Repeated Moral Hazard in Multi-Stage R&D Projects

(Job Market Paper One)

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Abstract

This paper studies a continuous-time principal-agent problem in which a principal hires an agent to do a multi-stage R&D project. The agent’s effort is unobservable and the transition from one stage to the next is modeled by a jump process with a constant Poisson arrival rate. We characterize the optimal dynamic contract that solves the repeated moral-hazard problem. In it, the agent’s payment decreases over time in case of failure and jumps up to a higher level after each success. We also provide an implementation of the optimal contract, in which a primary component of the agent’s compensation is a risky security and the principal lets the agent choose the consumption and effort levels. This implementation gives a theoretical justification for the wide-spread use of stock-based compensation by firms that rely on R&D.

Key words: Dynamic Contract, Repeated Moral Hazard, R&D, Employee Compensation

JEL: D82, D86, J33, O32

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1 Introduction

Over the last decade, the industries of information and communication technologies have become the engines of the U.S. economy: during this period, they have had an average share of 4.6 percent of gross GDP and have accounted for one-fourth of GDP-growth. A distinct feature of these so-called “new-economy” industries is a substantial investment in R&D, which constituted a third of business-sector expenditure on R&D in 2007 according to a recent report of the Bureau of Economic Analysis. Clearly, the success of firms in these industries depends crucially on the performance of the employees in their R&D units, and compensation schemes for these researchers become a major decision for these firms. This decision-problem shares some features with the standard problem of providing incentives to workers, but it also has some unique features. Like its standard counterpart, a moral-hazard phenomenon arises in this specific agency relationship. The outcome of research is uncertain, i.e. effort put into research today will not necessarily lead to a discovery tomorrow. However, the stochastic process governing the outcomes is influenced by how much effort is put into research: higher levels of effort increase the chance of making a discovery. Owing to task-complexity, effort exerted by researchers is difficult to monitor. Now, if the effort level is unobservable, then the imperfect monitoring of effort combined with the stochastic feature of innovation maps into a moral-hazard problem. Furthermore, since most R&D projects last a long period of time, the moral-hazard problem is dynamic in nature.

The point of departure from standard agency problems is the feature that some R&D projects progress through different phases, with research in each phase depending on the outcomes of previous phases. In these new-economy industries, this feature is particularly prominent. For example, in the software industry, Microsoft has released a sequence of Windows operating systems since 1985, from Windows 3.0, to Windows XP, and then to the most recent version, Windows 7. In each upgrade, Microsoft introduced a number of new features which make the use of computers easier and more convenient. In the hardware industry, the development of Intel’s CPU is an example of multi-stage R&D. From its earlier 8086 and 8088 processors to the advanced Intel Core processor family, besides the fast-growing initial clock speed (from 2MHz to 3GHz), Intel has also added new instructions to each new generation, which are specially optimized for the demand of new applications.

The agency problem faced by these new-economy firms combines the two features described
above, namely an imperfect correlation between outcome and effort, and the multistage nature of
the innovative process. Firms try to overcome this agency problem by adopting stock-based grants,
especially employee stock-options, which have become a primary component of compensation for
employees in R&D departments in the past two decades. Since the researchers’ actions have a
great impact on the performance of the firms, which in turn affects the return of their stocks,
stock-based compensation reduces the agency problem by providing a direct link between company
performance and researchers’ wealth, thereby providing incentive for researchers to put in effort
in research. Moreover, since the employee stock-options have a vesting period during which they
cannot be exercised, they are widely used by firms to provide long-term incentives. The question
then is whether these schemes are optimal.

We approach the problem by first studying the contracting problem in the abstract, deriving
the optimal contract and demonstrating an implementation of the optimal contract. Finally we
relate our implementation result with the observed business practices. Our finding is that the
optimal contract can be implemented by using a risky security, which shares features of the stock
of these firms, thereby providing a theoretical justification for the wide-spread use of stock-based
compensation in firms that rely on R&D.

Briefly, the setup of the paper is as follows. At any point in time, the agent can choose whether
to put in effort or shirk. Conditional on putting in effort, the transition from one stage to the next
is a Poisson-type process with a constant arrival rate. If the agent chooses to shirk, the Poisson
arrival rate is zero. The principal cannot observe the agent’s action. However, the whole history of
the innovation process is publicly observable, and the principal will use precisely this information to
provide incentives optimally. To overcome the repeated moral-hazard problem, the principal offers
the agent a long-term contract which specifies a flow of payments based on his observation of the
outcome of the project.

We use recursive techniques to characterize the optimal dynamic contract. First, we start with
a problem in which the R&D project has only one stage. After characterizing the optimal contract
in this problem, we use the results for this case to analyze the multi-stage problem by backward
induction. We find that in the optimal contract, the principal uses a compensation scheme that
combines punishments with rewards. If the agent fails to make a discovery, his continuation utility
and payment decrease over time until a discovery is made. If the agent completes a stage, the
principal rewards the agent by a discrete increase in the continuation utility.
We also provide a way to implement the optimal contract, in which a primary component of the agent’s compensation is a state-contingent security whose return in case of success is higher than that in case of failure. We assume that investing in this security is the only saving-technology for the agent to smooth consumption overtime. At any point in time, besides the effort-choice, the agent also chooses how much to consume and how much to invest in the security, subject to a minimum-holding requirement. Different from the optimal contract, in which the principal controls the agent’s consumption directly, the agent chooses the consumption process by himself in this implementation, which nonetheless generates the same effort and consumption process as the optimal contract. This implementation overcomes the problem pointed out by Rogerson (1985) which is that, if the agent is allowed access to credit, he would choose to save some of his wages, if he could, because of a wedge between the agent’s Euler equation and the inverse Euler equation implied by the principal’s problem. In our implementation, however, the return on savings is state contingent. When we choose the state-dependent rates of return appropriately, the agent’s Euler equation mimics the inverse Euler equation; put differently, the wedge between the Euler equation and the inverse Euler equation disappears.

This implementation is similar to the stock-based compensation scheme used in the real-world in two aspects. First, the return of the state contingent security and the stock price have a similar trend, with an notable increase after each breakthrough in R&D. Second, in the implementation, the agent is required to hold a certain amount of the state-contingent security until he completes the entire project. Similarly, stock options have a vesting period during which they cannot be exercised. Capturing these two main features, our implementation provides a theoretical explanation for the compensation scheme used in reality.

This paper is related to three strands of literature: memoryless patent races, management compensation and dynamic contracts. In the current paper, the innovation process is modeled by a memoryless process—the probability of making a discovery at a point of time depends only on the agent’s current action. This way of modeling the stochastic innovation process is commonly used in the patent-race literature, for example, Dasgupta and Stiglitz (1980), Lee and Wilde (1980).

In the management-compensation literature, there is extensive research on stock-based grants for CEO compensation. For researchers’ compensation, Anderson, Banker, and Ravindran (2000), Ittner, Lambert, and Larcker (2003), and Murphy (2003) have documented that executives and employees in new-economy firms receive more stock-based compensation than do their counterparts.
in old-economy firms. Sesil, Kroumova, Blasi, and Kruse (2002) compares the performance of 229 ‘New Economy’ firms offering broad-based stock options to that of their non-stock option counterparts, and shows that the former have higher shareholder returns. Our implementation contributes to this literature by giving a rationale for the use of stock-based compensation in new economy firms, from a theoretical point of view.

In terms of methodology, this paper relates to a rich and growing literature on dynamic contracts. Starting with Green (1987) and Spear and Srivastava (1987), using recursive techniques to characterize optimal dynamic contracts has become a standard approach in dynamic-contract theory. Finally, our use of a Poisson process is similar to Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2008). In these two papers, bad events happen with higher Poisson arrival rate when agents do not put enough effort to prevent such events. This current paper differs from these two papers mainly in the assumption of the agent’s preference, which leads to different dynamics of the agent’s payment. Both Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2008) assume that the agent is risk neutral, and hence he does not receive any payment until the continuation utility reaches a payment threshold. In our model the agent is risk-averse, and his payment decreases over time if he fails to make a discovery.

The rest of the paper is organized as follows. Section 2 describes the model. In the first part of section 3, a single stage innovation problem is studied as a benchmark. The results of the benchmark model are used in the second part of section 3 to analyze the finite-stage problem. In this section, we also discuss the infinite-stage problem. In section 4, we provide an implementation of the optimal dynamic contract. Section 5 concludes.

2 The model

We consider a dynamic principal-agent model in continuous time. At time 0, a principal hires an agent to do an R&D project. This project has \( N \) stages, which must be completed sequentially, i.e. to develop the stage \( n \) \( (0 < n \leq N) \) innovation, the agent must have finished the innovation of stage \( n - 1 \).

We model the transition from one stage to the next by a Poisson-type process, which is affected by the agent’s choice of effort. For simplicity, we assume that the agent has only two choices of effort: he can either put in effort or shirk. Conditional on putting in effort, the probability that
during a period of length $\Delta t$ the agent has not made a discovery is $e^{-\lambda \Delta t}$, where $\lambda$ is the Poisson arrival rate. If the agent chooses to shirk, the Poisson arrival rate is equal to zero.

Whether the agent puts in effort or shirks cannot be monitored by the principal. However, the principal can observe exactly when each stage of the R&D project is completed. Thus, at any point of time, the principal knows the current stage and the length of time it took the agent to finish each previous stage. Let $H_t$ denote the stage at time $t$. The stage-level process $H = \{H_t, 0 \leq t < \infty\}$ is stochastic and depends on the agent’s choice of effort. The history of $H$, denoted as $H^t = \{H_s, 0 \leq s \leq t\}$, is the realization of the stage-level process till time $t$. By assumption, $H^t$ is publicly observable, which is the only information that the principal can use to provide incentives to the agent.

At time 0, the principal offers the agent a contract that specifies a flow of consumption $c_t(H^t)$ based on the principal’s observation of the stage-level process. Let $T$ denote the stochastic stopping time when the agent finishes the last stage innovation. Note that the history of the stage-level process will not get updated after the agent finishes the last stage of the project. Thus, the principal can equivalently give the agent a lump-sum consumption transfer at $T$.

The agent’s utility is determined by his consumption flow and the effort put in research. The utility function is assumed to have a separable form $U(c) - L(a)$, where $U(c)$ is the utility from consumption, and $L(a)$ is the disutility of doing research. We assume that $U : [0, +\infty) \rightarrow [0, +\infty)$ is an increasing, concave and $C^2$ function with the property that $U'(c) \rightarrow +\infty$ as $c \rightarrow 0$. The agent’s choice of effort is binary, indicated by $a \in \{0, 1\}$. $a = 1$ means that the agent chooses to put in effort, and $a = 0$ means that the agent chooses to shirk. Moreover, we assume that the disutility of putting in effort equals some $l > 0$ and the disutility of shirking equals zero, i.e. $L(1) = l$ and $L(0) = 0$.

Given the contract, at any time $t$, the agent makes the effort choice based on the observation of $H^t$. Denote the effort process as $a = \{a_t(H^t), 0 \leq t < \infty\}$. The agent’s objective is to choose the effort process $a$ to maximize the total expected utility. Thus, the agent’s problem is

$$\max_{\{a, 0 \leq t < +\infty\}} E \left[ \int_0^T r e^{-rt}(U(c_t) - L(a_t)) dt + e^{-rT} U(c_T) \right],$$

where $r$ is the discount rate. Moreover, the agent has a reservation-utility $v_0$. If the maximum expected utility he can get from the contract is less than $v_0$, then the agent will reject the principal’s offer.
For simplicity, we assume that the agent and the principal have the same discount rate. Hence, the principal’s expected cost is given by

\[ E \left[ \int_0^T r e^{-rt} c_t dt + e^{-rT} c_T \right]. \]

The principal’s objective is to minimize the expected cost by choosing an incentive-compatible payment scheme subject to delivering the agent the requisite initial value of expected utility \( v_0 \). Therefore, the principal’s problem is

\[
\min_{c_t, 0 \leq t < +\infty} E \left[ \int_0^T r e^{-rt} c_t dt + e^{-rT} c_T \right]
\]

s.t.

\[
E \left[ \int_0^T r e^{-r(t)} (U(c_t) - l) dt + e^{-rT} U(c_T) \right] \geq v_0.
\]

Finally, to simplify the analysis, we could recast the problem as one where the principal directly transfers utility to the agent instead of consumption. In the transformed problem, the principal chooses a stream of utility transfers \( u_t(H_t) \) \((0 \leq t < +\infty)\) to minimize the expected cost of implementing positive effort. Then, the principal’s problem becomes

\[
\min_{\tilde{v}, \bar{v}} E \left[ \int_0^T r e^{-rt} S(u_t) dt + e^{-rT} S(u_T) \right]
\]

s.t.

\[
E \left[ \int_0^T r e^{-r(t)} (u_t - l) dt + e^{-rT} u_T \right] \geq v_0,
\]

where \( S(u) = U^{-1}(u) \), which is the principal’s cost of providing the agent with utility \( u \). It can be shown that \( S(u) \) is a decreasing and strictly convex function. Moreover, \( S(0) = 0 \) and \( S'(0) = 0 \).

3 The optimal dynamic contract

In this section, we derive the optimal dynamic contract and discuss its properties. In doing so, we follow the standard approach in the contracting literature: the optimal contract is written in terms of the agent’s continuation-utility \( v_t \), which is the total utility that the principal expects the agent to derive at any time \( t \). At any moment of time, given the continuation utility, the contract specifies the agent’s utility flow, the continuation utility if the agent makes a discovery, and the law of motion of the continuation utility if the agent fails to make a discovery.
### 3.1 Single-stage problem

Before analyzing the multi-stage case, we first look at a simple case where the R&D project has only one stage.

The continuous-time model can be interpreted as the limit of discrete-time models in which each period lasts $\Delta t$. When $\Delta t$ is small, conditional on putting in effort, the probability that the agent successfully finishes the innovation during $\Delta t$ is approximately $\lambda \Delta t$. For any given continuation-utility $v$, the principal needs to decide a triplet $(u, \underline{v}, \bar{v})$ in each period, where

- $u$ is the transferred-utility flow in the current period.
- $\underline{v}$ is the next-period continuation utility if the agent fails to make a discovery during this period of time.
- $\bar{v}$ is the next-period continuation utility if the agent completes the innovation during this period of time.

If the agent chooses to exert effort, his expected utility in the current period is

$$r(u - l)\Delta t + e^{-r\Delta t}((1 - \Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}),$$

where the first term is the current-period utility flow and the second term is the discounted expected continuation utility.

If the agent chooses to shirk, he does not incur any utility cost and will fail to make a discovery with probability 1. Thus, his expected utility in the current period is

$$ru\Delta t + e^{-r\Delta t}\underline{v}.$$  

The triplet $(u, \underline{v}, \bar{v})$ should satisfy two conditions. First, this policy should indeed guarantee that the agent gets the promised-utility $v$. That is

$$r(u - l)\Delta t + e^{-r\Delta t}((1 - \Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) = v.$$  

Second, the policy should implement positive effort, i.e. the expected utility of putting in effort should be higher than the expected utility of shirking. Thus,

$$r(u - l)\Delta t + e^{-r\Delta t}((1 - \Delta t\lambda)\underline{v} + \Delta t\lambda\bar{v}) \geq ru\Delta t + e^{-r\Delta t}\underline{v}.$$
Let $C(v)$ be the principal’s minimum expected cost of providing the agent with continuation-utility $v$. Then, the Bellman equation is

$$C(v) = \min_{u, \bar{v}} r(S(u)) \Delta t + e^{-r \Delta t}((1 - \lambda \Delta t)C(v) + \lambda \Delta t S(\bar{v}))$$

s.t.

$$r(u - l) \Delta t + e^{-r \Delta t}((1 - \Delta t \lambda) \bar{v} + \Delta t \lambda \bar{v}) = v,$$

$$r(u - l) \Delta t + e^{-r \Delta t}((1 - \Delta t \lambda) \bar{v} + \Delta t \lambda \bar{v}) \geq ru \Delta t + e^{-r \Delta t} \bar{v},$$

where $S(u)$ is the principal’s cost given the transferred-utility $u$ and $S(\bar{v})$ is the principal’s cost of providing the agent with the lump-sum utility-transfer $\bar{v}$ when the R&D project is completed. Equation (1) is the promise-keeping condition and equation (2) is the incentive-compatibility condition.

Multiplying both sides of the Bellman equation and the promise-keeping condition (1) by $(1 + r \Delta t)/\Delta t$ and letting $\Delta t$ converge to 0, we derive the following Hamilton-Jacobi-Bellman (HJB) equation in continuous time$^1$

$$rC(v) = \min_{u, \bar{v}} rS(u) + C'(v) \dot{v} + \lambda(S(\bar{v}) - C(v))$$

s.t.

$$\dot{v} = rv - r(u - l) - \lambda(\bar{v} - v),$$

$$\bar{v} \geq v + \frac{rl}{\lambda}.$$ 

The promise-keeping condition (1) becomes the evolution of the agent’s continuation utility (3). In the discrete-time case, after choosing $u$ and $\bar{v}$, $v$ is given by the promise-keeping condition. When $\Delta t$ converges to 0, $v$ converges to $v$. Hence, in continuous time, the continuation utility in the case of failure changes smoothly and its rate of change is determined by $u$ and $\bar{v}$. The continuation utility can be explained as the value that the principal owes the agent. It grows at the discount-rate $r$ and falls due to the flow of repayment $r(u - l)$ plus the expected repayment $\lambda(\bar{v} - v)$ if the agent completes the innovation.

$^1$In this paper, we derive the HJB equation, evolution of continuation utility, and the incentive-compatibility condition in continuous time by considering the limit of a discrete-time approximation. We can also derive these formally using stochastic-calculus techniques (see Biais et al. (2010)). The reason we choose this method is because it is more intuitive and generates the same result.
The incentive-compatibility constraint becomes a very simple expression (4). To get the agent to put in positive effort, the continuation utility should jump up by at least $rl$ in case of success. The term $rl$ is the minimum reward that the principal should give the agent when he completes the project. It is determined by three parameters: $r$, $l$, and $\lambda$, which have the following interpretations: (1) $r$ is discount rate. The agent discounts the future utility at higher rate when $r$ is bigger. (2) $l$ measures the cost of doing research. When $l$ is big, the cost of doing research is high. (3) $\lambda$ measures the difficulty of the R&D project. Small $\lambda$ implies a small chance of success. Thus, a big reward is associated with a high discount-rate, or a high cost of doing research, or a low chance of success.

Note that the continuation utility cannot be less than 0, because the agent can guarantee a utility level of 0 by not putting in any effort. Therefore, a negative continuation utility is not implementable.

To characterize the solution of the HJB equation, we do a diagrammatic analysis in the $v$-$C'(v)$ plane. The dynamics of $v$ and $C'(v)$ are determined by the sign of $dv/dt$ and $dC'(v)/dt$. The expression of $dv/dt$ is given by the evolution of the continuation utility, which is known. However, the expression of $dC'(v)/dt$ depends on whether the incentive-compatibility condition is binding or not. The following lemma gives the condition under which this condition binds.

Lemma 3.1 The incentive-compatibility condition binds if and only if $C'(v) \leq S'(v + rl)$.  

The next two lemmas determine the sign of $dC'(v)/dt$ and $dv/dt$ under these two different conditions.

Lemma 3.2 If $C'(v) < S'(v + rl)$, then $\frac{dC'(v)}{dt} < 0$ and

$$
\frac{dv}{dt} =
\begin{cases}
< 0, & \text{if } C'(v) > S'(v); \\
0, & \text{if } C'(v) = S'(v); \\
> 0, & \text{if } C'(v) < S'(v).
\end{cases}
$$

Lemma 3.3 If $C'(v) \geq S'(v + rl)$, then $\frac{dC'(v)}{dt} = 0$ and $\frac{dv}{dt} < 0$.

The proof of these lemmas can be found in the appendix.

Lemmas 3.1-3.3 characterize the dynamics of $v$ and $C'(v)$ in the $v$-$C'(v)$ plane. The $S'(v) = C'(v)$ locus determines the dynamics of $v$: $v$ is decreasing over time above it and increasing over
time below it. The $S'(v + \frac{rl}{\lambda}) = C'(v)$ locus determines the dynamics of $C'(v)$: $C'(v)$ is constant over time above it and decreasing over time below it. The dynamics are summarized in Figure 1.

The next step is to find the optimal path in the phase diagram. From the theorem regarding the existence of a solution to a differential equation, there is an unique path from the line $v = v_0$ to the origin (Path 1 in Figure 2). First, any path on which the state variable $v$ diverges to infinity could be ruled out (such as Path 2). This contains the area below Path 1. In the area above Path 1, the continuation-utility $v$ is decreasing over time. When $v$ hits the lower bound 0, it cannot decrease any further. Thus, we must have $dv/dt \geq 0$ at $v = 0$. This condition rules out any path above Path 1 (such as Path 3) because $dv/dt < 0$ when $v$ reaches 0 for any path in this area. Then, Path 1 is the only candidate path left in the phase diagram, and hence it is the optimal path that we are looking for. The final step is to pin down the boundary condition at $v = 0$. At this point, we have $u = 0$ and $\bar{v} = \frac{rl}{\lambda}$. Thus, when $v$ reaches 0, the agent’s continuation utility and transferred-utility flow remain at 0 until he makes a discovery. To force the agent to put in positive effort, the principal
needs to offer a lump-sum utility transfer of \( \frac{\lambda}{r} \) when the agent completes the single stage R&D project. We can pin down the boundary condition at \( v = 0 \)

\[
C(0) = \int_{t=0}^{\infty} e^{-rt} e^{-\lambda t} \lambda S\left(\frac{r}{\lambda}\right) dt = (r + \lambda)^{-1} \lambda S\left(\frac{r}{\lambda}\right).
\]

To summarize, starting at the initial point \((v_0, C'(v_0))\), the optimal path locates between the \( S'(v) = C'(v) \) locus and the \( S'(v + \frac{r}{\lambda}) = C'(v) \) locus and reaches the lower bound of the continuation utility at the origin (Figure 2).

![Figure 2: Optimal Path](image)

The optimal path and the boundary condition together determine the solution of the HJB equation. The properties of the optimal dynamic contract are summarized in Proposition 3.4.

**Proposition 3.4** The contract that minimizes the principal’s cost takes the following form:

(i) The principal’s expected cost at any point is given by an increasing and convex function \( C(v) \), which satisfies

\[
rC(v) = rS(u) + C'(v)(r(v - u)) + \lambda(S(\bar{v}) - C(v)),
\]

where \( \bar{v} \) and \( u \) denote the upper and lower bounds of the continuation utility, respectively.
and boundary condition $C(0) = \frac{\lambda S(\frac{r}{\lambda})}{r+\lambda}$.

(ii) The transferred-utility $u$ satisfies $S'(u) = C'(v)$.

(iii) When the agent completes the innovation, he receives a lump-sum transfer of $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$.

(iv) In case of failure to complete the innovation, the continuation-utility $v$ evolves according to $\dot{v} = r(v - u)$, which is decreasing over time and asymptotically goes to 0.

(v) $u$ and $\bar{v}$ have the same dynamics as $v$.

Proof of proposition 3.4: For part (i), it has been shown that $C(v)$ is determined by the HJB equation and the boundary condition. On the optimal path, $C'(v)$ is strictly increasing in $v$, which implies that $C(v)$ is strictly convex. In addition, $C'(0) = S'(0) = 0$. Thus $C'(v) > 0$ for all $v$. Consequently, $C(v)$ is an increasing function.

Part (ii) is due to the fact that the transferred-utility flow is determined by the first-order-utility condition $S'(u) = C'(v)$.

For part (iii), note that the optimal path locates in the area where the incentive-compatibility constraint binds. Hence, $\bar{v} = v + \frac{rl}{\lambda}$.

For part (iv), note that on the optimal path $v$ is decreasing over time and asymptotically converges to 0.

Finally, from part (ii), $S'(u) = C'(v)$. Because $S(u)$ and $C(v)$ are both convex, $u$ and $v$ are positively related. From part (iii), $\bar{v} = v + \frac{rl}{\lambda}$. Thus, $u$ and $\bar{v}$ have the same dynamics as $v$, which proves part (v). $Q.E.D.$

3.2 Multi-stage problem

When the innovation process has multiple but finite number of stages, the optimal dynamic contract can be derived by backward induction. When the project is at stage $n$ ($0 < n \leq N$), we mean that the agents have finished the $(n-1)$-th innovation and are working on the $n$-th innovation.

As in the last subsection, let $u$ be the transferred-utility flow, $S(u)$ be the principal’s cost flow given the agent’s utility flow $u$, and $C_n(v)$ be the principal’s minimum expected cost of providing the agent with continuation-utility $v$ in stage $n$. In each stage $n$, given continuation-utility $v$, the contract
specifies the agent’s current utility flow $u$, the continuation-utility $\bar{v}$ when the agent successfully completes the innovation of stage $(n+1)$, and the evolution of the continuation utility in case of failure.

The backward induction starts from the last stage. After the agent completes the last-stage innovation, no further research work needs to be done and the agent receives a lump-sum utility transfer of $\tau$. Therefore, $C_{N+1}(\bar{v}) = S(\bar{v})$, which is known. Then, the principal’s problem in the last stage is

$$rC_N(v) = \min_{u, \bar{v}} rS(u) + C'_N(v)\dot{\bar{v}} + \lambda(S(\bar{v}) - C_N(v))$$

s.t.

$$\dot{\bar{v}} = rv - r(u - l) - \lambda(\bar{v} - v),$$

$$\bar{v} \geq v + \frac{rl}{\lambda}.$$ 

This problem is the same as the single-stage problem. Thus, we have the following proposition.

**Proposition 3.5** The contract in the last stage takes the following form:

(i) The principal’s expected cost at any point is given by an increasing and convex function $C_N(v)$, which satisfies

$$rC_N(v) = rS(u) + C'_N(v)(r(v - u)) + \lambda(S(\bar{v}) - C_N(v)),$$

and boundary condition $C_N(0) = \frac{\lambda S(\bar{v})}{r + \lambda}$.

(ii) The transferred-utility $u$ satisfies $S'(u) = C'_N(v)$.

(iii) When the agent completes the last stage innovation, he receives a lump-sum utility transfer of $\bar{v}$, which satisfies $\bar{v} = v + \frac{rl}{\lambda}$.

(iv) In case of failure to complete the innovation, the continuation-utility $v$ evolves according to

$$\dot{v} = r(v - u),$$

which is decreasing over time and asymptotically goes to 0.

(v) $u$ and $\bar{v}$ have the same dynamics as $v$.

The proof is similar to that of the proof of Proposition 3.4 and is therefore omitted.
From the last-stage problem, we have figured out the principal’s minimum expected cost \( C_N(v) \) given the agent’s continuation-utility \( v \) in stage \( N \). Now, given \( C_{n+1}(v) \), the principal’s problem in stage \( n \) is

\[
\begin{align*}
  rC_n(v) &= \min_{u, \bar{v}} rS(u) + C'_n(v) \dot{v} + \lambda(C_{n+1}(\bar{v}) - C_n(v)) \\
  \dot{v} &= rv - r(u - l) - \lambda(\bar{v} - v), \\
  \bar{v} &\geq v + rl.
\end{align*}
\]

Similar to the single-stage problem discussed in section 3.1, the dynamics are determined by the \( C'_n(v) = C'_{n+1}(v + \frac{rl}{\lambda}) \) locus and the \( C'_n(v) = S'(v) \) locus in the phase diagram. It can be shown that the \( C'_n(v) = C'_{n+1}(v + \frac{rl}{\lambda}) \) locus is always above the \( C'_n(v) = S'(v) \) locus. By doing a similar phase-diagram analysis, we get the following proposition.

**Proposition 3.6** The optimal contract in an intermediate stage takes the following form:

(i) The principal’s expected cost at any point is given by an increasing and convex function \( C_n(v) \), which satisfies

\[
rC_n(v) = rS(u) + C'_n(v)(r(v - u)) + \lambda(C_{n+1}(\bar{v}) - C_n(v)),
\]

and boundary condition \( C_n(0) = \frac{\lambda C_{n+1}(\bar{v})}{r + \lambda} \).

(ii) The transferred-utility \( u \) satisfies \( S'(u) = C'_n(v) \).

(iii) When the agent completes stage \( n + 1 \) innovation, he enters stage \( n + 1 \) and starts with continuation-utility \( \bar{v} \), which satisfies \( \bar{v} = v + \frac{rl}{\lambda} \).

(iv) In case of failure to complete the innovation, the continuation-utility \( v \) evolves according to \( \dot{v} = r(v - u) \), which is decreasing over time and asymptotically goes to 0.

(v) \( u \) and \( \bar{v} \) have the same dynamics as \( v \).

The only difference between the last-stage problem and any of the intermediate-stage problems is that in the last-stage problem the agent receives a lump-sum utility transfer when he finishes the innovation of the last stage; while in an intermediate stage \( n \), the agent enters stage \( (n + 1) \) and starts with a higher continuation-utility after finishing the innovation of stage \( n \). Figure 3 is a sample path of the continuation utility for a 3-stage R&D project.
3.3 Infinite Stages

In this subsection, we consider the case in which the R&D project has an infinite number of stages. Now, the principal needs to solve the same problem after each success. Let $C(v)$ be the principal’s minimum cost of providing continuation-utility $v$. Then, the optimal contract is characterized by the following HJB equation

$$rC(v) = \min_{u,\bar{v}} rS(u) + C'(v)\dot{v} + \lambda(C(\bar{v}) - C(v))$$

s.t.

$$\dot{v} = rv - r(u - l) - \lambda(\bar{v} - v),$$

$$\bar{v} \geq v + \frac{rl}{\lambda},$$

where $u$ is transferred-utility flow and $\bar{v}$ is the agent’s continuation utility if he completes one innovation. Note that this differential equation is a delay differential equation. The derivative of
the cost function at a point \( v \) depends on the value of the cost function at another point \( \bar{v} \). Moreover, \( u \) is implicitly determined by \( C'(v) \) by first-order condition, which makes the problem even more complicated\(^2\). Shan (2010) provides a proof of the existence of a solution to this HJB equation under the assumption that the derivative of \( S \) is bounded. Unfortunately, we cannot prove the existence in more general case. A natural conjecture of the property of the cost function is that the cost function is twice-differentiable, strictly convex, and increasing. Suppose there exists a solution to the HJB equation that satisfies these properties. Due to strict convexity of the cost function, the incentive-compatibility condition binds, which implies \( \bar{v} = v + \frac{r}{\lambda} \) and \( \dot{v} = rv - r(u - l) - \lambda(\bar{v} - v) = r(v - u) \).

Substituting these into the HJB equation, we get

\[
     rC(v) = \min_{u, \bar{v}} rS(u) + C'(v)r(v - u) + \lambda(C(v + \frac{r}{\lambda}) - C(v)).
\]

Using envelop theorem,

\[
     rC'(v) = C''(v)r(v - u) + rC'(v) + \lambda(C'(v + \frac{r}{\lambda}) - C'(v)).
\]

Then,

\[
     \dot{v} = r(v - u) = \frac{-\lambda(C'(v + \frac{r}{\lambda}) - C'(v))}{C''(v)} < 0.
\]

Thus, if there exists a strictly convex and increasing function \( C(v) \) that solves the HJB equation, the optimal contract is similar to the optimal contract in finite-stage case. In the optimal contract, the agent’s continuation utility decreases over time in case of failure; after each success, it jumps up by \( \frac{r}{\lambda} \).

3.3.1 An Example

If the agent’s utility from consumption takes the logarithmic form \( U(c) = \log(c) \), then we can provide a closed form solution to the HJB equation\(^3\). For logarithmic utility, the cost of providing transferred utility flow \( u \) is \( S(u) = e^u \). Let \( X \) be the set of all differentiable functions. Define an

\(^2\)Biais, Mariotti, Rochet, and Villeneuve (2010) and Myerson (2008) analyze a similar version of this kind of problem. Under the assumption of risk-neutrality, the agent does not receive any payment until the continuation utility reaches a payment threshold at which he receives constant payment per unit time, such that his continuation utility remains constant. Then the HJB equation becomes a tractable delay differential equation.

\(^3\)Note that the logarithmic utility function is unbounded from below. Hence there is no lower bound the continuation utility.
operator $G : X \rightarrow X$ by
\[
(GC)(v) = \min_{u, \bar{v}} \frac{rS(u) + C'(v)\dot{v} + \lambda(C(\bar{v}) - C(v))}{r}
\]
s.t.
\[
\begin{align*}
\dot{v} &= rv - r(u - l) - \lambda(\bar{v} - v), \\
\bar{v} &\geq v + \frac{rl}{\lambda}.
\end{align*}
\]
Then, the solution to the HJB equation is a fixed point of this operator.

Consider a cost function in the form of $C(v) = qe^v$, where $q$ is a constant. Apply the operator $G$ to this cost function. Since $C(v)$ is strictly convex, $\bar{v} = v + \frac{rl}{\lambda}$ and $\dot{v} = rv - r(u - l) - \lambda(\bar{v} - v) = r(v - u)$. $u$ is determined by the first order condition $S'(u) = C'(v)$. Thus,
\[
e^u = qe^v \Rightarrow u = v + \log q,
\]
and
\[
\dot{v} = r(v - u) = -r \log q.
\]
Then we have
\[
(GC)(v) = \frac{rS(u) + C'(v)\dot{v} + \lambda(C(\bar{v}) - C(v))}{r}
= \frac{rqe^v + qe^v(-r \log q) + \lambda(qe^v + \frac{rl}{\lambda} - qe^v)}{r}
= \frac{rq - rq \log q + \lambda q e^{\frac{rl}{\lambda}} - \lambda q e^v}{r}
\]
This result shows that if operator $G$ is applied to $C$ of the form $qe^v$, $G(C)$ takes the same form as $C$—a constant times $e^v$. Thus, the solution to the HJB equation has the form $C(v) = q^* e^v$ where $q^*$ solves
\[
q^* = \frac{rq^* - rq^* \log q^* + \lambda q^* e^{\frac{rl}{\lambda}} - \lambda q^*}{r}
\]
Solving the equation, we get
\[
q^* = e^{\frac{1}{r}(e^{\frac{rl}{\lambda}} - 1)}.
\]

4 Implementation

The optimal contract derived in the previous sections is written in terms of continuation utility, which is highly abstract. Moreover, the principal controls the agent’s consumption directly, i.e.
the agent consumes all the payments from the principal at any point in time. In this section, we
provide an implementation of the optimal contract, in which a primary component of the agent’s
compensation is a state-contingent security. In this implementation, besides the decision of ex-
erting effort or shirking, the agent also chooses consumption by himself. Yet, the implementation
generates the same allocation as the original optimal contract. Finally, we briefly discuss how this
implementation relates to the compensation scheme used in reality.

To introduce the design of the state-contingent security, we first look at a discrete-time approx-
imation of the continuous-time setting. The security lasts for one period. When the project is at
stage \( n \), \( y \) shares of this security bought in period \( t \) pays \( y \) in period \( t + 1 \) if the agent fails to make
a discovery. If the agent succeeds, the payoff is \( Y_{n+1}(y) \), where \( Y_{n+1}(y) \) is a function of \( y \), which is
stage specific. The price of the security is determined by fair-price rule, i.e. the price of the security
equals the present value of this security. Let \( P_n(y) \) denote the price of \( y \) shares of the security when
the project is at stage \( n \). Then,

\[
P_n(y) = e^{-\gamma \Delta t}((1 - \lambda \Delta t)y + \lambda \Delta t Y_{n+1}(y)).
\]

**Remark:** In general, the pricing function \( P_n \) is non-linear. But, if the utility function is
logarithmic, then \( Y_{n+1}(y) \) is a linear function of \( y \), and hence the pricing function becomes linear
\( P_n(y) = p_n y \), where \( p_n \) is the price for each share of the security and is stage specific.

To implement the optimal contract, before the project starts, the principal provides the agent
with initial-wealth \( y_0 \), and \( y_0 \) of the initial wealth is paid in terms of this security. When
the project proceeds, in each period, the agent is required to hold a minimum amount of this
security until the whole project is completed. The minimum amount requirement, denoted by \( y_n \),
is also stage specific. We assume that investing in this security is the only saving technology for the
agent to smooth consumption overtime. Hence, in each period, besides effort choice, the agent also
decides how much to consume and how much to invest in the security. Let \( y_t \) denote the agent’s
wealth in period \( t \). Then, his budget constraint is

\[
r c_t \Delta t + e^{-\gamma \Delta t}((1 - \lambda \Delta t)y_{t+1} + \lambda \Delta t Y_{n+1}(y_{t+1})) \leq y_t,
\]

where the first term on the left-hand side is his consumption in the current period, and the second
term is his investment in the security. Note that \( y_{t+1} \) is the number of shares of the security that the
agent purchases in period \( t \), which is also his wealth in period \( t + 1 \) if he fails to make a discovery.
Let $\Delta t$ converges to 0, we can derive the evolution of the agent’s wealth in case of failure, which satisfies

$$\dot{y} = ry - rc - \lambda(Y_{n+1}(y) - y).$$

When the project is in stage $n$, the agent’s wealth in case of failure grows at rate $r$, and decreases due to the spending on consumption $c$ and the loss of the investment in the security $\lambda(Y_{n+1}(y) - y)$. If the agent succeeds, his wealth jumps to $Y_{n+1}(y)$.

The agent’s problem is to choose an effort process and a consumption process to maximize his discounted expected utility. Let $V_n(y)$ be the maximum expected utility that the agent can get in stage $i$, given income $y$. Then, in recursive form, the agent’s problem in stage $n$ is to solve the following HJB equation

$$V_n(y) = \max \left\{ \max_c \left[ rU(c) + V'_n(y) \dot{y} + \lambda(Y_{n+1}(y) - y), \max_c rU(c) + V'_n(y) \dot{y} \right] \right\}$$

s.t.

$$\dot{y} = ry - rc - \lambda(Y_{n+1}(y) - y),$$

$$y \geq y_n.$$

The next proposition shows that under certain conditions this implementation generates the same allocation as the original optimal contract. The proof is in the appendix.

**Proposition 4.1** Suppose the principal provides the agent with initial wealth $y_0$

$$y_0 = C_0(v_0),$$

and in stage $n$

$$Y_{n+1}(y) = C_{n+1}(C_n^{-1}(y) + \frac{rI}{\lambda}),$$

$$y_n = C_n(0).$$

Then, given income $y$, the highest discounted expected utility the agent can get is

$$V_n(y) = C_n^{-1}(y),$$

and he chooses consumption flow $c$ that satisfies

$$S'(U(c)) = C'_n(V_n(y)).$$
In addition, the agent always exerts effort until he completes the last-stage innovation.

In stage \( n \), given income \( y \), the highest expected utility that the agent can get is \( V_n(y) \), and he chooses consumption flow \( c \) which satisfies \( S'(U(c)) = C'_n(V_n(y)) \). In the optimal contract, the agent’s continuation utility is equal to \( V_n(y) \) at this point of time. Given this continuation utility, the transferred-utility flow satisfies \( S'(u) = C'_n(V_n(y)) \). This implies that \( U(c) = u \), or the consumption flow chosen by the agent in this implementation attains the same utility flow as what is chosen by the principal in the optimal contract for all possible histories. Hence, this implementation generates the same consumption allocation as the optimal contract.

The idea of this implementation comes from the fact that the agent’s utility maximization problem is the dual problem of the principal’s cost minimization problem in section 3. Given continuation-utility \( v \), \( C_n(v) \) is the minimum expected-cost to finance the incentive-compatible compensation scheme. From the dual perspective, given expected wealth \( y = C_n(v) \), the maximum expected utility that the agent can reach should equal \( v \). Furthermore, the consumption allocation should be the same.

In this implementation, the state-contingent security plays a key role in providing incentives. The gap between the payoff in case of success and that in case of failure guarantees that the agent is willing to exert effort. In fact, the required minimum amount is the lowest level that can provide an incentive for exerting effort. When the agent’s wealth drops to this level, the highest expected utility he can from the contract is zero, which is the lower bound of the continuation utility.

However, in the financial market, there does not exist such an exotic asset that has the exact same payoff structure as the state-contingent security used in this implementation. However, the stock of a company is a reasonable proxy for this security. Since these firms rely intensely on R&D, the performance of the employees in the R&D units have a great impact on these firms’ performance outcomes, which bring a close relationship between employees’ performance and the return of firms’ stocks. In particular, after each breakthrough in R&D, it always follows a notable increase in the firm’s stock price. When there is no arrival of such good news for a period time, its stock price tends to decline. Thus, among all available assets, the company’s stock has the closest payoff-pattern to that of the state-contingent security. Another feature of our implementation is the minimum amount holding requirement that the agent has to meet until he completes the project. In the real-world, this feature is mimicked by using employee stock-options, which has vesting period during which the options cannot be exercised. The time restriction provides long-term incentives
to overcome the repeated moral-hazard problem.

In the past two decades, stock-based grants, especially stock-options, have become the most popular compensation scheme used by new-economy firms. The similarities between this compensation scheme and our implementation of the optimal contract suggest that firms are getting as close to optimality as is allowed by the market structure. In other worlds, our implementation gives a justification for the wide-spread use of stock-based compensation in firms that rely on R&D from a theoretical point of view.

5 Conclusion

This paper constructed an optimal dynamic contract to solve the repeated moral-hazard problem when a principal hires an agent to do a multi-stage R&D project. The R&D process is modeled by a jump process (Poisson). In the optimal contract, incentive is provided in two ways: (1) the agent’s continuation utility jumps up to a higher value when he successfully completes an innovation (reward); (2) If the agent fails to make a discovery, his continuation utility decreases continuously over time (punishment). The evolution of the continuation utility depends on the entire history of the innovation process up to time $t$, i.e. it is based on how many innovations have been made before time $t$ and how long it takes the agent to complete each innovation.

We also show that the optimal contract could be implemented by a risky security, whose return depends on the outcome of the project. The agent is required to hold a minimum amount of this security until he completes the whole project. In this implementation, instead of the principal directly controlling the agent’s consumption as in the optimal contract, the agent chooses consumption level by himself. By a duality argument, we show that this implementation yields the same allocation as the optimal contract. This implementation provides a theoretical justification for the stock-based compensation used in reality.

Appendix

Proof of Lemma 3.1

In the HJB equation, the principal chooses $\bar{v}$ to minimize $-C'(v)\bar{v} + S(\bar{v})$ subject to the incentive-compatibility constraint $\bar{v} \geq v + \frac{d_l}{x}$. By assumption, $-C'(v)\bar{v} + S(\bar{v})$ is a convex and twice-
differentiable function of \( \tilde{v} \). The unconstraint minimum is reached at \( \tilde{v}' \) that satisfies the first order condition \( C'(v) = S'(\tilde{v}') \). If \( C'(v) > S'(v + \frac{r_l}{\lambda}) \), then \( S(\tilde{v}') > S(v + \frac{r_l}{\lambda}) \), which implies that \( \tilde{v}' > v + \frac{r_l}{\lambda} \). Thus, the optimal choice of \( \tilde{v} \) is \( \tilde{v}' \) and the incentive-compatibility constraint is not binding. \( C'(v) \leq S'(v + \frac{r_l}{\lambda}) \) implies that \( \tilde{v}' \leq v + \frac{r_l}{\lambda} \). In this case, the optimal choice of \( \tilde{v} \) is \( v + \frac{r_l}{\lambda} \) and the incentive-compatibility constraint binds.

**Proof of Lemma 3.2**

By Lemma 3.1, the incentive-compatibility constraint binds in this case. Using the equation \( \tilde{v} = v + \frac{r_l}{\lambda} \) in (3), the rate of change of \( v \) becomes \( \frac{dv}{dt} = r(v - u) \). Therefore, the HJB equation is

\[
 rC(v) = \min_u rS(u) + C'(v)(r(v - u)) + \lambda(S(v + \frac{r_l}{\lambda}) - C(v)).
\]

From the envelope theorem,

\[
 (r + \lambda)C'(v) = rC'(v) + \lambda S'(v + \frac{r_l}{\lambda} + C''(v)\frac{dv}{dt}).
\]

Thus,

\[
 \frac{dC'(v)}{dt} = \lambda(C'(v) - S'(v + \frac{r_l}{\lambda})).
\]

Since \( C'(v) < S'(v + \frac{r_l}{\lambda}) \), it follows that \( \frac{dC'(v)}{dt} < 0 \).

As \( \frac{dv}{dt} = r(v - u) \), the sign of \( \frac{dv}{dt} \) is determined by the values of \( v \) and \( u \). Note that \( u \) is chosen to minimize \( S(u) - C'(v)u \), which is a strictly convex function of \( u \). The first order condition implies that \( C'(v) = S'(u) \). If \( C'(v) = S'(v) \), then \( S'(v) = S'(u) \). Since \( S(v) \) is strictly convex, we have \( v = u \) and \( \frac{dv}{dt} = r(v - u) = 0 \). Similarly, \( C'(v) > S'(v) \) implies that \( \frac{dv}{dt} < 0 \) and \( C'(v) < S'(v) \) implies that \( \frac{dv}{dt} > 0 \).

**Proof of Lemma 3.3**

In this case, \( \tilde{v} \) is unconstrained optimal and \( \tilde{v} \geq v + \frac{r_l}{\lambda} \). Taking (3) into the HJB equation,

\[
 rC(v) = \min_{u, \tilde{v}} rS(u) + C'(v)(rv - r(u - l) - \lambda(\tilde{v} - v)) + \lambda (S(\tilde{v}) - C(v)).
\]

From envelope theorem

\[
 (r + \lambda)C'(v) = (r + \lambda)C'(v) + C''(v)\frac{dv}{dt}.
\]
Therefore,

\[ \frac{dC'(v)}{dt} = 0. \]

For the dynamics of \( v \), note that in this case

\[ \frac{dv}{dt} = rv - r(u - l) - \lambda(\bar{v} - v) \]

\[ = r(v - u) + (rl + \lambda v - \lambda \bar{v}). \]

Since \( \bar{v} \geq v + \frac{rl}{\lambda} \), the second term is non-positive. For the first term, \( u \) is determined by the first order condition \( C'(v) = S'(u) \). Since \( C'(v) \geq S'(v + \frac{rl}{\lambda}) \), we have \( S'(u) \geq S'(v + \frac{rl}{\lambda}) \), which implies that \( u \geq v + \frac{rl}{\lambda} > v \). Thus, the first term is strictly negative. It follows that \( \frac{dv}{dt} < 0. \)

**Proof of Proposition 4.1**

We first verify that \( V_n(y) = C_n^{-1}(y) \) solves the HJB equation under the conditions in Proposition 4. Then, we show that this implementation generates the same consumption allocation as the optimal contract. First note that

\[ rl - \lambda(V_{n+1}(Y_{n+1}(y)) - V_n(y)) = rl - \lambda(V_{n+1}(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})) - V_n(y)) \]

\[ = rl - \lambda(C_n^{-1}(y) + \frac{rl}{\lambda} - C_n^{-1}(y)) \]

\[ = 0. \]

This result implies that for any consumption flow \( c \), the agent is indifferent between exerting effort and shirking. Thus, we have

\[ RHS = rU(c) + V_n'(y)(ry - rc - \lambda(C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda}) - y)) \]

\[ = rU(c) + \frac{(r + \lambda)y - rc - \lambda C_{n+1}(C_n^{-1}(y) + \frac{rl}{\lambda})}{C'_n(C_n^{-1}(y))}, \]

where \( c \) is determined by the first-order condition \( U'(c) = \frac{1}{C'_n(C_n^{-1}(y))} \).

Since \( C_n(v) \) satisfies the following differential equation

\[ (r + \lambda)C_n(v) = rS(u) + C_n'(v)(r(v - u)) + \lambda C_{n+1}(v + \frac{rl}{\lambda}), \]

then

\[ \frac{1}{C'_n(v)} = \frac{r(v - u)}{(r + \lambda)C_n(v) - rS(u) - \lambda C_{n+1}(v + \frac{rl}{\lambda})}. \]
where \( u \) satisfies \( S'(u) = C'_n(v) \). Taking \( v = C^{-1}_n(y) \) into the equation above, we get

\[
\frac{1}{C'_n(C^{-1}_n(y))} = \frac{r(C^{-1}_n(y) - u)}{(r + \lambda)C_n(C^{-1}_n(y)) - rS(u) - \lambda C_{n+1}(C^{-1}_n(y) + \frac{r}{\lambda})}
\]

\[
= \frac{r(C^{-1}_n(y) - u)}{(r + \lambda)y - rS(u) - \lambda C_{n+1}(C^{-1}_n(y) + \frac{r}{\lambda})},
\]

where \( S'(u) = C'_n(C^{-1}_n(y)) \). Since \( S(u) = U^{-1}(u) \), it follows that \( \frac{1}{U'(S(u))} = C_n(C^{-1}_n(y)) \). Hence, \( S(u) = c \) and \( u = U(c) \) since \( c \) satisfies \( U'(c) = \frac{1}{C_n'(C^{-1}_n(y))} \). Therefore,

\[
\frac{1}{C'_n(C^{-1}_n(y))} = \frac{r(C^{-1}_n(y) - U(c))}{(r + \lambda)y - rc - \lambda C_{n+1}(C^{-1}_n(y) + \frac{r}{\lambda})}.
\]

Taking this expression for \( \frac{1}{C'_n(C^{-1}_n(y))} \) into the right-hand side of the HJB equation, we have

\[
RHS = rU(c) + \frac{(r + \lambda)y - rc - \lambda C_{n+1}(C^{-1}_n(y) + \frac{r}{\lambda})}{C'_n(C^{-1}_n(y))}
\]

\[
= rU(c) + r(C^{-1}_n(y) - U(c))
\]

\[
= rC^{-1}_n(y)
\]

\[
= rV_n(y)
\]

\[
= LHS.
\]

Thus, \( V_n(y) = C^{-1}_n(y) \) solves the following HJB equation.

The first order condition implies that \( S'(U(c)) = C'_n(C^{-1}_n(y)) = C'_n(V_n(y)) \). Moreover, since the agent is indifferent between exerting effort and shirking, he is always willing to put in effort.

References


