$ following any history, there exists a number, $d_t$, such that $d_t$ is weakly larger than the equilibrium loan amount at date $t$, and $s^i_{t,2}(\rho^{i,t}, (y_t, \ell_t), yes) = \ell_t$ if $\ell_t \leq d_t$ and if $\rho^{i,t}$ is consistent with equilibrium behavior.

We call a perfect Bayesian equilibrium, $(s^b, s^s)$, satisfying conditions (A1)-(A3) above a credit equilibrium. A few remarks are in order about these conditions. Our record keeping technology does not record all actions taken by the agents. Agents have private information about the number of matches they had, quantities they consumed, or offers that were rejected. Because of this private information about previous actions, beliefs about those are necessary to define a PBE.

\footnote{A PBE consists of a list of strategies, one for each player, and, at each information set, a belief system that specifies for each player a distribution over all possible histories consistent with their information. It requires sequential rationality and the
actions are observable, and PBE is reduced to subgame perfection. Although we prefer our environment, which is closer to the existing literature on monetary economics, our multiplicity result does not rely on the presence of private information. In fact, because of the belief-free requirement, (A1), any PBE we construct is also a subgame-perfect equilibrium (SPE) if all actions were observable. However, agents’ belief about how other agents will respond to deviations do matter but they are pinned down by equilibrium strategies. Conditions (A1) and (A2) imply that, for any credit equilibrium, its outcomes are characterized by \( \{ (y_t, \ell_t) \}_{t=0}^{\infty} \), the sequence of equilibrium offers made by buyers. Moreover, (A3) implies that \( x_t = \ell_t \) for each \( t \), and hence the sequence \( \{ (y_t, \ell_t) \}_{t=0}^{\infty} \) also determines the equilibrium allocation. Without (A1), equilibrium offers may depend on the buyer’s past matching histories. Condition (A3) is not vacuous either. It restricts sellers to believe that buyers will repay their debt when observing a deviating offer with obligations smaller than those in equilibrium.

Let \( \{ (y_t, \ell_t) \}_{t=0}^{\infty} \) be a sequence of equilibrium offers. Along the equilibrium path the lifetime expected discounted utility of a buyer at the beginning of period \( t \) is

\[
V^b_t = \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s}) - \ell_{t+s}] .
\]

In each period \( t + s \) the buyer is matched with a seller with probability \( \alpha \) in which case the buyer asks for \( y_{t+s} \) units of DM output in exchange for a repayment of \( \ell_{t+s} \) units of the numéraire in the following CM and the seller agrees. In any equilibrium \( -\ell_t + \beta V^b_{t+1} \geq 0 \), which simply says that a buyer must be better off repaying his debt and going along with the equilibrium rather than defaulting on his debt and offering no-trade in all future matches, \( (y_{t+s}, \ell_{t+s}) = (0, 0) \) for all \( s > 0 \). By a similar reasoning the lifetime expected utility of a seller along the equilibrium path is

\[
V^s_t = \sum_{s=0}^{\infty} \beta^s \alpha [-v(y_{t+s}) - \ell_{t+s}] .
\]

The seller’s participation constraint in the DM requires \( -v(y_t) + \ell_t \geq 0 \) since a seller can reject a trade without fear of retribution (due to condition A1). Given that buyers set the terms of trade unilaterally, and the output level is not part of the record \( \rho^i \), this participation constraint holds at equality. Our first proposition builds on these observations to characterize outcomes of credit equilibria.

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16 In such an equilibrium sellers’ beliefs about buyers’ private information are irrelevant for their decisions to accept or reject offers. Hence, actions that correspond to agents’ private trading histories would not matter even if they were publicly observable.

17 Obviously, when \( \alpha = 1 \), the matching-history-independence element in (A1) is vacuous. However, when \( \alpha < 1 \), it would be difficult to fully characterize all equilibrium outcomes without (A1) but it certainly adds many more equilibria.

18 Without this restriction one could sustain equilibria in which \( y_t > y^* \) for some \( t \); to do so, one can adopt a strategy that triggers a permanent autarky for the buyer if his offer \( \ell_t \) is smaller than the equilibrium one.
Proposition 1 A sequence, \( \{(y_t, x_t, \ell_t)\}_{t=0}^{\infty} \), is a credit equilibrium outcome if and only if, for each \( t = 0, 1, \ldots \),

\[
\ell_t \leq \sum_{s=1}^{\infty} \beta^s \alpha[u(y_{t+s}) - \ell_{t+s}] \tag{4}
\]

\[
\ell_t = x_t = v(y_t) \text{ and } y_t \leq y^* \tag{5}
\]

As mentioned earlier, a sequence of equilibrium offers, \( \{(y_t, \ell_t)\}_{t=0}^{\infty} \), also determines the sequence of allocations, \( \{(y_t, x_t)\}_{t=0}^{\infty} \), with \( x_t = \ell_t \) for each \( t \), and hence, Proposition 1 also gives a characterization of allocations that can be sustained in a credit equilibrium. In this sense, condition (4) is analogous to the participation constraint (IR) in KL, and the participation constraint in Proposition 2.1 in Kocherlakota (1996). However, while in KL the IR constraint is assumed from the outset as a primitive condition, condition (4) is derived as an equilibrium condition in our framework. It follows directly from (2) and the incentive constraint \(-\ell_t + \beta V_{t+1}^b \geq 0\).

The condition (5) is the outcome of the buyer take-it-or-leave-if offer. It also requires that the DM trade is pairwise Pareto efficient, \( y_t \leq y^* \). Proposition 1 shows that the conditions (4)-(5) are not only necessary but also sufficient for an equilibrium by constructing a simple strategy profile. Buyers can be in two states at the beginning of period \( t \), \( \chi_{i,t} \in \{G, D\} \), where \( G \) means "good standing" and \( D \) means "default", and each buyer’s initial state is \( \chi_{i,0} = G \). The law of motion of the buyer \( i \)'s state following a loan and repayment \((\hat{\ell}, \hat{x})\) are given by:

\[
\chi_{i,t+1}(\hat{\ell}, \hat{x}, \chi_{i,t}) = \begin{cases} 
D & \text{if } \hat{x} < \min(\hat{\ell}, \ell_t) \text{ or } \chi_{i,t} = D \\
G & \text{otherwise}
\end{cases}
\tag{6}
\]

where \((\hat{\ell}, \hat{x})\) might differ from the loan and repayment along the equilibrium path, \( \ell_t = x_t \). In order to remain in good standing, or state \( G \), the buyer must repay his loan, \( \hat{x} \geq \hat{\ell} \), if the size of the loan is no greater than the equilibrium loan size, \( \hat{\ell} \leq \ell_t \), and he must repay the equilibrium loan size, \( \hat{x} \geq \ell_t \), otherwise.\(^{20}\) The default state, \( D \), is absorbing: once a buyer becomes untrustworthy, he stays untrustworthy forever.

The strategies, \((s^b, s^s)\), depend on the buyer’s state as follows. The seller’s strategy, \( s^s \), consists in accepting all offers, \((y, \tilde{\ell})\), such that \( v(y) \leq \min\{\tilde{\ell}, \ell_t\} \) provided that the buyer’s state is \( \chi_{i,t} = G \). The buyer repays \( s^b_{t,2} = \min\{\ell_t, \tilde{\ell}\} \) if he is in state \( G \), and he does not repay anything otherwise, \( s^b_{t,2} = 0 \). These strategies are depicted in Figure 1 where \((y_t, \ell_t)\) is the offer made by a buyer in state \( G \) along the equilibrium path.

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\(^{19}\)To derive these conditions formally one has to use of the assumption that \( y_t \) is not publicly recorded—only the loan contract is—and the threshold property in (A3). See proof of Proposition 1.

\(^{20}\)Note that the buyer can remain in state \( G \) even if he does not pay his debt in full, and hence default is with respect to the common belief that buyers repay up to the size of the equilibrium loan. Also, notice that there are alternative strategy profiles represented by simple automata that deliver the same equilibrium outcome. For instance, an alternative automaton is such that the transition to state \( D \) only occurs if \( \hat{x} < \hat{\ell} \leq \ell_t \). If a loan such that \( \hat{\ell} > \ell_t \) is accepted, then the buyer can default without fear of retribution. Another automaton is one where any offer \( s^b_{t,2} \) such that \( \hat{\ell} > \ell_t \) triggers transition to state \( D \). If the buyer offers to borrow more than the debt limit, then he ceases to be trustworthy.
path and \((\tilde{y}, \tilde{\ell})\) is any offer. By the one-stage-deviation principle it is then straightforward to show that any \(\{(y_t, \ell_t)\}_{t=0}^{\infty}\) that satisfies (4)-(5) is an outcome for the strategy profile \((s^b, s^s)\).

![Figure 1: Automaton representation of the buyer’s strategy](image)

Following AJ we can reformulate a credit equilibrium by adding solvency constraints, \(\ell_t \leq d_t\), to an otherwise standard agent’s problem. These constraints specify that buyers cannot issue more than \(d_t\) units of debt in the DM of period \(t\). We call the threshold for the loan size, \(d_t\), a debt limit. While AJ introduces solvency constraints in the problem of an agent trading in a complete set of competitive markets, in our context such constraints will apply to the bargaining problem in the DM. Specifically, the buyer in a DM match sets the terms of the loan contract as follows:

\[
\max_{y_t, \ell_t} \{u(y_t) - \ell_t\} \quad \text{s.t.} \quad -v(y_t) + \ell_t \geq 0 \quad \text{and} \quad \ell_t \leq d_t.
\] (7)

The buyer maximizes his surplus, \(u(y_t) - \ell_t\), subject to the seller’s participation constraint and the solvency (or borrowing) constraint, \(\ell_t \leq d_t\). The solution to (7) is \(\ell_t = v(y_t)\) where \(y_t\) solves

\[
y_t = z(d_t) \equiv \min\{y^*, v^{-1}(d_t)\}.
\] (8)

The solvency constraint in the bargaining problem is reminiscent to the feasibility constraint in monetary models (e.g., Lagos and Wright, 2005) according to which buyers in bilateral matches cannot spend more than their real balances. The following corollary shows that a credit equilibrium reduces to a sequence of debt limits, \(\{d_t\}_{t=0}^{\infty}\), that satisfies a sequence of participation constraints.

**Corollary 1 (Debt limits)** The sequence of debt limits, \(\{d_t\}_{t=0}^{\infty}\), is part of a credit equilibrium outcome if
and only if

\[ d_t \leq \sum_{s=1}^{\infty} \beta^s \alpha [u(y_{t+s}) - v(y_{t+s})]; \tag{9} \]

and \( \ell_t = v(y_t) = \min\{d_t, v(y^*)\}. \tag{10} \]

While AJ introduce the solvency constraint as a primitive condition, we derive the debt limits endogenously as part of equilibrium strategies. AJ focus on solvency constraints that are "not-too-tight," meaning that \( d_t \) is the largest debt limit that solves the buyer’s CM participation constraint, (9), at equality, thereby preventing default while allowing as much trade as possible. A "too-tight" solvency constraint is such that (9) is slack. In contrast to AJ and G2MW, we do not impose the buyer’s participation constraint to bind, i.e., the solvency constraint to be "not-too-tight," as such restriction would reduce the equilibrium set dramatically and might eliminate equilibria with good welfare properties.\(^{21}\)

The next Corollary provides a sufficient condition for an equilibrium in recursive form.

**Corollary 2 (Recursive sufficient condition)** Any bounded sequence, \( \{d_t\}_{t=0}^{+\infty} \), that satisfies

\[ d_t \leq \beta \{\alpha [u(y_{t+1}) - v(y_{t+1})] + d_{t+1}\}, \tag{11} \]

where \( v(y_t) = \min\{d_t, v(y^*)\} \), is a credit equilibrium.

The sequence of inequalities, (11), are sufficient conditions for a credit equilibrium, but they are not necessary. We will provide examples later of credit equilibria that do not satisfy (11). One can rewrite (11) as follows:

\[ rd_t \leq \alpha [u(y_{t+1}) - v(y_{t+1})] + d_{t+1} - d_t. \tag{12} \]

The left side of (12) is the flow cost of repaying the debt while the right side of (12) is the flow benefit that has two components: the expected match surplus of a buyer who has access to credit and the capital gain (or loss) associated with the change in the debt limit. In Figure 2 we represent (11) holding at equality by a red curve. We plot a truncated sequence of debt limits, \( (d_{T-2}, d_{T-1}, d_T) \), that solves (11), i.e., \( (d_{T-2}, d_{T-1}) \) and \( (d_{T-1}, d_T) \) are located to the left of the red curve. In contrast, under the “not-too-tight” solvency constraints in AJ \( \{d_t\} \) solves a condition analogous to (11) where the weak inequality is replaced with an equality and hence any pair, \( (d_{t-1}, d_t) \), is on the red convex curve in Figure 2.

The AJ solvency constraints can be implemented by an automaton similar to the one in Figure 1 except that the transition from state \( G \) to state \( D \) occurs whenever \( \tilde{x} < \tilde{\ell} \) irrespective of whether \( \tilde{\ell} \) is smaller or larger than \( \ell_t \). To illustrate this difference suppose that \( \ell_t = x_t = $100 \) and an out-of-equilibrium

\(^{21}\)Allowing for such inequality is discussed in KL by a different way called “partial exclusion.”
loan is extended such that $\bar{t} = \$200 > \ell_t$. According to our candidate strategy profile, the buyer remains trustworthy to comply with his future obligations along the equilibrium path, as long as he repays at least $\$100. In contrast, under the AJ solvency constraints if the buyer fails to repay the loan in full he is no longer trustworthy to repay any debt in the future.

3.1 Steady-state equilibria

We characterize first steady states where debt limits and DM allocations are constant through time, $(d_t, y_t, \ell_t) = (d, y, \ell)$ for all $t$. Under such restriction the incentive-compatibility condition, (9), or, equivalently, (11), can be simplified to read:

$$rd \leq \alpha \{u[z(d)] - v[z(d)]\},$$

where $z$ given by (8) indicates the DM level of output as a function of $d$. The left side of (13) is the flow cost of repaying debt while the right side is the flow benefit from maintaining access to credit. This benefit is equal to the probability of a trading opportunity, $\alpha$, times the whole match surplus, $u(y) - v(y)$, where $y = z(d)$. Let $d^{\text{max}}$ denote the highest value of the debt limit that satisfies (13), i.e., $d^{\text{max}}$ is the unique positive root to $rd^{\text{max}} = \alpha \{u[z(d^{\text{max}})] - v[z(d^{\text{max}})]\}$. It is determined graphically in Figure 3 at the intersection of the left side of (13) that is linear and the right side of (13) that is concave. For all $d < d^{\text{max}}$ the gain from defaulting is less than the cost associated with permanent autarky. The reverse is true when $d > d^{\text{max}}$. The
next Proposition shows that any debt limit between \( d = 0 \) and \( d = d_{\text{max}} \) is part of an equilibrium.

**Proposition 2 (Steady-State Equilibria)** There exists a continuum of steady-state, credit equilibria indexed by \( d \in [0, d_{\text{max}}] \) with \( d_{\text{max}} > 0 \).

The two extreme debt limits, \( \{0, d_{\text{max}}\} \), correspond to the two steady-state equilibria under the AJ solvency constraints where the gain from defaulting is exactly equal to the cost of permanent autarky. Proposition 2 establishes that any debt limit in between these two extreme values is also part of an equilibrium. The intuition is as follows. For any debt \( d \) between 0 and \( d_{\text{max}} \) the gain from defaulting is strictly smaller than the cost associated with permanent autarky. So the buyer has incentives to repay such a loan. What about a slightly larger loan? If a buyer offers \( \ell > d \) then he only repays \( d \), which is the repayment that keeps him trustworthy to other sellers. As a result any loan such that \( \ell > d \) is treated like a promise to repay \( d \) and it is acceptable provided that \( v(y) \leq d \).

![Figure 3: Set of debt limits at steady-state, credit equilibria](image)

**3.2 Periodic equilibria**

Here we consider deterministic cycles where buyers’ access to credit changes over time. To simplify the presentation we start with 2-period cycles, \( \{d_0, d_1\} \), where \( d_0 \) is the debt limit in even periods and \( d_1 \) is the debt limit in odd periods. The incentive-compatibility condition, (9), becomes:

\[
rd_t \leq \alpha \left[ \frac{u(y_{(t+1) \mod 2}) - v(y_{(t+1) \mod 2})}{1 + \beta} \right] + \beta \alpha \left[ u(y_t) - v(y_t) \right], \quad t \in \{0, 1\},
\]

(14)
where we used that $\ell_t = v(y_t)$ from (10). The term on the numerator on the right side of (14) is the expected discounted utility for the buyer over the 2-period cycle. The steady-state equilibria considered above are special cases of two-period cycles; indeed, for any $d \in [0, d^{\text{max}}]$, $(d, d)$ satisfy (14). We define, for each $d_0 \in [0, d^{\text{max}}],$

$$\gamma(d_0) \equiv \max\{d_1 : (d_0, d_1) \text{ satisfies (14) with } t = 1\},$$

(15)

the highest debt limit in odd periods consistent with a debt limit equal to $d_0$ in even periods. A 2-period-cycle equilibrium, or simply a 2-period cycle, is a pair $(d_0, d_1)$ that satisfies $d_0 \leq \gamma(d_1)$ and $d_1 \leq \gamma(d_0)$.

**Lemma 1** The function $\gamma(d)$ is positive, non-decreasing, and concave. Moreover, $d^{\text{min}} \equiv \gamma(0) > 0$, $\gamma(d) > d$ for all $d \in (0, d^{\text{max}})$, and $\gamma(d^{\text{max}}) = d^{\text{max}}$. If $v(y^*) < d^{\text{max}}$, then $\gamma(d) = d^{\text{max}}$ for all $d \in [v(y^*), d^{\text{max}}]$.

The function $\gamma$ is represented in the two panels of Figure 4. It is non-decreasing because if the debt limit in even periods increases, then the punishment from defaulting gets larger and, as a consequence, higher debt limits can be sustained in odd periods. So there are complementarities between agents’ trustworthiness in odd periods and agents’ trustworthiness in even periods. The function $\gamma(d)$ is always positive because even if credit shuts down in even periods, credit can be sustained in odd periods by the threat of autarky. For a given $d_0$ we define the set of debt limits in odd periods that are consistent with a 2-period cycle by

$$\Omega(d_0) \equiv \{d_1 : d_0 \leq \gamma(d_1), d_1 \leq \gamma(d_0)\}.$$

(16)

In Figure 4 the set of credit cycles is the area between $\gamma$ and its mirror image with respect to the 45° line.

![Figure 4: Set of two-period credit cycles](image_url)
Proposition 3 (2-Period Credit Cycles) For all $d_0 \in [0, d^{\max})$ the set of 2-period credit cycles with initial debt limit, $d_0$, denoted $\Omega(d_0)$, is a nondegenerate interval.

G2MW restrict attention to equilibria such that $d_0 = \gamma(d_1)$ and $d_1 = \gamma(d_0)$, i.e., at the debt limit borrowers are exactly indifferent between repaying their debt and not repaying it and moving to autarky. Given the monotonicity and concavity of $\gamma(d)$ such equilibria do not occur outside of the $45^\circ$ line, i.e., there are no (strict) credit cycles. Indeed, if $d_0 \in (0, d^{\max})$ then the maximum debt limit in odd periods is $d_1 = \gamma(d_0) > d_0$. But given $d_1$ the maximum debt limit in even periods is $d_0' = \gamma(d_1) > d_1 > d_0$. Following this argument we obtain an increasing sequence, \{${d_0, \gamma(d_0), \gamma(\gamma(d_0))}$, \ldots\}, that converges to $d^{\max}$. In contrast we find a continuum of (strict) two-period cycle equilibria. Moreover, the set of steady-state equilibria is of measure 0 in the set of all 2-period equilibria. Indeed, for any $d_0$ in the interval $(0, d^{\max})$ there are a continuum of two-period cycles where $d_0$ is the debt limit in even periods.

The set of credit equilibria described in Proposition 3 contains equilibria with credit drying up periodically. In the left panel of Figure 4 such equilibria correspond to the case where $d_0 = 0 < d_1 < \gamma(0) = d^{\min}$, i.e., even-period IOUs are believed to be worthless while odd-period IOUs are repaid. If a seller extends a loan in an even period, the buyer defaults, in accordance with beliefs, but remains trustworthy in subsequent odd periods. Such outcomes are not consistent with any monetary equilibrium in pure-currency economies where a simple backward-induction argument rules out equilibria where fiat money loses its value and trades shut down periodically. In contrast a credit economy has IOUs issued at different dates (and by different agents), and hence agents can form different beliefs regarding the terminal value of these different assets. The maximum sustainable debt in active periods, $d^{\min}$, decreases with $r$ and increases with $\alpha$. So the amplitude of credit cycles increases when agents are more patient or when they trade frequently.

The result according to which there are a continuum of equilibria does not imply that everything goes. Fundamentals, such as preferences and matching technology, do matter for the outcomes that can emerge. The following corollary investigates how changes in fundamentals affect the equilibrium set.

Corollary 3 (Comparative statics) As $r$ decreases or $\alpha$ increases the set of 2-period cycles expands.

As shown in the left panel of Figure 4, if agents become more patient, i.e., $r$ decreases, then $\gamma$ shifts upward, as the discounted sum of future utility flows associated with a given allocation increases, and the set of 2-period cycle equilibria expands. The expansion of the equilibrium set is represented by the dark yellow area. Similarly, if the frequency of matches, $\alpha$, increases, then $d^{\max}$ increases as permanent autarky entails a larger opportunity cost, and the set of credit cycles expands.

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Corollary 4 (Credit tightness over the cycle) If $r \geq \alpha [u(y^*) - v(y^*)]/v(y^*)$ then $\ell_t \leq d_t$ binds for both $t \in \{0, 1\}$ in any 2-period cycle.

If $r < \alpha [u(y^*) - v(y^*)]/v(y^*)$, then there are 2-period cycles such that $\ell_t \leq d_t$ is slack for both $t \in \{0, 1\}$, and there are 2-period cycles where $\ell_t \leq d_t$ binds only periodically.

Corollary 4 shows that if agents are sufficiently impatient, as in the left panel of Figure 4, then the debt limit binds and output is inefficiently low in every period for all credit cycles. However, if agents are patient, then there are equilibria where the debt limit binds periodically. Such equilibria are represented by the blue and green areas in the right panel of Figure 4. There are also equilibria where the debt limit fluctuates over time but never binds, in which case $y_0 = y_1 = y^*$. These equilibria are represented by the red square, $[v(y^*), d_{\text{max}}]^2$, in the right panel of Figure 4.

One can generalize the above arguments to $T$-period cycles, $\{d_j\}_{j=0}^{T-1}$. The debt limits must solve the following inequalities:

$$d_t \leq \frac{\alpha \sum_{j=1}^{T} \beta^j \left\{ u \left[ y_{(t+j) \mod T} \right] - v \left[ y_{(t+j) \mod T} \right] \right\}}{1 - \beta^T}, \quad t = 0, \ldots, T - 1$$

(17)

The numerator on the right side of (17) is the expected discounted sum of utility flows over the $T$-period cycle. Following the same reasoning as above:

Proposition 4 (T-Period Credit Cycles) There exists a continuum of strict, T-period, credit cycle equilibria for any $T \in \{2, 3, \ldots\}$.

Our environment can lead to cycles of any periodicity, and for a given length of the cycle there are a continuum of equilibria. As an illustration, in Figure 5 we represent the set of 3-period for a given parametrization. One can see from the right panel that there are non-empty sets of 3-period cycles (the pink area) where credit shuts down periodically, once or twice every three periods. Also, for our parametrization the first best is implementable, i.e., there are equilibria with $d_t \geq 1$ for all $t \in \{0, 1, 2\}$.

3.3 Monetary vs credit equilibria

We show in the following that any allocation, $\{(x_t, y_t)\}_{t=0}^{+\infty}$, of a pure monetary economy, where $x_t$ is CM output and $y_t$ is DM output, is also an allocation of a pure credit economy. Consider a pure monetary economy along the lines of Lagos and Wright (2003, 2005) where all buyers are endowed with $M = 1$ units of fiat money at time $t = 0$. Money is perfectly divisible and its supply is constant over time. The CM price of money in terms of the numéraire good is denoted $\phi_t$. The buyer’s choice of money holdings in period $t$ is
the solution to the following problem:

$$\max_{m \geq 0} \{-\phi_t m + \beta \alpha [u(y_{t+1}) - v(y_{t+1})] + \beta \phi_{t+1} m\},$$

(18)

where, from buyers’ take-it-or-leave-it offers in the DM, sellers are indifferent between trading and not trading, \(v(y) = \phi_t m\). From (18) it costs \(\phi_t m\) to the buyer in the CM of period \(t\) to accumulate \(m\) units of money. In the following DM the buyer can purchase \(y_{t+1} = v^{-1}(\phi_{t+1} m)\) if he happens to be matched with probability \(\alpha\). Otherwise the buyer can resale his units of money at the price \(\phi_{t+1}\) in the CM of period \(t+1\).

From the first-order condition of (18), \(\{\phi_t\}_{t=0}^{+\infty}\) solves the following first-order difference equation,

$$\phi_t = \beta \phi_{t+1} \left[1 + \alpha \frac{u'(y_{t+1}) - v'(y_{t+1})}{v'(y_{t+1})}\right],$$

(19)

together with the transversality condition \(\lim_{t \to \infty} \beta^t \phi_t = 0\). According to (19) the value of fiat money in period \(t\) is equal to the discounted value of money in period \(t+1\) augmented with a liquidity term that captures the expected marginal surplus from holding an additional unit of money in a pairwise meeting in the DM.

**Proposition 5 (Monetary vs Credit Equilibria)** Let \(\{(x_t, y_t, \phi_t)\}_{t=0}^{+\infty}\) be an equilibrium of the pure monetary economy. Then, \(\{(x_t, y_t)\}_{t=0}^{+\infty}\) is a credit equilibrium with \(\{\phi_t\}\) as the sequence of debt limits in a pure credit economy.

Proposition 5 establishes that the set of (dynamic) allocations in pure credit economies encompasses the set of allocations of pure monetary economies taking as given the trading mechanism. This result is related
to those in Kocherlakota (1998), but with a key difference: while Kocherlakota (1998) shows that the set all implementable outcomes (allowing for arbitrary trading mechanisms) using money is contained in the set of all implementable outcomes with memory, we compare the equilibrium outcomes for the two economies under a particular trading mechanism. Later on we discuss the robustness to other trading mechanisms.

We illustrate this result in Figure 6 where the green, backward-bending line represents the first-order difference equation for a monetary equilibrium, (19), while the red area is the first-order difference inequality for a credit equilibrium, (11). Starting from some initial condition, $d_0$, we represent by a dashed line a sequence $\{d_t\}$ that satisfies the equilibrium condition for a monetary equilibrium. This sequence also satisfies the conditions for a credit equilibrium, i.e., all pairs $(d_t, d_{t+1})$ are located in the red area. Therefore, if the equilibrium set of a pure monetary economy contains cycles and chaotic dynamics, the same must be true for the equilibrium set of the same economy with no money but record-keeping. It is also easy to see from Figure 6 that credit equilibria in G2MW, under the requirement that (11) holds as an equality, do not coincide with the monetary equilibria in Lagos-Wright. The phase line for credit economies is located to the right of the phase line for monetary economies. The reason for this discrepancy is as follows. Under the A-J solvency constraints the payment capacity of buyers, $d_t$, is the discounted sum of all future match surpluses,

$$d_t = \beta \{ \alpha [u(y_{t+1}) - v(y_{t+1})] + d_{t+1} \}.$$
In LW the payment capacity of buyers, $\phi_t$, is the discounted sum of all future marginal surpluses multiplied by the value of money,

$$\phi_t = \beta \left\{ \alpha \frac{\partial [u \circ v^{-1}(\phi_{t+1}) - \phi_{t+1}]}{\partial \phi_{t+1}} \phi_{t+1} + \phi_{t+1} \right\}.$$  

From the concavity of the match surplus, if $\phi_t = d_t$, then $\phi_{t+1} > d_{t+1}$.

The reverse of Proposition 5 does not hold; There are equilibria of pure credit economies that are not equilibria of pure monetary economies. As we saw above there are credit equilibria where trades shut down periodically, and such equilibria cannot be captured by Figure 6. (Recall that the recursive condition in Corollary 2 is only sufficient but not necessary.) As another example, one can construct equilibria where the debt limit, $d_t$, increases in a monotonic fashion over time as buyers become more and more trustworthy. Such equilibria would not be sustainable in monetary economies.

### 3.4 Sunspot equilibria

So far we have focused on deterministic equilibria. One can also construct sunspot equilibria where the DM allocation, $\{(y_\chi, \ell_\chi)\}$, depends on the realization of a sunspot state, $\chi \in \mathbb{X}$, at the beginning of the DM.

Suppose that $\mathbb{X}$ is finite and the process driving the sunspot state is iid with distribution $\pi$. The value of a buyer along the equilibrium path solves

$$V^b_{\chi} = \alpha [u(y_\chi) - v(y_\chi)] + \beta \tilde{V}^b_{\chi}$$

$$\tilde{V}^b_{\chi} = \int V^b_{\chi'} d\pi(\chi'),$$

for all $\chi \in \mathbb{X}$. As before, the lifetime utility of a buyer is the expected discounted sum of the surpluses coming from DM trades. It follows that a sunspot credit equilibrium is a vector, $\langle d_\chi; \chi \in \mathbb{X} \rangle$, that satisfies $d_\chi \leq \beta \tilde{V}^b_{\chi}$. Hence, $\{d_\chi\}$ satisfies

$$rd_\chi \leq \alpha \int \{u[z(d_\chi')] - v[z(d_\chi')]\} d\pi(\chi') \quad \forall \chi \in \mathbb{X}$$

**Proposition 6 (Sunspot equilibria)** Suppose that $\mathbb{X}$ has at least two elements. Then, for each $d \in (0, d^{\max})$, there exists a continuum of sunspot credit equilibria, $\{d_\chi; \chi \in \mathbb{X}\}$, indexed by $d$ in the sense that $\max\{d_\chi; \chi \in \mathbb{X}\} = d$ and $d_\chi \neq d_{\chi'}$ for any $\chi \neq \chi'$.

### 4 Alternative trading mechanisms

So far we assumed that buyers set the terms of the loan contract in the DM by making take-it-or-leave-it offers and we showed that under the AJ "not-too-tight" solvency constraint there exist no periodic equilibria whereas there are a continuum of credit cycle equilibria once this restriction on strategies is removed. In
the following we show that our results are robust to the adoption of alternative trading mechanisms. We also extend our model in order to parametrize buyers’ temptation to renege on their debt. This extension adds a new parameter that plays a key role for both the emergence of credit cycles under alternative trading mechanisms and the normative results in Section 5.

![Figure 7: Timing of the extended model with temptation to renege](image)

Suppose from now on that a buyer who promises to deliver \( \ell \) units of goods in the next CM incurs the linear disutility of producing at the time he is matched in the DM. This new timing is illustrated in Figure 7.\(^{22}\) The effort exerted by the buyer in the DM, \( \ell \), is perfectly observable to the seller. Hence, the seller can condition his own production to the buyer’s effort. At the time of delivery, at the beginning of the CM, the disutility of production has been sunk and the buyer has the option to renege on his promise to deliver the good. The buyer’s utility from consuming his own output is \( \lambda \ell \) with \( \lambda \leq 1 \). A buyer has no incentive to produce more good than the amount he promises to repay to the seller since the net utility gain from producing \( x \) units of the good for oneself is \( (\lambda - 1)x \leq 0 \). Although the physical environment is different, mathematically speaking, the model of the previous section can be regarded as a special case with \( \lambda = 1 \).

As before we will focus on belief-free, symmetric Perfect Bayesian Equilibria with no default.\(^{23}\)

Let \( \{(d_t, y_t, \ell_t)\}_{t=0}^{\infty} \) be the sequence of equilibrium debt limits and trades. A necessary condition for the repayment of \( d_t \) to be incentive feasible is \( \beta V_{t+1}^B \geq \lambda d_t \), where the left side is the buyer’s continuation value from delivering the promised output and the right side of the inequality is the expected utility of the buyer if he keeps the output for himself, in which case he enjoys a utility flow \( \lambda d_t \), and goes to autarky. Following

\(^{22}\)The description of the buyer’s incentive problem is taken from Gu et al. (2013a,b).

\(^{23}\)G2MW also introduce an imperfect record-keeping technology as follows. At the end of the CM of period \( t \) the repayments are recorded for a subset of buyers, \( B_r^t \subset B \), chosen at random among all buyers. The set, \( B_r^t \), of monitored buyers is of measure \( \pi \), and the draws from \( B \) are independent across periods. So in every period, while his promise is always recorded, a buyer has a probability \( \pi \) of having his repayment decision being recorded. The parameter \( \pi \) does not play any role in our analysis so we chose to set it equal to one.
the same reasoning as before, a credit equilibrium is a sequence, \( \{(d_t, y_t, \ell_t)\}_{t=0}^{\infty} \), that satisfies

\[
\lambda d_t \leq \beta V_{t+1}^h = a \sum_{s=1}^{+\infty} \beta^i [u(y_{t+s}) - \ell_{t+s}], \quad t \in \mathbb{N}_0, \tag{23}
\]

where the relationship between \( y_t, \ell_t, \) and \( d_t \) will depend on the assumed trading mechanism. For instance, if the buyer makes a take-it-or-leave-it offer, then \( \ell_t = v(y_t) \) and \( v(y_t) = \min\{v(y^*), d_t\} \).

### 4.1 Bargaining

It is standard in the literature on markets with pairwise meetings to determine the outcome of a meeting by an axiomatic bargaining solution. In this section we consider two such solutions: (i) the proportional bargaining solution given by Kalai (1977) and (ii) the generalized Nash solution. If the buyer has all of the bargaining power in trade, then the outcome under both proportional and Nash bargaining corresponds to the outcome of the take-it-or-leave-it game considered in Section 3. However, in general, when sellers have a positive weight in bargaining, the two solutions have different implications for the relationship between the buyer’s surplus in trade and the debt limit. Namely, the proportional solution requires that the buyer’s surplus increase with the total trade surplus and therefore the debt limit, while the Nash solution makes no such restriction.

In order to apply axiomatic bargaining solutions it is necessary to have a convex bargaining set, which is obtained by focusing on buyer’s repayment strategies that follow a threshold rule: for a given sequence of debt limits, \( \{d_t\}_{t=0}^{+\infty} \), the buyer repays \( \min\{\ell_t, d_t\} \) if his date-\( t \) obligation from his DM trade is \( \ell_t \). Due to the linearity of the CM value functions, the buyer’s surplus from a DM trade, \( (y_t, \ell_t) \) with \( \ell_t \leq d_t \), is \( u(y_t) - \ell_t \) and the seller’s surplus is \( -v(y_t) + \ell_t \).

**Kalai proportional bargaining** We amend the buyer take-it-or-leave-it offer game by restricting the set of buyers’ feasible offers: the buyer can only make offers such that the fraction of the match surplus he receives is no greater than a given parameter \( \theta \in [0, 1] \), i.e.,

\[
u(y_t) - \ell_t \leq \theta[u(y_t) - v(y)]. \tag{24}\]

Thus, the buyer’s offer in the DM, assuming he is in state \( G \), solves

\[
(y_t, \ell_t) = \arg \max_{y, \ell} [u(y) - \ell] \text{ s.t. } (24) \text{ and } \ell \leq d \tag{25}\]

According to (25) the buyer maximizes his utility of consumption net of the cost of repaying his debt subject to the feasibility constraint, (24), and the repayment constraint, \( \ell \leq d \). The solution to (25) is

\[
y_t = z(d_t) \equiv \min\{y^*, \eta^{-1}(d_t)\} \text{ and } \ell_t = \eta[z(d_t)]. \tag{26}\]
where \( \eta(y) = (1 - \theta)u(y) + \theta v(y) \).

In our proposed equilibrium, the buyer offers \((y_t, \ell_t)\) given by (26) and the seller accepts it. The seller rejects any offer from a buyer with state \(D\). As before an outcome for a credit equilibrium can be characterized by the sequence of debt limits, \(\{d_t\}\), that solves (23).

**Proposition 7 (Credit equilibrium under proportional bargaining)** A sequence, \(\{(y_t, \ell_t, d_t)\}\), is the outcome of a credit equilibrium under proportional bargaining if and only if \((y_t, \ell_t)\) is given by (26) and, for all \(t\),

\[
\lambda_d \leq \alpha \theta \sum_{i=1}^{+\infty} \beta^i [u(y_{t+i}) - v(y_{t+i})]. \tag{27}
\]

Proposition 7 describes the set of all debt limits, \(\{d_t\}_{t=0}^{+\infty}\), and allocations, \(\{(y_t, \ell_t)\}_{t=0}^{+\infty}\), that are generated by credit equilibria under bargaining weight \(\theta\). The right side of (27) takes into account that buyers only receive a fraction \(\theta\) of the match surplus. Note that Corollary 1 is a special case of Proposition 7 by taking \(\theta = \lambda = 1\).

A steady state is a solution \(d\) to \(r \lambda d \leq \alpha \theta [u(y) - v(y)]\) where \(y = z(d)\). We can generalize Proposition 2 by showing that the set of steady-state equilibria is the interval \([0, d_{\text{max}}]\), where \(d_{\text{max}}\) is the largest nonnegative root to \(r \lambda d = \alpha \theta [u[z(d)] - v[z(d)]]\), and that \(d_{\text{max}} > 0\) if and only if \(\lambda r < \alpha \theta / (1 - \theta)\). To see this, notice that the right side of the equality is concave and its right-derivative at \(d = 0^+\) is \(\alpha \theta / (1 - \theta)\). If buyers do not have all the bargaining power, then an active steady-state credit equilibrium exists only if buyers are sufficiently patient.\(^{24}\) The lower the value of \(\theta\) the lower the rate of time preference that is required for credit to emerge. Indeed, if \(\theta\) decreases buyers get a lower share in current and future match surpluses and, for a given \(d\), the amount of DM consumption they can purchase is lower. Both effects reduce the gains from participating in the DM and hence reduce the maximum sustainable debt limit. It can also be checked that a higher \(\lambda\) reduces \(d_{\text{max}}\) since the temptation to renege on one’s debt is higher. As a result any allocation under \(\lambda = 1\) is also an allocation under \(\lambda < 1\). We now move to equilibria with endogenous fluctuations.

**Proposition 8 (2-Period Credit Cycles under proportional bargaining)** If \(\lambda r < \alpha \theta / (1 - \theta)\), then there exists a continuum of strict, 2-period, credit cycle equilibria. Moreover, if \(r < \sqrt{1 + \alpha \theta / |\lambda (1 - \theta)|} - 1\), then there exist equilibria where credit shuts down periodically.

Proposition 8 establishes a condition for the the existence of a continuum of credit-cycle equilibria under proportional bargaining. The set of equilibria is represented by Figure 4 where the outer envelope shifts...
outward as $\theta$ increases. Moreover, if agents are sufficiently patient then there are equilibria where credit shuts down periodically. In contrast, if we impose the AJ solvency constraints, then there are no periodic equilibrium under proportional bargaining, irrespective of the buyer’s bargaining share. Propositions 4 and 5 regarding the existence of $N$-period credit cycles and the relationship between monetary and credit equilibria can be generalized to proportional bargaining in a similar fashion.

**Generalized Nash bargaining** The most commonly-used bargaining solution in environments with pairwise meetings is the generalized Nash bargaining solution that has both axiomatic and strategic foundations. If $\eta$ denotes the bargaining weight of the buyer, then the terms of the loan contract are

$$ (y_t, \ell_t) \in \arg \max [u(y) - \ell]^\theta [\ell - v(y)]^{1-\theta} \text{ s.t. } \ell \leq d_t. $$

The solution is given by (26) where

$$ \eta(y) = \Theta(y)u(y) + [1 - \Theta(y)] v(y) \text{ and } \Theta(y) = \theta v'(y)/ [\theta v'(y) + (1 - \theta) u'(y)]. $$

(28)

A sequence, $\{(y_t, \ell_t, d_t)\}_{t=0}^{+\infty}$, is the outcome of a credit equilibrium under generalized Nash bargaining if and only if $(y_t, \ell_t)$ is given by (26) with $\eta(y)$ given by (28), and for all $t$,

$$ \lambda d_t \leq \alpha \sum_{i=1}^{+\infty} \beta^i [u(y_{t+i}) - \eta(y_{t+i})]. $$

(29)

Let us denote $\hat{y} = \arg \max \{u(y) - \eta(y)\}$. A key difference with respect to the proportional solution is the fact that $\hat{y} < y^*$ under Nash for all $\theta < 1$ whereas $\hat{y} = y^*$ under proportional bargaining. As a result the buyer’s surplus, $u(y) - \eta(y)$, that appears in the right side of the participation constraint, (29), is non-monotonic with the debt limit provided that $\theta < 1$. It follows that the function $\gamma(d)$ is non-monotonic, reaching a maximum at $d = \hat{d} = \eta(\hat{y})$ and it is constant for $d > \eta(y^*)$. In Figure 8 we represent the function $\gamma$ and the set of pairs, $(d_0, d_1)$, consistent with a 2-period credit cycle equilibrium. One can see that the results are qualitatively unchanged except for the fact that the credit limits at a periodic equilibrium can be greater than the highest debt limit at a stationary equilibrium. This result will have important normative implications.

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25 One could implement the Nash bargaining solution as the outcome of a strategic game with alternating offers. Suppose a buyer and a seller who are matched in the DM engage in an alternating offer bargaining game with an infinite number of rounds. The buyer makes the first offer. If the seller rejects the buyer’s offer, then he has the opportunity to make a counteroffer provided that the negotiation does not break down with probability $\theta \Delta \in [0, 1]$, where $\Delta$ can be interpreted as the length of a time period between two consecutive rounds of the bargaining game. Similarly, if the buyer rejects a seller’s offer he has the opportunity to make a counteroffer if the negotiation does not break down with probability $(1 - \theta) \Delta$. As $\Delta$ approaches 0 this game admits as an outcome the one generated by the generalized Nash solution.

26 This non-monotonicity property of the Nash bargaining solution and its implications for monetary equilibria is discussed at length in Aruoba, Rocheteau, and Waller (2007).

27 In the Appendix we formally prove that any 2-period cycle under proportional bargaining is also a 2-period cycle under Nash bargaining.
Figure 8: 2-period cycles under Nash bargaining

In the top panels of Figure 9 we plot the numerical examples in Gu and Wright (2011) under generalised Nash bargaining. The following functional forms and parameter values are

\[ u(y) = \frac{(x + b)^{1-a} - b^{1-a}}{1-a} \]

with \( a = 2 \) and \( b = 0.082 \), \( v(y) = Ay, \) \( \beta = 0.6, \) \( \alpha = 1, \) \( \theta = 0.01, \) and \( \lambda = 3/40. \) In the top left panel \( A = 1.1 \) and there are two 2-period cycles under AJ solvency constraints such that the borrowing constraint binds periodically. In the bottom right panel, \( A = 1.5 \) and the borrowing constraint binds in all periods. In contrast, for both examples there exists a continuum of PBE 2-period cycles, a fraction of which feature borrowing constraints that bind periodically and a fraction such that the borrowing constraint binds in all periods.

### 4.2 Competitive pricing

Here we follow KL and AJ and assume that the terms of the loan contract in the DM are determined by price-taking behavior. We reinterpret matching shocks as preference and productivity shocks, i.e., only \( \alpha \) buyers want to consume and only \( \alpha \) sellers can produce. As in the previous sections, buyer’s repayment strategy follows a threshold rule: for a given sequence of debt limits, \( \{d_t\}_{t=0}^{\infty} \), the buyer repays \( \min\{\ell_t, d_t\} \) if his date-\( t \) obligation from his DM trade is \( \ell_t \). Hence, if \( p_t \) denotes the price of DM output in terms of the numéraire, buyer’s demand is subject to the borrowing constraint, \( p_t y \leq d_t \). For a given \( \{d_t\}_{t=0}^{\infty} \) the consistent market-clearing price is given by \( p_t = v'(y_t) \), where

\[ y_t = z(d_t) \equiv \min\{y^*, \eta^{-1}(d_t)\} \] and \( \ell_t = \eta[z(d_t)] \),

(30)
with $\eta(y) = v'(y)y$. The buyer’s surplus is $u(y) - py = u(y) - v'(y)y$. For a given $p$, the buyer’s surplus is non-decreasing in his borrowing capacity, $d_t$. However, once one takes into account the fact that $p = v'(y)$ then the buyer’s surplus is non-monotone in his capacity to borrow, $d_t$. Provided that $v$ is strictly convex, the buyer’s surplus reaches a maximum for $y = \hat{y} < y^*$. Using similar arguments as in previous sections, a sequence, $\{(y_t, \ell_t, d_t)\}$, is the outcome of a credit equilibrium under competitive pricing if and only if $(y_t, \ell_t)$ is given by (30) and for all $t$, (29) holds. A steady state is a $d$ such that

$$r\lambda d \leq \alpha \{u[z(d)] - v'[z(d)]z(d)\}. \quad (31)$$

Under some weak assumptions on $v$ (for example, $\eta(y) = v'(y)y$ is convex), $d_{\text{max}} > 0$, i.e., there exists a continuum of steady-state equilibria. This also implies that there exist a continuum of strict, 2-period, credit cycle equilibria.\(^{29}\) Note that, under competitive pricing, the corresponding function $\gamma$ (analogous to (15)) may not be monotone or concave, but the logic for Proposition 3 does not depend on those properties. This result can be contrasted with the ones in G2MW (Corollary 1-3) where conditions on parameter values are needed to generate a finite number (typically, two) of cycles. The right panel of Figure 8 illustrates these differences. Under "not-too-tight" solvency constraints credit cycles are determined at the intersection between $\gamma(d)$ and its mirror image with respect to the 45° line. These cycles are marked by a red star. If we do not impose the buyer’s participation constraint to hold at equality, cycles are at the intersection of the area underneath $\gamma(d)$ and its mirror image with respect to the 45° line. This intersection corresponds to the blue area in the figure. Finally, Proposition 5 on the equivalence result between monetary equilibria and credit equilibria holds for Walrasian pricing as well. (See the Appendix for a formal proof).

We now review the numerical examples in G2MW in the case where the DM market is assumed to be competitive. The functional forms are $u(y) = y$, $v(y) = y^{1+\gamma}/(1 + \gamma)$, and there are no idiosyncratic shocks, $\alpha = 1$. The first example in the bottom left panel of Figure 9 is obtained with the following parameter values: $\gamma = 2.1$, $\beta = 0.4$, $\lambda = 1/6$. Under the AJ solvency constraints G2MW identify two (strict) two-period cycles, $(d_0, d_1) = (0.477, 0.936)$ and its converse, marked by red dots in the figure. The second example in the bottom right panel is obtained with the following parameter values: $\gamma = 0.5$, $\beta = 0.9$, $\lambda = 1/10$. The credit cycles under the AJ solvency constraints, $(d_0, d_1) = (0.933, 1.037)$ and its converse, are such that period allocations fluctuate between being debt-constrained and unconstrained. We find a much bigger set

\(^{28}\)The buyer’s problem is $\max_y \{u(y) - py\}$ s.t. $py \leq d_t$. The solution is $y_t = \min \{u^{-1}(pt), d_t/p_t\}$. Using that there is the same measure, $\alpha$, of buyers and sellers participating in the market, market clearing implies $p_t = v'(y_t)$. As a result $y_t = y^*$ if $y^*v'(y^*) \geq d_t$ and $y_tv'(y_t) = d_t$ otherwise. For a detailed description of this problem in the context of a pure monetary economy, see Rocheteau and Wright (2005, Section 4).

\(^{29}\)To see this notice that the derivative of the right side of (31) is $u'(y)/[v''(y)y + v'(y)] - 1$. Using that $u'(0) = +\infty$ and $v'(0) = 0$, this derivative is equal to $+\infty$ when evaluated at $d = y = 0$. See also the Appendix for a formal proof for the existence of 2-period cycles.
of PBE credit cycles represented by the blue colored region composed of a continuum of cycles in which the allocations fluctuate between being debt-constrained and unconstrained and a continuum of cycles in which agents are debt-constrained in all periods. In the second example, the credit cycle under AJ solvency constraints is such that \((y_0, y_1) = (0.96, 1.00)\) with a standard deviation of 2.3 percent. In contrast the most volatile PBE, \((y_0, y_1) = (1.00, 0.45)\), has a standard deviation equal to 38.3 percent.

5 Normative analysis

We now turn to the normative implications of our model. We will characterize constrained-efficient allocations under two alternative market structures: pairwise meetings versus large-group meetings. We will establish conditions on fundamentals under which the Second Welfare Theorem of AJ holds—constrained-efficient allocations are implemented with "not-too-tight" solvency constraints. We will show that when this theorem fails to hold constrained-efficient allocations are non-stationary and feature slack participation constraints.
5.1 Optimal mechanism with pairwise meetings

We now study a mechanism design problem according to which a planner chooses the allocations, \( \{y_t, \ell_t\}_{t=0}^{+\infty} \), subject to incentive-feasibility conditions in DMs and CMs for both buyers and sellers. The planner’s problem is

\[
\max_{\{y_t, \ell_t\}} \sum_{t=0}^{+\infty} \beta^t \alpha [u(y_t) - v(y_t)]
\]

s.t. \( M_t \leq \sum_{s=1}^{+\infty} \beta^s \alpha [u(y_{t+s}) - \ell_{t+s}] \)

\( v(y_t) \leq \ell_t \leq u(y_t) \). \hspace{1cm} (34)

The objective function, (32), is the discounted sum of the match surpluses. \(^{30}\) The inequality, (33), is the participation constraint guaranteeing that buyers prefer to repay their debt rather than going to permanent autarky. The conditions in (34) make sure that both buyers and sellers receive a positive surplus from their DM trades. To respect the pairwise core requirement we have to add the additional constraint that \( y_t \leq y^\ast \), but this requirement is satisfied endogenously and hence we ignore it here. We call a solution to the problem (32)-(34) a constrained-efficient outcome. In the following we define \( y^{\text{max}} \) as the highest, stationary level of output consistent with both the seller’s and buyer’s participation constraints, i.e., \( y^{\text{max}} \) is the highest solution to

\[
\lambda r v(y^{\text{max}}) = \alpha [u(y^{\text{max}}) - v(y^{\text{max}})]. \hspace{1cm} (35)
\]

**Proposition 9** (Constrained-efficient allocations under pairwise meetings)

1. If \( y^\ast \leq y^{\text{max}} \), then any constrained-efficient outcome is such that \( y_t = y^\ast \) and \( \ell_t \in [v(y^\ast), \overline{\ell}] \) for all \( t \in \mathbb{N}_0 \), where \( \overline{\ell} = \alpha [u(y^\ast) - v(y^\ast)] / \lambda r \).

2. If \( y^\ast > y^{\text{max}} \), then the constrained-efficient outcome is such that \( y_t = y^{\text{max}} \) and \( \ell_t = v(y_t) \) for all \( t \in \mathbb{N}_0 \).

If \( \lambda r \leq \alpha [u(y^\ast)/v(y^\ast) - 1] \), then the first-best allocation is implementable. \(^{31}\) This condition holds if agents are sufficiently patient (\( r \) low) and if the temptation to renege is not too large (\( \lambda \) low). In contrast, if \( \lambda r > \alpha [u(y^\ast)/v(y^\ast) - 1] \), then the constraint-efficient outcome is \( y_t = y^{\text{max}} < y^\ast \), which corresponds to the highest steady state.

\(^{30}\) Kocherlakota (1996) and Gu et al. (2013a, Section 7) study a Pareto problem to determine a contract curve linking the expected discounted utilities of buyers and sellers. In contrast the planner’s objective in our model is a social welfare function that aggregates the buyers’ and sellers’ utilities. One can interpret this social welfare function as the ex ante expected utility of a representative agent in a version of the model where the role of an agent in the DM is determined at random in each period. \(^{31}\) Hu, Kenaan, and Wallace (2009) derive the same condition for pure monetary economies in the case where \( \lambda = 1 \). A difference, however, is that the game where buyers make take-it-or-leave-it offers is not the optimal mechanism in monetary economies.
The constrained-efficient outcome characterized in Proposition 9 can be implemented by having buyers set the terms of the loan contract unilaterally, in which case \( t = y_t \) for all \( t \). In the case where \( y_{\text{max}} < y^* \), it is the only mechanism that implements the constrained-efficient allocation. By giving all the bargaining power to buyers the planner relaxes participation (repayment) constraints in the CM, which allows for more borrowing and higher levels of output. Moreover, the solvency constraint in the buyer’s bargaining problem that implements the constrained-efficient allocation is "not-too-tight," in accordance with AJ’s welfare theorem. We summarize this implementation result in the following Corollary.

**Corollary 5 (Second Welfare Theorem under pairwise meetings)** The constrained-efficient allocation is implemented with buyers making take-it-or-leave-it offers and with “not-too-tight” solvency constraints.

### 5.2 Optimal mechanism with centralized meetings

We now turn to the optimal credit equilibrium allocation when agents meet in a centralized location in the DM. To prevent renegotiations, we impose the core requirement in the market for goods and IOUs (see Hu, Kennan, and Wallace, 2009; and Wallace, 2013 for a related assumption), which is equivalent to imposing the competitive equilibrium outcome in the DM.\(^{32}\) From (30) the terms of the loan contract are given by \( \eta(y) = v'(y)y \). If \( v'' > 0 \) this core requirement rules out the optimal mechanism under pairwise meetings described in Proposition 9. Indeed, if \( \eta(y) = v(y) \) then a buyer and two sellers can form a deviating coalition in which each seller produces \( y_t/2 \) at a total cost of \( 2v(y_t/2) < v(y_t) \) and the buyer compensates the sellers by offering them a positive surplus, \( \ell_t/2 - v(y_t/2) > 0 \).

The planner’s problem that selects the constrained-efficient allocation is similar to (32)-(34) where the buyer’s participation constraint in the CM, (33), is replaced with

\[
\lambda \eta(y_t) \leq \alpha \sum_{s=1}^{+\infty} \beta^s \left[ u(y_{t+s}) - \eta(y_{t+s}) \right],
\]

and the DM participation constraints, (34), are satisfied by definition of \( \eta \) for all \( y \in [0, y^*] \). The CM participation constraint, (36), states that the loan a buyer takes to finance his DM consumption, \( \ell_t = \eta(y_t) \), cannot be greater than the discounted sum of the buyer’s future surpluses adjusted by the factor \( 1/\lambda \) to take into account the temptation to renege.

We define two critical values for DM output:

\[
\hat{y} = \arg \max_{y \in [0, y^*]} \left[ u(y) - \eta(y) \right]
\]

\[
y_{\text{max}} = \max \{ y > 0 : \alpha[u(y) - \eta(y)] \geq r \lambda \eta(y) \}.
\]

\(^{32}\)For equivalence between the core and competitive equilibrium in our economy, see the Appendix. This result is similar to Aumann (1964) and Anderson (1978).
The quantity $\hat{y}$ is the output level that maximizes the buyer’s surplus in the DM. The quantity $y_{\text{max}}$ is the highest, stationary level of output that is consistent with the buyer’s participation constraint in the CM. We assume that both $\hat{y}$ and $y_{\text{max}}$ are well-defined and, for all $0 \leq y \leq y_{\text{max}}$, $\alpha[u(y) - \eta(y)] \geq r\lambda\eta(y)$.

**Proposition 10** *(Constrained-efficient allocations under centralized meetings: Part I)*

1. If $y^* \leq y_{\text{max}}$, then the constrained-efficient allocation is such that $y_t = y^*$ and $\ell_t = \eta(y^*)$ for all $t \in \mathbb{N}_0$.
2. If $y_{\text{max}} \leq \hat{y} \leq y^*$, then the constrained-efficient allocation is such that $y_t = y_{\text{max}}$ and $\ell_t = \eta(y_{\text{max}})$ for all $t \in \mathbb{N}_0$.

In accordance with "Folk theorems" for repeated games, provided that agents are sufficiently patient, $r \leq \alpha[u(y^*) - \eta(y^*)]/\lambda\eta(y^*)$, the first-best allocation is an equilibrium outcome. Under the condition $y_{\text{max}} \leq \hat{y}$ in the second part of Proposition 10 the first best is not implementable but the buyer’s surplus is monotone increasing over the range of output levels that are incentive feasible in steady-state equilibria, $[0, y_{\text{max}}]$. As a result, and as shown in the left panel of Figure 10, the buyer’s welfare and society’s welfare covary positively with $y$. Hence, there is no trade-off between social efficiency and incentives: the highest steady state is also the incentive-feasible allocation that maximizes the welfare of all agents.

![Figure 10: Let panel: No trade-off between efficiency and incentives over $[0, y_{\text{max}}]$; Right panel: A trade-off between efficiency and incentives over $[\hat{y}, y^*]$.](image-url)

We now turn to the case where $\hat{y} < y_{\text{max}} < y^*$. For all $y \in (\hat{y}, y_{\text{max}})$ the buyer’s surplus, $u(y) - \eta(y)$, and society’s surplus, $u(y) - v(y)$, covary negatively with $y$, as shown in the right panel of Figure 10. This
negative relationship gives rise to a trade-off between social efficiency and incentives for debt repayment. As a result of this trade-off the highest steady state, $y^{\text{max}}$, might no longer be the PBE outcome that maximizes social welfare.

In order to analyze this trade-off formally we write the problem recursively by introducing a state variable, $\omega_t$, called buyer’s “promised utility”. Society’s welfare, denoted $V(\omega)$, solves the following Bellman equation,

$$V(\omega) = \max_{y, \omega'} \left\{ \alpha [u(y) - v(y)] + \beta V(\omega') \right\}$$  \hspace{1cm} (39)

s.t. \hspace{1cm} $-\eta(y) + \frac{\omega'}{\lambda} \geq 0$ \hspace{1cm} (40)

$\omega' \geq (1 + r) \left\{ \omega - \alpha [u(y) - \eta(y)] \right\}$ \hspace{1cm} (41)

$y \in [0, y^*], \omega' \in [0, \bar{\omega}]$, \hspace{1cm} (42)

where $\bar{\omega} = [u(\hat{y}) - \eta(\hat{y})] / (1 - \beta)$ is the maximum lifetime expected utility a buyer can expect across all PBEs. The novelty is the promise-keeping constraint, (41), according to which the lifetime expected utility promised to the buyer along the equilibrium path, $\omega$, is implemented by generating an expected surplus in the current period equal to $\alpha [u(y) - \eta(y)]$ and by making a promise for future periods worth $\beta \omega'$. In the appendix, we show that there is a unique $V$ solution to (39)-(42) in the space of continuous, bounded and concave functions, and this solution is non-increasing. As a result, the maximum value for society’s welfare is $V(0) = \max_{\omega \in [0, \bar{\omega}]} V(\omega)$, as the initial promised utility to the buyer is a choice variable. The next Proposition provides a full analytical characterization of this dynamic contracting problem.

**Proposition 11 (Constrained-efficient allocations under centralized meetings: Part II)** Suppose that $\hat{y} < y^{\text{max}} < y^*$ and that $\eta$ is a convex function.

1. If $\lambda \geq \alpha \left\{ 1 - u'(y^{\text{max}})/\eta'(y^{\text{max}}) \right\}$, then the constrained-efficient allocation is such that $y_t = y^{\text{max}}$ and $t_t = \eta(y^{\text{max}})$ for all $t \in \mathbb{N}_0$.

2. If $\lambda < \alpha \left\{ 1 - u'(y^{\text{max}})/\eta'(y^{\text{max}}) \right\}$, then the constrained-efficient allocation is such that $y_t \in (y^{\text{max}}, y^*)$ and $y_t = y_t(y^{\text{max}})$ for all $t \geq 1$, where $(y_0, y_1)$ is the unique solution to

$$\max_{y_0, y_1} \left\{ u(y_0) - v(y_0) + \frac{u(y_1) - v(y_1)}{r} \right\}$$ \hspace{1cm} (43)

s.t. \hspace{1cm} $\eta(y_0) = \frac{\alpha [u(y_1) - \eta(y_1)]}{\lambda r}$ \hspace{1cm} (44)

The constrained-efficient allocation corresponds to the highest steady state when the temptation to renege is sufficiently large. This condition holds in our benchmark economy when $\lambda = 1$ since $\alpha \leq 1$ and $u'(y^{\text{max}})/\eta'(y^{\text{max}}) < 1$. When $\lambda$ is large, one would have to implement a large drop in future output below
In order to raise current output by a small amount above $y_{\text{max}}$ in a way that is consistent with the buyer’s incentive to repay his debt. In order to see this we differentiate the buyer’s participation constraint, (44), obtained under the assumption that future output is constant, in the neighborhood of the highest steady state:

$$\frac{dy_1}{dy_0} \bigg|_{y_{\text{max}}} = \frac{\lambda \eta'(y_{\text{max}})}{\alpha [u'(y_{\text{max}}) - \eta'(y_{\text{max}})]} < 0.$$ 

When $\lambda \geq \alpha [1 - u'(y_{\text{max}})/\eta'(y_{\text{max}})]$ exploiting this trade-off is harmful to society’s welfare.

Remarkably, the solution to the dynamic contracting problem, (39)-(42), is fully characterized analytically by (43)-(44). In Figure 11 we illustrate the determination of $(y_0, y_1)$. The red curve labelled IR corresponds to (44), the debt limit at $t = 0$ given the output level in subsequent periods. As $y_1$ increases above $\bar{y}$ the debt limit decreases and hence $y_0$ decreases as well. By definition the IR curve intersects the 45°-line at $y_{\text{max}}$.

The blue curve labelled EULER corresponds to the first-order condition of the problem (43)-(44). Given the strict concavity of the surplus function it is optimal to smooth consumption by increasing $y_0$ when $y_1$ increases. When $\lambda$ is low the EULER curve is located above the IR curve at $y_1 = y_{\text{max}}$. As a result, the optimal solution, denoted $(y_0^{**}, y_1^{**})$ is such that $y_0^{**} > y_{\text{max}}$ and $y_1^{**} < y_{\text{max}}$.

**Corollary 6 (Second Welfare Theorem under large-group meetings)** Assume that $\eta$ is a convex function.

1. If either $y_{\text{max}} \leq \bar{y} \leq y^*$ or $\bar{y} < y_{\text{max}} < y^*$ and $\lambda \geq \alpha [1 - u'(y_{\text{max}})/\eta'(y_{\text{max}})]$, then the constrained-efficient allocation is implemented with "not-too-tight" solvency constraints.

2. If $\bar{y} < y_{\text{max}} < y^*$ and $\lambda < \alpha [1 - u'(y_{\text{max}})/\eta'(y_{\text{max}})]$, then the constrained-efficient allocation is implemented with repayment constraints that are slack in all future periods, $t \geq 1$.

The first part of Corollary 6 establishes conditions under which the AJ Second Welfare Theorem applies to our credit economy, namely, the constrained-efficient allocation is implemented with "not-too-tight" solvency constraints. A sufficient condition is $y_{\text{max}} \leq \bar{y}$ so that there is no trade-off between efficiency and incentive.

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KL provide an example where partial exclusion leads to a welfare-improving outcome. See their Example 2 on p. 875.
If $\hat{y} < y^\text{max} < y^*$, it is optimal to keep solvency constraints "not-too-tight" provided that $\lambda$ is sufficiently large.

If $\hat{y} < y^\text{max} < y^*$ and $\lambda$ is small, then the constrained-efficient allocation is implemented with participation constraints that are slack in some periods (second part of Corollary 6). The failure of the AJ Welfare Theorem is surprising as one would conjecture that higher debt limits allow society to generate larger gains from trade. By analogy, a larger value of money raises society's welfare in monetary economies. This reasoning is valid in a static sense. If $d_t$ increases, the sum of all surpluses in period $t$, $\alpha [u(y_t) - v(y_t)]$, increases. Moreover, for a given $p_t$, buyers cannot be made worse-off by an increase in their debt limit. However, there is a general equilibrium effect associated with a higher $d_t$, i.e., more IOUs are competing for DM goods. This increased competition raises the price of DM goods, $p_t = v'(y_t)$. If the economy is close enough to the first best, this pecuniary externality lowers the buyers' welfare (even though society as a whole is better off) and worsens their incentive to repay their debt in earlier periods.

So far we allowed the planner to pick the equilibrium among the full set of PBE. The results in Proposition 11 are robust if we restrict the planner to choose among 2-period cycles. To see this we adopt the numerical example from the left panel of Figure 9, $\gamma = 2.1$, $\beta = 0.4$, $\lambda = 1/6$. For these parameter values $\lambda < \alpha [1 - u'(y^\text{max})/q'(y^\text{max})]$. Society’s welfare over a 2-period cycle is measured by $u [y(d_0)] - v [y(d_0)] + \beta \{u[y(d_1)] - v[y(d_1)]\}$. In the left panel of Figure 12 we highlight in red the set of 2-period cycles, $(d_0, d_1)$, that dominate the equilibria under AJ "not-too-tight" solvency constraints. There exist a continuum of such cycles and they have the property that the buyer’s participation constraint is slack periodically. Moreover,
Figure 12: The blue area is set of all 2-period cycles. The red star is the 2-period cycle in G2MW and the green star is the highest steady state. Left panel: $\lambda = 1/6$; Right panel: $\lambda = 1/4$.

we represent by a green area the set of 2-period cycles that dominate the highest steady state, $y^{\text{max}}$. We also find a continuum of such cycles. Hence, the imposition of "not-too-tight" solvency constraints eliminates good equilibria. Finally, we represent with a black indifference curve society’s welfare at the constrained-efficient allocation and by a grey area the set of debt limits that implement the first best. Figure 12 shows that the first best is not implementable and, in accordance with the second part of Proposition 11, the constrained-efficient allocation is not a 2-period cycle.

The right panel of Figure 12 considers the same example as above when $\lambda$ is reduced from $\lambda = 1/6$ to $\lambda = 1/4$. The condition in Part 2 of Proposition 11 holds so that the highest steady state is not the constrained-efficient allocation. Moreover, there is no credit cycle under the "not-too-tight" solvency constraints. However, there are a continuum of PBE cycles, a fraction of which dominate the highest steady state.

Finally, G2MW show that a sufficient condition for the emergence of endogeneous credit cycles under the "not-too-tight" solvency constraint is $\lambda < \alpha [1 - u'(y^{\text{max}})/\eta'(y^{\text{max}})]/(2 + r)$. As illustrated in Figure 13 this condition is incompatible with the condition in Proposition 11 for "not-too-tight" solvency constraints to implement constrained-efficient allocations, i.e., $\lambda \geq \alpha [1 - u'(y^{\text{max}})/\eta'(y^{\text{max}})]$. Hence, when the sufficient condition for endogeneous cycles under "not-too-tight" solvency constraints holds, the normative rationale for imposing such constraints is no longer valid.

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5.3 Optima under arbitrary trading mechanism

We now turn to the optimal credit equilibrium allocation taking the mechanism to set the terms of the loan contract, $\eta$, as given. Although Propositions 10-11 are obtained under competitive pricing, they hold for any arbitrary trading mechanism, $\eta$. For example, if $\eta$ is determined by proportional bargaining, then $\hat{y} = y^*$, and Proposition 10 implies that the best PBE corresponds to the highest steady state, $y_t = y_{max}$ for all $t$.

Similarly, Proposition 10 also applies to generalized Nash bargaining: if $y_{max} \leq \hat{y} \leq y^*$ or $y^* \leq \hat{y} \leq y_{max}$, then the best PBE is the highest steady state (in the proof of the proposition we only use the fact that $\hat{y}$ is the unique maximizer). However, under Nash bargaining, the loan contract $\eta$ may not be convex in general, and hence Proposition 11 may not apply. Nevertheless, we showed that (39)-(42) defines a contraction mapping so that we can easily solve for the best PBE allocation numerically.

In Figure 14 we adopt the same functional forms and parameter values as the ones in the bottom left
panel of Figure 9. The left panel plots the DM output level over time while the right panel plots the buyer’s lifetime expected discounted utility. It can be seen that the allocation that maximizes social welfare is non-stationary. The output level is high in the first period and low and constant in all subsequent periods, in accordance with the second part of Proposition 11. Conversely, the buyer’s lifetime expected utility is low initially and high afterwards. The logic for why the solution is non-stationary is similar to the one described in the case of price taking. Given that the buyer’s surplus is hump-shaped, one can implement a high level of output in the initial period by promising a high utility in the future, which is achieved by lowering future output. In Figure 15 we represent the set of 2-period cycles under this parametrization. There are a continuum of cycles that dominate the periodic equilibria obtained under AJ solvency constraints (the red area) and the the highest steady state (the green area).

Figure 15: 2-period cycles and their welfare properties under Nash bargaining

6 Conclusion

We have characterized the set of outcomes of a pure credit economy and their welfare properties. The economy features intertemporal gains from trade that can be exploited with simple intra-period loan contracts. Such contracts and their execution are publicly recorded. Agents interact either through random, pairwise meetings under various trading mechanisms, as in the New-Monetarist literature, or in competitive spot markets, as in AJ. In sharp contrast with the existing literature (e.g., G2MW) we have shown that such economies exhibit a continuum of steady states, a continuum of endogenous cycle equilibria of any periodicity, and a wide variety of non-stationary equilibria. We have shown that any outcome of a pure monetary economy with no record-keeping but fiat money—which includes cycles and chaotic dynamics—is also an
outcome of the pure credit economy. The reverse is not true. There are outcomes of pure credit economies that cannot be implemented by monetary equilibria. For instance, there are equilibria where credit shuts down periodically while there are no equilibria where fiat money is valueless periodically. We also provided examples where steady states are dominated in terms of social welfare by cycles where borrowing constraints are slack periodically. Finally, we have characterized the PBEs that maximize social welfare and we showed that in contrast to the second welfare theorem in AJ constrained-efficient allocations are not implemented with "not-too-tight" solvency constraints.
References


7 Appendix

Proof of Proposition 1 \((\Rightarrow)\) Here we prove necessity. Suppose that \(\{ (y_t, \ell_t) \}_{t=0}^\infty\) is an equilibrium outcome in a credit equilibrium, \((s^b, s^a)\).

(i) Here we show condition (4). By condition (A3) in the credit equilibrium, buyers are willing to repay their obligations, \(\{ \ell_t \}_{t=0}^\infty\), at each period \(t\). Because the worst payoff to buyers at each period is 0 while the equilibrium payoff at period \(t\) is \(u(y_t) - \ell_t\), condition (4) is necessary for buyers to repay their promises at each period.

(ii) Here we show condition (5). First we show that \(\ell_t = v(y_t)\) for all \(t\). Obviously, if \(\ell_t < v(y_t)\), then the seller would not accept the offer. Suppose, by contradiction, that \(\ell_t > v(y_t)\). Then, the buyer may deviate and offer \((y', \ell_t)\) with \(v(y') \in (v(y_t), \ell_t)\). Because this deviation does not affect the buyer’s public record, it is dominant for the seller to accept it. It then is a profitable deviation.

Next, to show that \(y_t \leq y^*\) for all \(t\), suppose, by contradiction, that \(y_t > y^*\) and hence \(u(y') \geq \ell_t > v(y_t) > v(y^*)\). Let \((y', \ell')\) be such that \(u(y') - \ell' > u(y_t) - \ell_t\) and \(-v(y') + \ell' > -v(y_t) + \ell_t\) and \(\ell' \leq \ell_t\). We show that \((y', \ell')\) is a profitable deviation for the buyer. It suffices to show that it is dominant for the seller to accept the offer. The seller’s payoff at the current period is 0 if he rejects. However, if the seller accepts, then by (A3), the buyer will repay his promise \(\ell'\). Then, for the seller, by accepting the offer she obtains \(-v(y') + \ell' > 0\), which is strictly better than the payoff by rejecting the offer. Thus, \((y', \ell')\) is a profitable deviation for the buyer.

\((\Leftarrow)\) Here we show sufficiency. Let \(\{ (y_t, \ell_t) \}_{t=0}^\infty\) be a sequence satisfying (4) and (5). Consider \((s^b, s^a)\) given as follows: Buyers can be in two states, \(\chi_{i,t} \in \{ G, D \}\), where \(G\) means "good standing" and \(D\) means "default", and each buyer’s initial state is \(\chi_{i,0} = G\). The law of motion of the buyer \(i\)’s state is given by:

\[
\chi_{i,t+1}(\rho_{i,t}, \chi_{i,t}) = \begin{cases} 
D & \text{if } \rho_{i,t} = (\ell'_t, x_t, i) \text{ and } x_t < \min(d_t, \ell'_t) \text{ or } \chi_{i,t} = D \\
G & \text{otherwise} \end{cases}
\]

with \(d_t = \ell_t\) for each \(t \in \mathbb{N}\) and with \(s^b_{1,1}(\rho^i,t) = (y_t, \ell_t)\) if the state for \(\rho^i,t\) is \(G\) and \(s^b_{1,1}(\rho^i,t) = (0,0)\) otherwise; \(s^b_{1,2}(\rho^i,t, (\mathbf{y}_t, \ell_t), \text{yes}) = \min\{\ell_t, d_t\}\) if the state for \(\rho^i,t\) is \(G\) and \(s^b_{1,2}(\rho^i,t, (\mathbf{y}_t, \ell_t), \text{yes}) = 0\) otherwise; \(s^a(\rho^i,t, (\mathbf{y}_t, \ell_t)) = \text{yes}\) if the state for \(\rho^i,t\) is \(G\), \(v(\mathbf{y}_t) \leq \ell_t \leq d_t\), and \(s^a(\rho^i,t, (\mathbf{y}_t, \ell_t)) = \text{no}\) otherwise. We show that \((s^b, s^a)\) is a credit equilibrium.

Given \(s^b\) it is easy to see that \(s^a\) is optimal. Here we show that \(s^b\) is optimal given \(s^a\). Consider a buyer with state \(D\) at the beginning of period \(t\). Under \(s^b\), the buyer will not repay any promises in the CM and hence any positive offer to the seller will be rejected. Hence, it is optimal for him to offer \((0,0)\). Similarly, for such a buyer at the CM stage at period \(t\) with a promise \(\ell'\), his state will remain in \(D\), independent of his
repayment decisions and hence it is optimal to repay nothing.

Now consider a buyer with state $G$ at the CM stage of period $t$, with a promise $\ell'$ made to the seller. The buyer has to pay $\min\{\ell_t, \ell'\}$ to maintain state $G$. By (4), paying this amount is better than becoming a $D$ person, whose continuation value is 0. Finally, consider a buyer with state $G$ at the beginning of period $t$. Note that under $s^b$, his continuation value from period $t+1$ onward is independent of his offer at period $t$. Moreover, for any offer $(y, \ell)$, the seller accepts the offer if and only if $v(y) \leq \min\{\ell, \ell_t\}$. Thus, the buyer problem is

$$\max_{(y, \ell)} u(y) - \min\{\ell, \ell_t\} \text{ s.t. } v(y) \leq \min\{\ell, \ell_t\}.$$ 

Because $\ell_t = v(y_t) \leq v(y^*)$, $(y_t, \ell_t)$ is a solution to the problem. □

**Proof of Corollary 1** We have proved the “if” direction in Proposition 1 for the case where $d_t = \ell_t$ for all $t$. Suppose that $d_t > \ell_t = v(y^*)$ for some $t$. To show that the strategy constructed in Proposition 1 is an equilibrium, we only need to show that it is optimal to offer $(y^*, v(y^*))$, which follows directly from the buyer problem.

Now we prove the “only if” part. Let $(s^b, s^*)$ be a credit equilibrium that uses the strategies proposed in the proof of Proposition 1 with debt limits $\{d_t\}_{t=0}^\infty$ and has $\{(y_t, \ell_t)\}_{t=0}^\infty$ as its equilibrium outcome. Because $\ell_t$ is the equilibrium amount of debt at period $t$, $\ell_t \leq d_t$ for all $t$. Condition (9) is necessary for buyers to be willing to stay at state $G$. Suppose that $v(y_t) = \ell_t < v(y^*)$. If $d_t > \ell_t$, then buyer could offer some $(y', \ell')$ at date $t$ with $\ell_t < \ell' < d_t$, $\ell' > v(y')$ but $u(y') - \ell' > u(y_t) - \ell_t$. By (9) the buyer will repay his debt $\ell'$, and the seller will accept the offer. This leads to a contradiction so $d_t = \ell_t$. However, if $\ell_t = v(y^*)$ and hence $y_t = y^*$, then for any $d_t \geq \ell_t$ that satisfies (9), the buyer’s optimal offer is $(y_t, \ell_t) = (y^*, v(y^*))$.

**Proof of Corollary 2** Rewrite the incentive-compatibility constraint (11) at time $t+1$ and multiply it by $\beta$ to obtain:

$$\beta d_{t+1} \leq \beta^2 \{\alpha[u(y_{t+2}) - v(y_{t+2})] + d_{t+2}\}. \tag{46}$$

Combining (11) and (46) we get:

$$d_t \leq \beta \{\alpha[u(y_{t+1}) - v(y_{t+1})]\} + \beta^2 \alpha[u(y_{t+2}) - v(y_{t+2})] + \beta^2 d_{t+2}.$$ 

By successive iterations we generalize the inequality above as follows:

$$d_t \leq \sum_{s=1}^{T} \beta^s \{\alpha[u(y_{t+s}) - v(y_{t+s})]\} + \beta^{t+T} d_{t+T}. \tag{47}$$

Note that for each $s \geq 1$, $[u(y_{t+s}) - v(y_{t+s})]$ is bounded and, because $\{d_t\}$ is bounded, $\lim_{T \to \infty} \beta^{t+T} d_{t+T} = 0$. Hence, by taking $T$ to infinity, it follows from (47) that $\{d_t\}$ satisfies (9).
Proof of Proposition 2  To calculate the threshold, \( d^{\text{max}} \), define the right side of (13) as a function

\[
\Psi(d) = \alpha \{ u[z(d)] - v[z(d)] \}.
\]

Note that \( \Psi \) is continuous in \( d \) with \( \Psi(0) = 0 \) and \( \Psi(d) = \alpha [u(y^*) - v(y^*)] \) for all \( d \geq v(y^*) \). Moreover, it is differentiable with

\[
\Psi'(d) = \alpha \left[ \frac{u'[z(d)] - v'[z(d)]}{v'[z(d)]} \right] \quad \text{if} \quad d \in (0, v(y^*)], \quad \text{and} \quad \Psi'(d) = 0 \quad \text{if} \quad d > v(y^*).
\]

It is easy to check that this derivative is decreasing in \( d \) for all \( d \in (0, v(y^*)) \). Hence, \( \Psi \) is a concave function of \( d \). There exists a \( d > 0 \) solution to (13) if and only if \( \Psi'(0) > r \), which is always satisfied since \( \Psi'(0) = \infty \) by assumption on preferences. It follows that there exists a unique \( d^{\text{max}} > 0 \) such that \( \Psi(d^{\text{max}}) = rd^{\text{max}} \).

Therefore, for all \( d \in [0,d^{\text{max}}] \), the condition (13) is satisfied.

Proof of Lemma 1  Define the correspondence \( \Gamma: \mathbb{R}_+ \supset \mathbb{R}_+ \) as follows:

\[
\Gamma(d) = \{ x \in \mathbb{R}_+ : r (1 + \beta) x \leq \alpha \{ u[z(d)] - v[z(d)] \} + \beta \alpha \{ u[z(x)] - v[z(x)] \} \}.
\]  (48)

Then, \( \gamma(d) = \max \Gamma(d) \). First we show that \( \Gamma(d) \) is a closed interval and \( \gamma \) is well-defined. By definition, \( x \in \Gamma(d) \) if and only if

\[
 r(1 + \beta) x \leq \Psi(d) + \beta \Psi(x).
\]

Using a similar argument to that in Proposition 2, \( \Gamma(d) \) is a closed interval with zero as the lower end point. Thus, \( \gamma \) is well-defined, and \( \gamma(d) \) is the unique \( x > 0 \) that satisfies

\[
 r (1 + \beta) x = \Psi(d) + \beta \Psi(x). \]  (49)

Moreover, if \( d > d' \), then \( \Gamma(d') \subseteq \Gamma(d) \), and hence \( \gamma \) is an increasing function. It is also easy to show that

\[
\gamma(0) > 0, \quad \gamma(d^{\text{max}}) = d^{\text{max}},
\]

where \( d^{\text{max}} \) is given in Proposition 2. We denote \( \gamma(0) \) by \( d^{\text{min}} > 0 \). Because \( \Psi(d) \) is constant for all \( d \geq v(y^*) \), \( \gamma \) is constant for all \( d \geq v(y^*) \), but it is strictly increasing for \( d < v(y^*) \). Now we show that \( \gamma \) is a concave function. Because \( \Psi(x) \) is everywhere differentiable, by the implicit function theorem, for all \( d < v(y^*) \),

\[
\gamma'(d) = \Psi'(d) \{(1 + \beta) r - \beta \Psi'[\gamma(d)]\}^{-1},
\]

which is decreasing in \( d \) as \( \Psi' \) is decreasing. Note that \( \beta \Psi'[\gamma(0)] < (1 + \beta)r \). Hence, \( \gamma \) is a concave function.
Proof of Proposition 3  Notice that, by definition, any pair \((d_0, d_1)\) that satisfies \(d_0 \leq \gamma(d_1)\) and \(d_1 \leq \gamma(d_0)\) also satisfies (14) with \(y_0 = z(d_0)\) and \(y_1 = z(d_1)\), and hence \((d_0, d_1)\) is a 2-period credit cycle. To characterize the set of such cycles, we define the “inverse” of \(\gamma\), denoted by \(\lambda\), with domain \([0, \gamma[v(y^*)]]\), as follows:

\[
\lambda(d) = \begin{cases} 
0 & \text{if } d \leq d_{\text{min}} \\
\gamma^{-1}(d) & \text{if } d_{\text{min}} < d \leq \gamma[v(y^*)].
\end{cases}
\]

So \(\lambda\) is a convex function.

Consider two cases.

First suppose that \(d_{\text{max}} \leq v(y^*)\). Then, \(d_{\text{max}} = \gamma(d_{\text{max}}) \leq \gamma[v(y^*)]\) and hence \(\lambda(d_{\text{max}}) = d_{\text{max}} = \gamma(d_{\text{max}}) > 0\). Moreover, for any \(d \in [0, d_{\text{max}}]\), \(\lambda(d) < \gamma(d)\) but for any \(d > d_{\text{max}}\), \(\lambda(d) > \gamma(d)\). Then, for each \(d_0 \in [0, d_{\text{max}}]\) and any \(d_1 \in (\lambda(d_0), \gamma(d_0))\), \((d_0, d_1)\) satisfies \(d_0 \leq \gamma(d_1)\) and \(d_1 \leq \gamma(d_0)\) and hence can sustain a 2-period, credit-cycle equilibrium. In this case, let \(\Omega(d_0) = (\lambda(d_0), \gamma(d_0))\), and, because \(\lambda(d_0) < \gamma(d_0)\), it is a nondegenerate interval. Recall that \(\gamma(0) = d_{\text{min}} > 0\) and \(\lambda(0) = 0\). Moreover, for any such credit equilibrium, the constraint \(\ell_t \leq d_t\) never binds.

Next, suppose that \(d_{\text{max}} > v(y^*)\). Then, \(d_{\text{max}} = \gamma(d_{\text{max}}) = \gamma[v(y^*)]\) because \(\gamma\) is constant for \(d \geq v(y^*)\). For \(d_0 \in [0, v(y^*)]\), the construction of \(\Omega(d_0)\) is the same as the first case. However, for \(d \in (v(y^*), d_{\text{max}})\), \(\lambda(d) < v(y^*) < d < \gamma(d)\). Hence, for any \(d_0 \in (v(y^*), d_{\text{max}})\) and \(d_1 \in (v(y^*), \gamma(d_0))\), \((d_0, d_1)\) satisfies \(d_0 \leq \gamma(d_1)\) and \(d_1 \leq \gamma(d_0)\). In this case, let \(\Omega(d_0) = (v(y^*), \gamma(d_0))\), and, because \(v(y^*) < \gamma(d_0)\), it is a nondegenerate interval. Note that for such credit equilibrium with \(d_0 \in (v(y^*), d_{\text{max}})\), the borrowing constraint \(\ell_t \leq d_t\) never binds.

Proof of Corollary 3  Fix a pair \((\alpha, r)\). As shown earlier, a pair \((d_0, d_1)\) is a 2-period cycle if and only if \(d_0 \leq \gamma(d_1; \alpha, r)\) and \(d_1 \leq \gamma(d_0; \alpha, r)\), where \(\gamma\) is given by Lemma 1. Note that here we make the parameters \((\alpha, r)\) explicit. By Proposition 3, for all \(d_0 \in [0, d_{\text{max}}]\), there exists a continuum of \(d_1\) such that \((d_0, d_1)\) is a 2-period cycle. Now, \(d_1 \leq \gamma(d_0; \alpha, r)\) if and only if

\[
r(1 + \beta)d_1 \leq \Psi(d_0; \alpha) + \beta \Psi(d_1; \alpha), \tag{50}
\]

where \(\Psi\) is given in the proof of Proposition 2, and it is easy to verify that \(\Psi(d; \alpha)\) is increasing in \(\alpha\). Moreover, the left-side of (50) is increasing in \(r\) but the right-side is decreasing in \(r\). Thus, if \(\alpha' > \alpha\), then (50) implies that the same inequality holds for \(\alpha'\) as well, but it is easy to see that there exists pairs that are equilibria under \(\alpha'\) but not under \(\alpha\). Similar arguments hold for \(r\) as well.
Proof of Corollary 4  Note that \( d_{\text{max}} > v(y^*) \) if and only if \( r < \alpha \left[ u(y^*) - v(y^*) \right] / v(y^*) \). The proof then follows from that of Proposition 3.

Proof of Proposition 4  Define the mapping \( \gamma : \mathbb{R}^{(T-1)+} \to \mathbb{R}^+ \) as follows:

\[
\gamma(d_1, \ldots, d_{T-1}) = \max \left\{ x \in \mathbb{R}^+ : \alpha \sum_{j=1}^{T-1} \beta^j \left[ u[z(d_j)] - v[z(d_j)] \right] + \alpha \beta^T \left[ u[z(x)] - v[z(x)] \right] \right\}.
\]

It is easy to show that the function \( \gamma \) satisfies properties listed in Lemma 1. A \( T \)-period cycle equilibrium is list, \((d_0, \ldots, d_{T-1})\), that satisfies \( d_j \leq \gamma(d_{j+1}, d_{j+2}, \ldots, d_{j+T-1}) \) where the index for each debt limit is modulo \( T \). The rest of the proof is similar to the proof of Proposition 3.

Proof of Proposition 5  Replace \( d_t = \phi_t \) in (19) to get

\[
d_t = \beta d_{t+1} \left[ 1 + \alpha \frac{u'(y_{t+1}) - u'(y_{t+1})}{v'(y_{t+1})} \right]. \tag{51}
\]

The right side of (51), \( \left[ u'(y_{t+1}) - u'(y_{t+1}) \right] / v'(y_{t+1}) \), is the derivative of the concave function, \( u(y_{t+1}) - v(y_{t+1}) \), with respect to \( d_{t+1} \). Consequently,

\[
\frac{u'(y_{t+1}) - u'(y_{t+1})}{v'(y_{t+1})} d_{t+1} < u(y_{t+1}) - v(y_{t+1}). \tag{52}
\]

From (51) and (52),

\[
d_t \leq \beta \alpha \left[ u(y_{t+1}) - v(y_{t+1}) \right] + \beta d_{t+1}. \tag{53}
\]

Iterating (53),

\[
d_t \leq \sum_{j=1}^{J} \beta^j \alpha \left[ u(y_{j+j}) - v(y_{j+j}) \right] + \beta^J d_{t+j}. \tag{54}
\]

By using the transversality condition, \( \lim_{J \to \infty} \beta^J d_{t+j} = 0 \), we prove that the sequence, \( \{d_t\} \), solution to (51) satisfies (9), and hence it is part of a credit equilibrium.

Proof of Proposition 6  Because \( 0 < d < d_{\text{max}} \), it follows that

\[
rd < \alpha \{ u[z(d)] - v[z(d)] \}. \tag{55}
\]

Fix an element \( \chi_0 \in \mathbb{X} \) and let \( \mathbb{X}_0 = \mathbb{X} - \{ \chi_0 \} \). Define the set

\[
\Phi(d) = \{ \langle d_{\chi} ; \chi \in \mathbb{X}_0 \rangle : d_{\chi} < d, \ rd < \sum_{\chi \in \mathbb{X}_0} \pi(\chi) \alpha \{ u[z(d_{\chi})] - v[z(d_{\chi})] \} + \pi(\chi_0) \alpha \{ u[z(d)] - v[z(d)] \} \}.
\]

By continuity and by (55), the set \( \Phi(d) \) is nonempty and is open. So, by (22), for any \( \langle d_{\chi} ; \chi \in \mathbb{X}_0 \rangle \in \Phi(d) \), \( \langle d_{\chi} ; \chi \in \mathbb{X} \rangle \) with \( d_{\chi_0} = d \) is a sunspot credit equilibrium. Note that because \( \Phi(d) \) is nonempty and is open, there are a continuum of points in it with \( d_{\chi} \neq d_{\chi'} \) for all \( \chi \neq \chi' \).
Proof of Proposition 7  The proof is almost the same as that of Proposition 1, as seen from (23), except for that we need to add \( \lambda \) before \( d_t \) for the temptation to default, and to add \( \theta \) before the future surpluses for the buyer.

Proof of Proposition 8  First we show that \( d^{\text{max}} > 0 \) if and only if \( \lambda \theta < \alpha \theta / (1 - \theta) \). Recall that \( d^{\text{max}} \) is defined as the highest solution to \( r \lambda d = \alpha \theta \{ u[z(d)] - v[z(d)] \} \). Define the function

\[
\Psi(d) = \alpha \theta \{ u[z(d)] - v[z(d)] \}.
\]

Then, \( d^{\text{max}} > 0 \) if and only if \( \Psi'(0) > r \lambda \). Now,

\[
\Psi'(d) = \alpha \theta \left[ \frac{u'[z(d)] - v'[z(d)]}{(1 - \theta) u'[z(d)] + \theta v'[z(d)]} \right]
\]
and hence \( \Psi'(0) = \alpha \theta / (1 - \theta) \), which shows that \( d^{\text{max}} > 0 \) if and only if \( \lambda \theta < \alpha \theta / (1 - \theta) \).

Define the mapping \( \gamma : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) as follows:

\[
\gamma(d) = \sup \{ x \in \mathbb{R}^+ : r \lambda (1 + \beta) x = \alpha \theta \{ u[z(d)] - v[z(d)] \} + \beta \alpha \theta \{ u[z(x)] - v[z(x)] \} \}.
\]

The left side of the equation in (56), \( r \lambda (1 + \beta) x \), is linear and increasing while the right side, \( \beta \alpha \theta \{ u[z(x)] - v[z(x)] \} \), is non-decreasing and concave. Given that the first term on the right side is non-negative, \( \gamma(d) \) is well-defined. Moreover, it is easy to verify that for a pair \( ((y_0, d_0), (y_1, d_1)) \) to be 2-period cycle credit equilibrium outcome, i.e., for it to satisfy (27), it is necessary and sufficient to have \( y_t = z(d_t) \) and \( d_0 \leq \gamma(d_1) \) and \( d_1 \leq \gamma(d_0) \).

Hence, we may define a 2-period cycle equilibrium as a pair, \( (d_0, d_1) \), that satisfies \( d_0 \leq \gamma(d_1) \) and \( d_1 \leq \gamma(d_0) \). Note that \( \gamma(0) = 0 \) and \( \gamma(d^{\text{max}}) = d^{\text{max}} \). Moreover, use the similar arguments to Lemma 1, we can show \( \gamma \) is concave. Moreover, when \( d^{\text{max}} > 0 \), i.e., when \( r \lambda < \alpha \theta / (1 - \theta) \), for each \( 0 < d_0 < d^{\text{max}} \), \( d_0 < \gamma(d_0) \), and, for each \( d_1 \in (d_0, \gamma(d_0)) \), \( d_0 < \gamma(d_0) \leq \gamma(d_1) \). So any such \( (d_0, d_1) \) is a 2-period cycle and there are continuum of them.

To show the existence of 2-period cycles without periodic credit shutdowns, we need to show that \( \gamma(0) > 0 \). Recall that

\[
\Psi(d) = \alpha \theta \{ u[z(d)] - v[z(d)] \}.
\]

Then, the right-side inside the set in (56) is \( \Psi(d) + \beta \Psi(x) \), and its derivative is

\[
\beta \Psi'(x) = \beta \alpha \theta \left[ \frac{u'[z(x)] - v'[z(x)]}{(1 - \theta) u'[z(x)] + \theta v'[z(x)]} \right].
\]
It follows that \( \gamma(0) > 0 \) if and only if \( r \lambda (1 + \beta) \beta \Psi'(0) = \beta \alpha \theta / (1 - \theta) \), which corresponds to the condition
\( r < \sqrt{1 + \alpha \theta / \lambda (1 - \theta)} - 1 \). Given that \( \gamma(0) > 0 \), for any pair \( (d_0, d_1) \) with \( d_0 = 0 \) and \( d_1 \in (0, \gamma(0)) \) is a credit equilibrium where the credit is shut down every two periods.
Equivalence between monetary and credit equilibria  Here we extend the equivalence result, Proposition 5, to other trading mechanisms. We first consider pairwise bargaining and then consider Walrasian pricing. Note that even in the environment introduced in Section 4, the monetary trades follow a similar pattern to that in Section 3.3: buyers use money to buy DM goods from sellers in the DM; while buyers produce CM goods in the DM, they do so in anticipation of selling them in the CM; sellers use the money obtained from DM to buy CM goods in the CM. Because $\lambda \leq 1$, buyers never produce CM goods for self-consumption. As a result, the parameter $\lambda$ plays no role in monetary equilibria.

Bargaining  Under a general bargaining solution represented by the function $\eta(y)$, the sequence for the values of money, $\{\phi_t\}$, solves

$$\phi_t = \beta \phi_{t+1} \left\{ \alpha \left[ \frac{u'(y_{t+1})}{\eta'(y_{t+1})} - 1 \right] + 1 \right\},$$  (57)

where $\eta(y_t) = \phi_t$ for all $t$. In the credit economy, the debt limits, $\{d_t\}$, solves

$$d_t = \beta \left\{ \alpha [u(y_{t+1}) - \eta(y_{t+1})] + d_{t+1} \right\}.$$  (58)

If the buyer’s surplus, $u \circ \eta^{-1}(\phi_t) - \phi_t$ is convex in terms of the value of money, then the same reasoning as before applies. This condition is satisfied for the proportional bargaining solution and for the general Nash bargaining solution under the functional forms for $u$ and $v$ that guarantee the concavity of the buyer’s surplus.

Walrasian pricing  Suppose the DM is competitive and $p_t$ denotes the price of DM goods in terms of CM goods. In a monetary economy the buyer chooses money holdings solution to:

$$\max_{m \geq 0} \left\{ -\phi_t m + \beta \alpha [u(y_{t+1}) - p_{t+1}y_{t+1}] + \beta \phi_{t+1} m \right\},$$  (58)

where, $\phi_{t+1} m = p_{t+1}y_{t+1}$. The first-order condition for (58) is

$$\phi_t = \beta \phi_{t+1} \left\{ \alpha \left[ \frac{u'(y_{t+1})}{p_{t+1}} - 1 \right] + 1 \right\}.$$  (59)

From the seller’s maximization problem, $p_{t+1} = u'(y_{t+1})$ so that $\{\phi_t\}$ solves

$$\phi_t = \beta \phi_{t+1} \left\{ \alpha \left[ \frac{u'(y_{t+1})}{u'(y_{t+1})} - 1 \right] + 1 \right\}.$$  (59)

It should be noticed that it is the same first-order difference equation than the one obtained under buyer’s TIOLI bargaining. The difference equation for the debt limit in a credit economy under AJ constraints is

$$d_t = \beta \left\{ \alpha [u(y_{t+1}) - u'(y_{t+1})y_{t+1}] + d_{t+1} \right\}.$$  (59)
Notice that
\[
\phi_{t+1} \left[ \frac{u'(y_{t+1})}{v'(y_{t+1})} - 1 \right] = u'(y_{t+1})y_{t+1} - v'(y_{t+1})y_{t+1} < u(y_{t+1}) - v'(y_{t+1})y_{t+1},
\]
from the concavity of \( u \). This proves that the phase of the monetary equilibrium is located to the left of the phase line of the credit equilibrium. Hence, by the same reasoning as before, any outcome of the monetary economy is an outcome of the credit economy.

**Existence of 2-period cycles under alternative mechanisms**

**Walrasian pricing** Under Walrasian pricing, the function \( \eta(y) \) is given by \( \eta(y) = v'(y)y \). Here we show existence of a continuum of 2-period cycles when \( \eta(y) \) is convex. Recall that \( z(d) = \min \{ \eta^{-1}(d), y^* \} \).

**Lemma 2** Suppose that \( \eta(y) \) is convex. Then, \( d_{\max} \), the positive solution to
\[
r \lambda d = \alpha \{ u[z(d)] - \eta[z(d)] \}
\]
is unique and for each \( d_0 \in [0, d_{\max}] \), there is a nondegenerate interval, \( \Omega(d_0) \), such that for any \( d_1 \in \Omega(d_0) \), \((d_0, d_1)\) is a (strict) 2-period cycle.

**Proof.** Because \( \eta(y) \) is convex, there is a unique positive number, denoted \( y_{\max} \), such that
\[
r \lambda \eta(y_{\max}) = \alpha \{ u[y_{\max}] - \eta(y_{\max}) \}.
\]
We claim that \( d_{\max} \) given by
\[
d_{\max} = \begin{cases} 
\eta(y_{\max}) & \text{if } y^* \geq y_{\max} \\
\frac{\alpha \{ u(y^*) - \eta(y^*) \}}{r} & \text{otherwise}
\end{cases}
\]
is the unique positive solution to (60). We consider two cases. First suppose that \( y^* \geq y_{\max} \). Then, \( z(d) = \eta^{-1}(d) \) for all \( d \leq d_{\max} \) and hence \( d_{\max} \) is the unique solution to (60). Next, suppose that \( y^* < y_{\max} \). Then, \( r \lambda \eta(y^*) < \alpha \{ u(y^*) - \eta(y^*) \} \) and hence \( d_{\max} > \eta(y^*) \). Thus, \( z(d_{\max}) = y^* \) and so \( d_{\max} \) is the unique solution to (60). Note that any \( d \in [0, d_{\max}] \) corresponds to a steady-state equilibrium. Let us turn to 2-period cycles. For each \( d_0 \in [0, d_{\max}] \), we claim that the equation,
\[
r \lambda d_1 = \frac{\alpha \{ u[z(d_0)] - \eta[z(d_0)] \} + \beta \alpha \{ u[z(d_1)] - \eta[z(d_1)] \}}{1 + \beta},
\]
has a unique positive solution, denoted \( \gamma(d_0) \), and that \( \gamma(d_0) > d_0 \) for all \( d \in [0, d_{\max}] \), with \( \gamma(d_{\max}) = d_{\max} \). To see this, let \( \hat{y} = \arg \max [u(y) - \eta(y)] \) and let \( \hat{d} = \eta(\hat{y}) \). Then, \( \alpha \{ u[z(d)] - \eta[z(d)] \} \) is increasing and concave for \( d_0 < \hat{d} \) and is decreasing for \( d_0 > \hat{d} \) (and becomes flat after reaching \( \eta^{-1}(y^*) \)) with \( \lim_{d \to \hat{d}} \alpha \{ u[z(d)] - \eta'[z(d)]z'(d) = \infty \). Hence, (61) has a unique positive solution. The fact that \( \gamma(d_0) > d_0 \) for all \( d \in [0, d_{\max}] \).
Proof of Proposition 9

(1) Suppose that \( y^* \leq y^\text{max} \). Then, the sequence with \( \{y_t\}_{t=0}^{\infty} \) with \( y_t = y^* \) for all \( t \) is implementable by setting \( \ell_t = v(y_t) \) for all \( t \).

(2) Suppose that \( y^* > y^\text{max} \). We show that the optimal sequence that has \( y^\text{max} \) at each date and \( \ell_t = v(y_t) \) for all \( t \). Suppose, by contradiction, that there is another sequence \( \{z_t, \ell_t\}_{t=0}^{\infty} \) satisfying (33) and (34) with a strictly higher welfare.

It then follows that \( y^* \geq z_t > y^\text{max} \) for some \( t \). Let \( t_0 \) be the first \( t \) such that \( u(z_t) - v(z_t) > u(y^\text{max}) - v(y^\text{max}) \). Without loss of generality, we may assume that \( t_0 = 0 \).

Now, if \( z_t \leq z_0 \) for all \( t > 0 \), then, by (33) and (34),
\[
v(z_0) \leq \ell_t \leq \lambda^{-1} \sum_{s=1}^{\infty} \beta^s \alpha [u(z_s) - \ell_s] \leq \lambda^{-1} \sum_{s=1}^{\infty} \beta^s \alpha [u(z_0) - v(z_0)],
\]
where the second inequality follows the fact that \( \ell_t \leq v(z_t) \leq v(z_0) \) for all \( t \), but this implies \( z_0 \leq y^\text{max} \), a contradiction. So \( y^* \geq z_t > z_0 \) for some \( t \). Again, without loss of generality we may assume that \( z_1 > z_0 \).

By induction, we can then assume that the sequence \( \{z_t\} \) is strictly increasing and is bounded from above. So there exists a limit \( \bar{z} = \lim_{t \to \infty} z_t > y^\text{max} \). Hence, by monotonicity, we have for all \( t \),
\[
rv(z_t) \leq r\ell_t \leq \frac{\alpha |u(\bar{z}) - v(\bar{z})|}{\lambda},
\]
and, by taking \( t \) to infinity, we have
\[
rv(\bar{z}) \leq \frac{\alpha |u(\bar{z}) - v(\bar{z})|}{\lambda},
\]
a contradiction.
Proof of Proposition 10  The program that selects the best PBE is

\[
\max_{\{y_t\}_{t=0}^{\infty}} +\infty \sum_{t=0}^{+\infty} \beta^t \alpha [u(y_t) - v(y_t)]
\]

s.t.  \(\lambda \eta(y_t) \leq \alpha \sum_{s=1}^{+\infty} \beta^s [u(y_{t+s}) - \eta(y_{t+s})]\) \hspace{1cm} (62)

\(y_t \leq y^* \text{ for all } t = 0, 1, 2, \ldots\) \hspace{1cm} (64)

(1) Suppose that \(y^* \leq y^{\max}\). Then, the sequence with \(\{y_t\}_{t=0}^{\infty}\) with \(y_t = y^*\) for all \(t\) is implementable.

(2) Suppose that \(y^* > y^{\max}\) but \(y^{\max} \leq \hat{y}\). We show that the sequence that has \(y^{\max}\) at each date is the optimum. Suppose, by contradiction, that there is another sequence \(\{z_t\}_{t=0}^{\infty}\) satisfying (63) and (64) with a strictly higher welfare.

Given that \(u(y) - v(y)\) is increasing in \(y\) for all \(y < y^*\), \(y^* \geq z_t > y^{\max}\) for some \(t\). Let \(t_0\) be the first \(t\) such that \(u(z_t) - v(z_t) > u(y^{\max}) - v(y^{\max})\). Without loss of generality, we may assume that \(t_0 = 0\).

First we show that \(z_t \leq \hat{y}\) for all \(t\). Suppose, by contradiction, that there is a \(t\) such that \(z_t > \hat{y}\). Then, because \(\hat{y} \geq y^{\max}\),

\[
\lambda \eta(z_t) > \lambda \eta(\hat{y}) \geq \sum_{s=1}^{+\infty} \beta^s \alpha [u(\hat{y}) - \eta(\hat{y})] \geq \sum_{s=1}^{+\infty} \beta^s \alpha [u(z_{t+s}) - \eta(z_{t+s})],
\]

a contradiction to (63).

Now, if \(z_t \leq z_0\) for all \(t > 0\), then, by (63),

\[
\lambda \eta(z_0) \leq \sum_{s=1}^{+\infty} \beta^s \alpha [u(z_s) - \eta(z_s)] \leq \sum_{s=1}^{+\infty} \beta^s \alpha [u(z_0) - \eta(z_0)],
\]

where the second inequality follows the fact that \(z_t \leq \hat{z}\) for all \(t\), but this implies \(z_0 \leq y^{\max}\), a contradiction. So \(y^* \geq z_t > z_0\) for some \(t\). Again, without loss of generality we may assume that \(z_1 > z_0\). By induction, we can then assume that the sequence \(\{z_t\}\) is strictly increasing and is bounded from above. So there exists a limit \(\bar{z} = \lim_{t \to \infty} z_t > y^{\max}\). Hence, by monotonicity, we have for all \(t\),

\[
r \lambda \eta(z_t) \leq \alpha [u(\bar{z}) - \eta(\bar{z})],
\]

and, by taking \(t\) to infinity, we have

\[
r \lambda \eta(\bar{z}) \leq \alpha [u(\bar{z}) - \eta(\bar{z})],
\]

a contradiction to \(\bar{z} > y^{\max}\).

Core and competitive equilibrium  Recall that an allocation, \(L = \{(y(i), x(i)), (y(j), x(j)) : i \in B, j \in S\}\), where \((y(i), x(i))\) denotes buyer \(i\)'s DM and CM consumptions and \((y(j), x(j))\) denotes seller \(j\)'s DM and
CM consumptions, is in the core if there is no blocking (finite) coalition, \( I \subset B \cup S \), such that each agent in \( I \) enjoys at least the same utility as his allocation in \( L \), but at least one of them is strictly better off. Now we show that the only core allocation is the competitive outcome, with debt limit, \( d \), is given by the symmetric allocation, \((y, \ell)\), such that \( \ell = \eta(y) \equiv v'(y)y \) and \( y = \min\{y^*, \eta^{-1}(d)\} \).

First notice that, by standard arguments, the competitive outcome is in the core. For necessity, we restrict ourselves to symmetric allocations. For a justification of such assumption, see Mas-Colell et al. Note that to be in the core, \( u(y) \geq \ell \geq v(y) \). First we show that \( \ell = v'(y)y \). Suppose, by contradiction, \( \ell \neq v'(y)y \). Assume that \( \ell < v'(y)y \). The other direction has a similar proof. Let \( \varepsilon \) be so small that

\[
[v'(y) - \varepsilon]y > \ell. \tag{65}
\]

Consider a coalition with \( m \) buyers and \( n \) sellers such that with \( \delta = m/n < 1 \), we have

\[
\frac{v(y) - v(\delta y)}{(1 - \delta)y} > v'(y) - \varepsilon. \tag{66}
\]

Consider the following allocation: each buyer consumes \( y \) and issues an IOU with face value \( \ell \), and each seller produces \( \delta y \) and receives an IOU with face value \( \delta \ell \). Note that such allocation is feasible:

\[
m y = n \delta y \quad \text{and} \quad m \ell = n \delta \ell.
\]

Now, each buyer enjoys the same utility as before, but each seller has a higher utility: combining (65) and (66),

\[
v(y) - v(\delta y) > [v'(y) - \varepsilon](1 - \delta)y > (1 - \delta)\ell,
\]

and hence

\[
\delta \ell - v(\delta y) > \ell - v(y).
\]

This proves \( \ell = v'(y)y = \eta(y) \). Finally, if \( y < \min\{y^*, \eta^{-1}(d)\} \), then a buyer and a seller can form a coalition to increase surplus.

**Recursive formulation of best PBE** Here we show that we can solve the problem (62)-(64) recursively. First we show that the use the promised utility is justified.

**Lemma 3** A sequence \( \{y_t\}_{t=0}^{\infty} \) satisfies (63) and (64) if and only if there is a sequence \( \{\omega_t\}_{t=0}^{\infty} \) such that,
for all \( t = 0, 1, 2, \ldots \),

\[
\omega_t \leq \alpha [u(y_t) - \eta(y_t)] + \beta \omega_{t+1},
\]

\( \eta(y_t) \leq \beta \omega_{t+1}/\lambda, \) \( \tag{67} \)

\( y_t \in [0, y^*], \) \( \tag{68} \)

\( \omega_t \in [0, \bar{\omega}]. \) \( \tag{69} \)

\( \) \( \) \( \) \( \) \( \) \( \)

**Proof.** Suppose that \( \{y_t\}_{t=0}^{\infty} \) satisfies (63) and (64). Then, define, for each \( t = 0, 1, 2, \ldots, \)

\[
\omega_t = \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})].
\]

(71)

Because the right side of (63) is equal to \( \beta \omega_{t+1}/\lambda \) for each \( t, \{\omega_t, y_t\}_{t=0}^{\infty} \) satisfies (68). Because, by definition of \( \hat{y} \), for all \( t = 0, 1, 2, \ldots, \)

\[
u(y_t) - \eta(y_t) \leq u(\hat{y}) - \eta(\hat{y}),
\]

and because of (64), \( \{\omega_t, y_t\}_{t=0}^{\infty} \) satisfies (70). Finally, by (71), for each \( t = 0, 1, 2, \ldots, \)

\[
\omega_t = \alpha [u(y_t) - \eta(y_t)] + \beta \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s+1}) - \eta(y_{t+s+1})] = \alpha [u(y_t) - \eta(y_t)] + \beta \omega_{t+1},
\]

and hence \( \{\omega_t, y_t\}_{t=0}^{\infty} \) satisfies (67).

Conversely, suppose that \( \{\omega_t, y_t\}_{t=0}^{\infty} \) satisfies (67)-(70). Then, \( \{y_t\}_{t=0}^{\infty} \) satisfies (64) by (70). To show (63), define, for each \( t = 0, 1, 2, \ldots, \)

\[
\omega'_t = \sum_{s=0}^{\infty} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})].
\]

(72)

By (68), it suffices to show that \( \omega_t \leq \omega'_t \) for all \( t \geq 0 \). Let \( t \) be given. We show by induction on \( T \) that

\[
\omega_t \leq \sum_{s=0}^{T} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \omega_{T+1}.
\]

(73)

When \( T = 0 \), this follows from (67). Suppose that it holds for \( T \). Then,

\[
\omega_t \leq \sum_{s=0}^{T} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \omega_{T+1}
\]

\[
= \sum_{s=0}^{T} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+1} \{ \alpha [u(y_{T+1}) - \eta(y_{T+1})] + \beta \omega_{T+2} \}
\]

\[
= \sum_{s=0}^{T+1} \beta^s \alpha [u(y_{t+s}) - \eta(y_{t+s})] + \beta^{T+2} \omega_{T+2}.
\]

This proves (73). Now, because, by (70), \( \omega_{T+1} \leq \bar{\omega} \) for all \( T \), it follows from the limit by taking \( T \) to infinity in (73) that \( \omega_t \leq \omega'_t. \)
Because of Lemma 3, we may replace the constraints (63) and (64) by (67)-(70). Note that the constraints (67)-(70) are in the recursive form with the state variable \( \omega_t \), but the initial condition \( \omega_0 \) is also a choice variable.

Thus, for any given \( \omega_0 \in [0, \bar{\omega}] \), our problem is a standard recursive problem, and can be solved by the standard Bellman equation. We may then define the value function \( V(\omega) \) as follows:

\[
V(\omega) = \max_{(y_t) \in \mathbb{R}^+} \sum_{t=0}^{+\infty} \beta^t \alpha [u(y_t) - v(y_t)]
\]

subject to (67)-(70) with \( \omega_0 = \omega \). Standard arguments show that the value function \( V \) satisfies the following Bellman equations,

\[
\begin{align*}
V(\omega) &= \max_{y,\omega'} \{ \alpha [u(y) - v(y)] + \beta V(\omega') \} \\
\text{s.t.} \quad &-\eta(y) + \beta \frac{\omega'}{\lambda} \geq 0 \tag{75} \\
&\beta \omega' \geq \{ \omega - \alpha [u(y) - \eta(y)] \} \tag{76} \\
&y \in [0, y^*], \ \omega' \in [0, \bar{\omega}] \tag{77}
\end{align*}
\]

The proposition below shows that the above Bellman equation is well-defined and that \( V \) is uniquely determined. As a result, the maximization problem (62)-(64) is reduced to

\[
\max_{\omega_0 \in [0, \bar{\omega}]} V(\omega_0).
\]

Moreover, we can find the optimal policy function by the use of Bellman equation.

**Proposition 12** Suppose that \( y^* > y^{\max} > \hat{y} \).

1. The value function \( V \) is the unique solution to (74)-(77), and is continuous and weakly decreasing in \( \omega \).
2. The function \( V \) is concave in \( \omega \) if \( \eta \) is convex.

**Proof.** (1) First we show that, for any \( \omega \in [0, \bar{\omega}] \), the set of \((y, \omega')\)'s satisfying (75)-(77) is nonempty and hence the maximization problem is well-defined. Let \( \omega \in [0, \bar{\omega}] \) be given. Let \( y \leq \hat{y} \leq y^* \) be the least \( y \) such that

\[
\omega = \frac{\alpha}{1 - \beta} [u(y) - \eta(y)]. \tag{78}
\]

Note that as \( u(0) - \eta(0) = 0 \), such \( y \) exists by the Intermediate Value Theorem. We claim that \((y, \omega')\) satisfies (75)-(77) for any \( \omega' \in [\omega, \bar{\omega}] \). First (77) holds by construction. Moreover, rearranging (78), we have

\[
(1 - \beta) \omega = \alpha[u(y) - \eta(y)]
\]
and hence
\[ \omega = \alpha[u(y) - \eta(y)] + \beta \omega, \]
which implies (76) for any \( \omega' \geq \omega \). Finally, by (78) and the fact that \( y \leq \hat{y} \leq y^{\text{max}}, \)
\[ \eta(y) \leq \beta \frac{\omega'}{\lambda} \]
for any \( \omega' \geq \omega \).

To show that the Bellman equation (75)-(77) has a unique solution, we show that it defines a contraction mapping. Let \( C[0, \hat{\omega}] \) be the set of continuous functions over \([0, \hat{\omega}] \) equipped with the sup norm. Define \( T : C[0, \hat{\omega}] \to C[0, \hat{\omega}] \) by
\[ T(W)(\omega) = \max_{y, \omega'} \{ \alpha [u(y) - v(y)] + \beta W(\omega') \}, \]
subject to (75)-(77). It is easy to verify that \( T \) satisfies the Blackwell sufficient condition, and hence \( T \) is a contraction mapping, which admits a unique fixed point. Because \( V \) satisfies \( T(V) = V \), it follows that \( V \) is the unique solution to the Bellman equation and hence is continuous.

Notice that by decreasing \( \omega \) we increase the set of \( (y, \omega') \) that satisfies (75)-(77), but without affecting the objective function. Hence, \( V^* \) is weakly decreasing.

(2) Assume now that \( \eta \) is convex. To show that \( V \) is concave, we show that if \( W \) is concave, then \( T(W) \) is also concave. Let \( \omega_0, \omega_1 \in [0, \hat{\omega}] \) be given. Let \( (y_0', \omega_0') \) and \( (y_1', \omega_1') \) solves (75)-(77) for \( \omega_0 \) and \( \omega_1 \), respectively. let \( \epsilon \in (0, 1) \) be given. Then,
\[
T(W)(\epsilon \omega_0 + (1 - \epsilon) \omega_1) \\
\geq \alpha [u(\epsilon y_0 + (1 - \epsilon)y_1) - v(\epsilon y_0 + (1 - \epsilon)y_1)] + \beta W(\epsilon \omega_0' + (1 - \epsilon) \omega_1') \\
\geq \alpha [u(y_0) - v(y_0)] + \alpha (1 - \epsilon)[u(y_0) - v(y_0)] + \beta [\epsilon W(\omega_0') + (1 - \epsilon) W(\omega_1')] \\
= \epsilon T(W)(\omega_0) + (1 - \epsilon) T(W)(\omega_1).
\]
Notice that the first inequality follows from the fact that \( (\epsilon y_0 + (1 - \epsilon)y_1, \epsilon \omega_0' + (1 - \epsilon) \omega_1') \) also satisfies (75)-(77) for \( \omega = \epsilon \omega_0 + (1 - \epsilon) \omega_1 \) because \( \eta \) is convex. ■

**Proof of Proposition 11**  The Lagrangian associated with the Bellman equation (74)-(77) is:
\[
\mathcal{L} = \alpha [u(y) - v(y)] + \beta V(\omega') + \xi \left( \frac{\beta \omega'}{\lambda} - \eta(y) \right) + \nu \left\{ \alpha [u(y) - \eta(y)] + \beta \omega' - \omega \right\},
\] (79)
where the Lagrange multipliers, $\xi$ and $\nu$, are non-negative. The first-order conditions with respect to $y$ and $\omega'$ are
\begin{align}
\alpha [u'(y) - v'(y)] - \xi \eta'(y) + \nu \alpha [u'(y) - \eta'(y)] &= 0, \quad (80) \\
\beta V'(\omega') + \beta \frac{\xi}{\lambda} + \beta \nu &= 0, \quad (81)
\end{align}
and the envelope condition is
\[V'(\omega) = \nu. \quad (82)\]

**Proof of part 1** By Proposition 12, $V$ is decreasing, and hence we may set the initial promised utility to be $\omega = 0$. The following lemma shows that, for $\omega = 0$, the optimal choice to (74)-(77) is $(\omega_{\max}, y_{\max})$, and, for $\omega = \omega_{\max}$, the optimal choice to (74)-(77) is still $(\omega_{\max}, y_{\max})$. This shows that the best PBE has $y_t = y_{\max}$ for all $t$.

**Lemma 4** Suppose that $\hat{y} < y_{\max} < y^*$ and $\lambda \geq \alpha [1 - u'(y_{\max})/\eta'(y_{\max})]$. Then, $\tilde{\omega} = \omega = \alpha [u(\hat{y}) - \eta(\hat{y})] / (1 - \beta)$ and the unique $V$ that solves (74)-(77) is given by
\begin{align}
V(\omega) &= \frac{\alpha [u(y_{\max}) - v(y_{\max})]}{1 - \beta} \quad \text{if } \omega \in [0, \omega_{\max}], \\
&= \frac{\alpha [u(g(\omega)) - v(g(\omega))]}{1 - \beta} \quad \text{if } \omega \in (\omega_{\max}, \tilde{\omega}), \\
\end{align}
where $g(\omega)$ is the unique solution for $y$ to $\alpha [u(y) - \eta(y)] = (1 - \beta)\omega$ for all $\omega \in (\omega_{\max}, \tilde{\omega})$.

**Proof.** First we establish a claim.

**Claim 1.** Let $V$ be given by (83)-(84). Then, $V$ is a concave function and is strictly concave in $(\omega_{\max}, \tilde{\omega})$. Moreover, $V$ is differentiable except at $\omega_{\max}$ and $\tilde{\omega}$, with derivatives given by
\[V'(\omega) = \frac{u'(g(\omega)) - v'(g(\omega))}{u'(g(\omega)) - \eta'(g(\omega))} \quad (85)\]
for all $\omega \in (\omega_{\max}, \tilde{\omega})$ with
\[V'(\omega_{\max}) \equiv \lim_{\omega \downarrow \omega_{\max}} V'(\omega) = \frac{u'(y_{\max}) - v'(y_{\max})}{u'(y_{\max}) - \eta'(y_{\max})}\]
and
\[V'(\tilde{\omega}) \equiv \lim_{\omega \uparrow \tilde{\omega}} V'(\omega) = -\infty. \quad (86)\]

**Proof.** Let $\omega \in (\omega_{\max}, \tilde{\omega})$. Note that $g(\omega_{\max}) = y_{\max}$, $g(\tilde{\omega}) = \hat{y}$, and that $g$ is strictly decreasing in $(\omega_{\max}, \tilde{\omega})$. By Implicit Function Theorem, we have
\[g'(\omega) = \frac{1 - \beta}{\alpha [u'(g(\omega)) - \eta'(g(\omega))]} \quad (87)\]
Then, for all $\omega \in (\omega^\text{max}, \hat{\omega})$, we have
\[ V'(\omega) = \frac{\alpha}{1 - \beta} \left( u'(g(\omega)) - v'(g(\omega)) \right) g'(\omega) \]
(86)
\[ = \frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} \]
(87)
Because $g(\omega^\text{max}) = y^\text{max}$,
\[ V'_+(\omega^\text{max}) = \frac{u'(y^\text{max}) - v'(y^\text{max})}{u'(y^\text{max}) - \eta'(y^\text{max})}. \]
Because $g(\hat{\omega}) = \hat{y}$ and $u'(\hat{y}) = \eta'(\hat{y})$, $V'(\hat{\omega}) = -\infty$.

Moreover, for all $\omega \in (\omega^\text{max}, \hat{\omega})$,
\[ V''(\omega) = \frac{\{u''[g(\omega)] - v''[g(\omega)]\} \{u'[g(\omega)] - \eta'[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) \]
\[ + \frac{-\{u'[g(\omega)] - v'[g(\omega)]\} \{u''[g(\omega)] - \eta''[g(\omega)]\}}{\{u'[g(\omega)] - \eta'[g(\omega)]\}^2} g'(\omega) < 0. \]
Note that, for all $\omega \in (\omega^\text{max}, \hat{\omega})$, $u'[g(\omega)] - \eta'[g(\omega)] < 0$ as $g(\omega) > \hat{y}$ and that $u'[g(\omega)] - v'[g(\omega)] > 0$ as $g(\omega) \leq y^\text{max} < \hat{y}^*$.

We consider two cases.

(a) Suppose that $\omega \in [0, \omega^\text{max}]$. First we claim that any $(\omega', y)$ with $\eta(y) \leq \beta \omega' < \beta \omega^\text{max}$ is dominated by $(\omega^\text{max}, y^\text{max})$. Because $\eta(y^\text{max}) = \beta \omega^\text{max}$, we have $y < y^\text{max}$. Because $u(y) - v(y)$ is increasing in $y$ for such $y$'s and $V(\omega^\text{max})$, the result follows.

Therefore, we may add the condition $\omega \geq \omega^\text{max}$ to (77), and the FOC’s are
\[ \alpha \left[ u'(y) - v'(y) \right] - \xi \eta'(y) + \nu \alpha \left[ u'(y) - \eta'(y) \right] = 0, \]
(88)
\[ \beta V'(\omega') + \beta \frac{\xi}{\lambda} + \beta \nu \leq 0, \]
(89)
with (89) at equality if $\omega' > \omega^\text{max}$. Because $V$ is strictly concave in $(\omega^\text{max}, \hat{\omega})$, these conditions are sufficient as well.

Note that $(\omega^\text{max}, y^\text{max})$ satisfies (76) for all $\omega \leq \omega^\text{max}$ and it satisfies (75) at equality. Now we show that $(\omega^\text{max}, y^\text{max})$ satisfies (88) and (89) with $\nu = 0$ and
\[ \xi = \frac{\alpha [u'(y^\text{max}) - v'(y^\text{max})]}{\eta'(y^\text{max})} > 0. \]
The condition (88) holds by the definition of $\xi$. For (89), it holds if and only if
\[ V'_+(\omega^\text{max}) + \frac{\xi}{\lambda} \]
\[ = \frac{1}{\lambda} \left[ u'(y^\text{max}) - v'(y^\text{max}) \right] \frac{\eta'(y^\text{max})}{u'(y^\text{max}) - \eta'(y^\text{max})} \]
\[ = \frac{1}{\lambda} \left[ u'(y^\text{max}) - v'(y^\text{max}) \right] \left\{ \alpha + \lambda \frac{\eta'(y^\text{max})}{u'(y^\text{max}) - \eta'(y^\text{max})} \right\} \leq 0, \]

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and, because $u'(y^{\text{max}}) - v'(y^{\text{max}}) > 0$, the last inequality holds iff

$$
\alpha \left[ \frac{u'(y^{\text{max}})}{\eta'(y^{\text{max}})} - 1 \right] \geq -\lambda,
$$

that is, $\lambda \geq \alpha [1 - u'(y^{\text{max}})/\eta'(y^{\text{max}})]$.

(b) Suppose that $\omega \in (\omega^{\text{max}}, \hat{\omega})$. We claim that $(\omega', y) = (\omega, g(\omega))$ is the optimal solution to (74)-(77). Because $V$ is strictly concave in $(\omega^{\text{max}}, \hat{\omega})$, it is sufficient to show that $(\omega, g(\omega))$ is a unique local maximum. To this end, we show that $(\omega, g(\omega))$ satisfies the feasibility condition, (75) and (76), and the first-order conditions, (80) and (81) for some $\xi, \nu \geq 0$. Note that these FOC’s are sufficient for optimality b/c of strict concavity of $V$.

To check feasibility, first note that (76) holds by construction. To show (75), note that because $y = g(\omega) \leq y^{\text{max}}$,

$$
\lambda \eta(y) \leq \beta \alpha [u(y) - \eta(y)]/(1 - \beta)
$$

and hence, noting that $\omega = \alpha [u(y) - \eta(y)]/(1 - \beta)$, (75) follows.

Now we show that the FOC’s are satisfied by $(\omega, g(\omega))$ with $\xi = 0$ and

$$
\nu = -\frac{u'[g(\omega)] - v'[g(\omega)]}{u'[g(\omega)] - \eta'[g(\omega)]} > 0.
$$

The FOC for $y$, (80), holds by the definition of $\nu$. The FOC for $\omega'$, (81), iff

$$
\nu + V'(\omega) = 0,
$$

which holds by (85).

Finally, for $\omega = \hat{\omega}$, the only feasible $(\omega', y)$ that satisfies (75)-(77) is $(\hat{\omega}, \hat{y})$ so it has to be the solution. Note that $g(\hat{\omega}) = \hat{y}$. □

**Proof of part 2** Because $V$ is decreasing, we may set the initial promised utility to be $\omega = 0$. The following lemma shows that, for $\omega = 0$, the optimal choice to (74)-(77) is $(\omega_1, y_0)$ with $\omega_1 = \alpha [u(y_1) - \eta(y_1)]/(1 - \beta)$ and $(y_0, y_1)$ given by (92)-(93). Moreover, for $\omega = \omega_1$, the optimal choice to the problem (74)-(77) is $(\omega_1, y_1)$. Thus, the best PBE has $(y_0, y_1)$ given by (92)-(93) and $y_t = y_1$ for all $t > 1$.

**Lemma 5** Suppose that $\hat{y} < y^{\text{max}} < y^*$ and $\lambda < \alpha [1 - u'(y^{\text{max}})/\eta'(y^{\text{max}})]$. Then, $\check{\omega} = \hat{\omega} = \alpha [u(\hat{y}) - \eta(\hat{y})]/(1 - \beta)$. Moreover, there exists a unique $(y_0, y_1)$ with $\hat{y} < y_1 < y^{\text{max}} < y_0 < y^*$ that satisfies (92)-(93) and the unique $V$ that solves (74)-(77) satisfies

$$
V(\omega) = \alpha [u(y_0) - v(y_0)] + \frac{\beta}{1 - \beta} \alpha [u(y_1) - v(y_1)] \text{ if } \omega = 0,
$$

and

$$
\frac{\alpha}{1 - \beta} [u[g(\omega)] - v[g(\omega)]] \text{ if } \omega \in [\omega^{\text{max}}, \hat{\omega}],
$$

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where \( g(\omega) \) is given in Lemma 4.

**Proof.** We prove by two claims.

**Claim 1.** There exists a unique pair \((y_0, y_1)\) with \( \hat{y} < y_1 < y^{\max} < y_0 < y^* \) such that

\[
\frac{u'(y_0) - v'(y_0)}{\eta'(y_0)} = -\frac{\lambda}{\alpha} \left[ \frac{u'(y_1) - v'(y_1)}{\eta'(y_1)} \right]
\]

\[\alpha[u(y_1) - \eta(y_1)] = r \lambda \eta(y_0),\]  

(92)  

(93)

which is also the unique solution to (43)-(44).

**Proof.** First note that (92) corresponds to the FOC for the problem (43)-(44). Now we show that there is a unique pair satisfying (92)-(93). For each \( y_1 \in (\hat{y}, y^{\max}] \), define

\[
h(y_1) = \eta^{-1}\left[ \frac{\alpha}{r \lambda} [u(y_1) - \eta(y_1)] \right].
\]

By definition, the pair \((h(y_1), y_1)\) satisfies (93) and \( h(y^{\max}) = y^{\max} \). For any \( y_1 \in (\hat{y}, y^{\max}] \),

\[
h'(y_1) = \frac{\alpha}{r \lambda} \left[ \frac{u'(y_1) - \eta'(y_1)}{\eta'[h(y_1)]} \right] < 0.
\]

After substitute \( y_0 \) by \( h(y_1) \) in the left side of (92) and moving the right side to the left, we can define, for each \( y_1 \in (\hat{y}, y^{\max}] \),

\[
H(y_1) = \frac{u'[h(y_1)] - v'[h(y_1)]}{\eta'[h(y_1)]} + \frac{\lambda}{\alpha} \left[ \frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)} \right].
\]

Then, \((h(y_1), y_1)\) satisfies (92) iff \( H(y_1) = 0 \). The function \( H(y_1) \) is continuous and strictly increasing in \((\hat{y}, y^{\max}] \) with

\[
\lim_{y_1 \downarrow \hat{y}} H(y_1) = -\infty,
\]

and, at \( y_1 = y^{\max} \), we have

\[
H(y^{\max}) = \frac{u'(y^{\max}) - v'(y^{\max})}{\eta'(y^{\max})} + \frac{\lambda}{\alpha} \left[ \frac{u'(y^{\max}) - \eta'(y^{\max})}{u'(y^{\max}) - \eta'(y^{\max})} \right]
\]

\[= \frac{u'(y^{\max}) - v'(y^{\max})}{\eta'(y^{\max})} \left( \frac{1}{\eta'(y^{\max})} + \frac{\lambda}{\alpha} \left[ \frac{1}{u'(y^{\max}) - \eta'(y^{\max})} \right] \right) > 0
\]

because \( \lambda < \alpha \left[ 1 - u'(y^{\max})/\eta'(y^{\max}) \right] \).

Thus, by Intermediate Value Theorem, there exists a unique \( y_1 \in (\hat{y}, y^{\max}) \) such that \( H(y_1) = 0 \) and hence (92) holds for \((h(y_1), y_1)\), and \( h(y_1) > y^{\max} \) as \( h \) is strictly decreasing with \( h(y^{\max}) = y^{\max} \). \( \square \)

**Claim 2.** Let \( W : [0, \hat{\omega}] \rightarrow \mathbb{R} \) be a bounded concave function that satisfies (91). Let \( U : [0, \hat{\omega}] \rightarrow \mathbb{R} \) solve

\[
U(\omega) = \max_{y, \omega'} \{ \alpha[u(y) - v(y)] + \beta W(\omega') \}
\]

(94)
subject to (75)-(77). Then, $U$ is also a bounded concave function that satisfies (91).

Proof. The fact that $U$ is concave and hence continuous follows from standard arguments. We claim that for all $\omega > \omega^{\text{max}}$, the solution to (94) is $(\omega, g(\omega))$, where $g(\omega)$ is given in Lemma 4. The proof is identical to the proof of the second case in Lemma 4. Indeed, because $W$ is concave, a unique local maximum is also the global maximum. Note that by (91), $W'(\omega)$ is given by (85) for $\omega > \omega^{\text{max}}$. Thus, for all $\omega > \omega^{\text{max}},$

$$U(\omega) = \alpha \{ u[g(\omega)] - v[g(\omega)] \} + \beta W(\omega) = \frac{\alpha}{1-\beta} \{ u[g(\omega)] - v[g(\omega)] \}$$

because $W$ satisfies (91). So $U$ also satisfies (91) for all $\omega > \omega^{\text{max}}$. Finally, $U$ satisfies (91) at $\omega^{\text{max}}$ by continuity.

By Claim 2 and the Contraction Mapping Theorem it follows that $V$ is concave and satisfies (91). Now we show that $V$ satisfies (90). Let $(y_0, y_1)$ be given by Claim 1 and let

$$\omega_1 = \frac{1}{1-\beta} \alpha [u(y_1) - \eta(y_1)].$$

First note that $(\omega_1, y_0)$ satisfies (75) with equality by (93) and it satisfies (76) for $\omega = 0$. Moreover, because $y_1 < y^{\text{max}}, \omega_1 > \omega^{\text{max}}$.

Now we show that $(\omega_1, y_0)$ satisfies the (80) and (81) with $\nu = 0$ and

$$\xi = \frac{\alpha [u'(y_0) - v'(y_0)]}{\eta'(y_0)} > 0.$$ 

Note that (80) holds by definition of $\xi$. For (81), it holds if and only if (note that $y_1 = g(\omega_1)$ and $\omega_1 > \omega^{\text{max}}$)

$$V'(\omega_1) + \frac{\xi}{\lambda} \frac{u'(y_1) - v'(y_1)}{u'(y_1) - \eta'(y_1)} + \frac{1}{\lambda} \frac{\alpha [u'(y_0) - v'(y_0)]}{\eta'(y_0)} = 0$$

by (92). Note that the FOC’s are also sufficient for local maximum as $V$ is strictly concave near the proposed solution; by concavity it is also a global maximum. □