Market Design under Distributional Constraints: Diversity in School Choice and Other Applications*†

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Abstract

Distributional constraints are important in many market design settings. Prominent examples include the minimum manning requirements at each branch in military cadet matching and diversity in school choice, whereby school districts impose constraints on the demographic distribution of students at each school. Standard assignment mechanisms implemented in practice are unable to accommodate all of these constraints. This leads policymakers to resort to ad-hoc solutions that eliminate blocks of seats ex-ante (before agents submit their preferences) to ensure that all constraints are satisfied ex-post.

We show that these solutions ignore important information contained in the submitted preferences, resulting in avoidable inefficiency. We introduce a new class of dynamic quotas mechanisms that allow the institutional quotas to dynamically adjust to the submitted preferences of the agents. We show how a wide class of mechanisms commonly used in the field can be adapted to our dynamic quotas framework. Focusing in particular on a new dynamic quotas deferred acceptance (DQDA) mechanism, we show that DQDA Pareto dominates current solutions. While it may seem that allowing the quotas to depend on the submitted preferences would compromise the strategyproofness of deferred acceptance, we show that this is not the case: as long as the order in which the quotas are adjusted is determined exogenously to the preferences, DQDA remains strategyproof. Thus, policymakers can be confident that efficiency will be improved without introducing perverse incentives. Simulations with school choice data are used to quantify the potential efficiency gains.

JEL Classification: C78, D61, D63, I20

Keywords: minimum quotas, floors, ceilings, affirmative action, school choice, diversity, strategyproofness, deferred acceptance

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1 Introduction

In many market design settings, distributional goals are important to policymakers. For instance, the geographic distribution of doctors is an issue in many medical residency markets, with hospitals in rural areas often having difficulty attracting enough residents (the so-called “rural hospital” problem). In military cadet matching, each Army branch has a minimum manning requirement that must be met. In cities such as New York, Chicago, and Boston, among others, centralized mechanisms are used to assign tens of thousands of students to schools each year based on submitted student preferences. An important additional consideration of many school choice plans is achieving demographically diverse student bodies.\footnote{See Roth (1986) for an introduction to the rural hospital problem, and Sönmez and Switzer (2013) for details on military cadet matching. For school choice from a mechanism design perspective, the seminal paper is Abdulkadiroğlu and Sönmez (2003). See also the Institute for Innovation in Public School Choice (http://www.iipsc.org), a collaboration between economists and educators, for details on many school choice markets in the United States, and http://www.matching-in-practice.eu for examples in Europe.} The question we address in this paper is how to design mechanisms for use in markets with constraints like those listed above. We identify inefficiencies present in commonly-used solutions caused by their insensitivity to agent demand. Our main contribution is to propose a new class of mechanisms that increase efficiency without compromising fairness or incentive properties.

The constraints we consider take the form of floors and ceilings on the number of agents assigned to each institution. We consider two distinct cases. In the single type case, floors and ceilings are imposed on the total number of agents, as in military branch assignment. In the multiple type case, agents may be of different types, and floors and ceilings are imposed on each type separately. For example, school districts concerned about diversity often classify students based on demographic characteristics such as socioeconomic status (SES) and then impose floors and ceilings on the number of students of each SES type at each school (for concreteness, we use the language of “students” and “schools”, keeping in mind that our model can be applied to many other settings; Section 2 discusses specific markets in detail).

Most commonly-used mechanisms (e.g., the serial dictatorship, deferred acceptance, top trading cycles) can easily handle ceilings but are unable to accommodate floors, leading many markets to adopt ad-hoc policies to attempt to satisfy all constraints. One popular approach taken by policymakers is the following: run one of the standard mechanisms (denoted $\psi$), but in place of the true ceilings, use some artificially lower ceilings. For example, the Japanese medical match caps the number of doctors who can be assigned to hospitals in urban areas and then runs deferred acceptance, a policy designed to force more assignments to rural hospitals (Kamada and Kojima (2013)). We call this approach \textit{artificial caps}. Artificial caps mechanisms are appealing because (i) many of the good properties of the underlying mechanism $\psi$ will carry over to $\psi$ with artificial caps, and (ii) imposing sufficiently stringent artificial caps guarantees that all floors will be implicitly
satisfied.

In addition, artificial caps is a natural solution because given an underlying mechanism $\psi$, it is trivial to implement; policymakers need only decide on a choice for the artificial caps. However, this approach can lead to final assignments that are inefficient. We begin by illustrating this with a stylized example of 60 students and 3 schools with floors of 15 and ceilings of 30.\footnote{For simplicity, this example considers only the single type case, but a similar example can be constructed with multiple types. The formal model will encompass both cases.} Using a standard mechanism under the true ceilings of 30 may leave the floors at some schools unfilled (e.g., if two schools take 30 students each, leaving none for the third school). On the other hand, consider imposing artificial caps of 20 at each school. In this case, each school is assigned 20 students for any submitted student preferences, and the floors are implicitly satisfied. However, ex-post it may be possible to move some students to schools they prefer without violating the true constraints.\footnote{For example, if 30 students have school $s_1$ as a first choice while 15 students each have $s_2/s_3$ as a first choice, then giving every student her first choice does not violate the true floors or ceilings, but would violate the artificial caps of 20. Hence, artificial caps is inefficient. Section 2 discusses this example in further detail.} Intuitively, the mechanism is too rigid: to guarantee that all floors are satisfied ex-post, seats are eliminated ex-ante, ignoring important information contained in the student preferences.

We propose a new class of dynamic quotas mechanisms that increase efficiency by allowing the school quotas to dynamically adjust to the submitted student preferences. Briefly, given some underlying mechanism $\psi$, we first run $\psi$ under high ceilings. If the outcome satisfies all floors, the mechanism ends with this matching. If not, we lower some ceiling and run $\psi$ again. After a sufficient number of iterations, all constraints will be satisfied. By lowering the ceilings gradually and stopping when the floors are met, our mechanism ends under higher ceilings than artificial caps, making the students better off.

We show how any mechanism $\psi$ that takes ceilings as an input can be adapted to our dynamic quotas framework to achieve the aforementioned efficiency gains. We pay particular attention to the case when the underlying mechanism $\psi$ is the deferred acceptance (DA) mechanism of Gale and Shapley (1962). Without floors, DA is a popular mechanism because it satisfies two key properties, namely: (i) strategyproofness (it is a dominant strategy for students to always report their preferences truthfully), and (ii) elimination of justified envy (if student $i$ prefers student $j$’s school, then $j$ must have higher priority than $i$ at that school). Strategyproofness guarantees that when we analyze welfare, we are doing so with respect to the true preferences. In addition, strategyproofness is highly valued by school districts because it prevents parents from having to play a complicated strategic game, which may put parents who are unwilling or unable to strategize at a disadvantage. Elimination of justified envy is a fairness criterion that is important to many school districts that distribute priorities based on factors such as the distance a student lives from a school or whether a student has a sibling attending a school. While standard DA satisfies these good properties, it may leave floors unfilled, leading to artificial caps DA (ACDA) as an ad-hoc
way to ensure that all constraints are met.

In the single type case, we show that our dynamic quotas DA (DQDA) mechanism Pareto dominates ACDA, while also eliminating justified envy. On incentives, it may seem that modifying the DA algorithm to allow the school quotas to change depending on the submitted student preferences will result in a loss of strategyproofness. However, it turns out that this is not the case. In particular, we show that as long as the order in which the quotas are reduced is determined exogenously to the submitted preferences, DQDA remains strategyproof. In fact, we further show that DQDA is on the Pareto frontier of strategyproof mechanisms, thus providing a sense in which it is impossible to improve upon DQDA without compromising incentives. With multiple types, the problem becomes more complicated because a school may reduce either its overall capacity or a ceiling for a specific demographic type. Nevertheless, we are able to show that DQDA once again Pareto dominates ACDA without compromising fairness or strategyproofness. Thus, a school district that switches to a dynamic quotas mechanism can be confident that doing so will not introduce perverse incentives and that the theoretical efficiency gains will actually be realized in practice.

While incentives are important, strong incentive constraints can limit efficiency. We move to identify features of DQDA that were needed for full strategyproofness but that do indeed limit efficiency. We introduce one final mechanism, endogenous-reduction DQDA (EDQDA), that allows the order in which quotas are reduced to be determined more endogenously by the submitted student preferences. While this results in a loss of strategyproofness, we show that EDQDA is at least strategyproof in the large in the sense of Azevedo and Budish (2013): as the market size increases, the incentives for any student to manipulate go to zero. From a practical perspective, large school districts may opt for EDQDA to achieve greater efficiency at a small cost to incentives.

Lastly, while our formal results demonstrate Pareto improvements, they do not speak to the magnitude of the efficiency gains. To answer this question, we turn to simulations. We structure the simulations around kindergarten assignment in Cambridge, MA. The simulation results show that the rank distribution of ACDA is stochastically dominated by that of DQDA, which itself is stochastically dominated by that of EDQDA, with on average over 20% more students receiving their first choice school under EDQDA than under ACDA for some specifications. In addition, we conduct comparative statics and find that these results are sensitive to two important parameters: the correlation in the student preferences and the relative sizes of the floors and ceilings, i.e., the degree of “flexibility”. Though the dominance relations hold for all parameter values, less correlation and more flexibility lead to larger improvements from dynamic quotas.

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4Cambridge makes a limited amount of data publicly available which we use to choose many of the simulation parameters, but for full implementation, we must make some additional assumptions. See Section 6 for details.

5The rank distribution $F(x)$ gives the number of students who receive their $x^{th}$ or better choice.
The remainder of this paper is organized as follows. Section 2 presents a stylized example to highlight our approach and discusses some real-world markets and related literature. Section 3 describes the formal model and gives definitions of deferred acceptance and artificial caps. Section 4 introduces DQDA, while Sections 5 and 6 characterize DQDA in the single type and multiple type cases, respectively. Section 7 discusses some extensions, including how to apply our dynamic quotas techniques to a broad class of mechanisms used in practice. Section 8 concludes. All proofs can be found in the appendix.

2 Motivating examples and related literature

Before discussing real-world applications in detail, we first present a stylized example to motivate our main idea.

2.1 A stylized example

For simplicity, the example we construct focuses on the case where all students are of the same type, but it is possible to construct an analogous example with multiple types as well.

Example 1. Consider a market of 3 schools \( S = \{A, B, C\} \), each of capacity 30, and 60 students \( I = \{i_1, \ldots, i_{60}\} \). Let the distributional constraints of the school district be to assign at least 15, but no more than 30, students to each school.

Let the preferences of the students be \( P_i: A, B, C \) for all \( i \) (i.e., each student prefers \( A \) to \( B \) to \( C \)), and let \( P_I = (P_i)_{i \in I} \). For now, consider a very simple serial dictatorship (SD) mechanism that works as follows: Without loss of generality, order the students \( i_1 \gg \cdots \gg i_{60} \). Then, starting from the top of the ordering, let each student pick her most preferred school that has not yet reached its capacity. Under the given preferences, students \( i_1 \cdots i_{30} \) are assigned to \( A \), students \( i_{31} \cdots i_{60} \) are assigned to \( B \) and no students are assigned to \( C \). This is illustrated graphically in Figure 1. As can be seen in the figure, the matching is not feasible, since no students apply to school \( C \) and so it does not reach its floor of 15.

Consider instead running the same serial dictatorship mechanism, but under lower capacities of 20 at each school. Figure 2 shows that this results in a feasible matching. Figure 2 is the motivation behind the idea of artificial caps. Any matching that satisfies the artificial caps of 20 (the red line in the figure) is also guaranteed to satisfy the floors of 15. This can be seen by noting that the largest number of students that can be accommodated by any two schools is 40, which leaves 20 students remaining to go to the third school. Because of this, running the serial dictatorship under these capacities will be feasible for any submitted preference profile \( P_I \).\(^6\)

\[^6\text{We impose symmetric artificial caps of 20 in this example for illustrative purposes, but in general there will}\]
Figure 1: The number of students assigned to each school under the serial dictatorship with capacities of 30 and preference profile $P_i : A, B, C$ for all $i \in I$.

Figure 2: The number of students assigned to each school under the serial dictatorship with capacities of 20 and preference profile $P_i : A, B, C$ for all $i \in I$.

Now, consider an alternative preference profile $P'_i$ defined by:

- $P'_i : A, B, C$ for $i = i_1, \ldots, i_{30}$
- $P'_i : B, C, A$ for $i = i_{31}, \ldots, i_{45}$
- $P'_i : C, A, B$ for $i = i_{46}, \ldots, i_{60}$.

In this case, running the serial dictatorship under the true capacities of 30 gives everyone their first choice, which also happens to be a feasible matching (see Figure 3). However, this matching be multiple choices of artificial caps that ensure a feasible matching, including some that are not symmetric. The inefficiencies identified here will be present for any choice of the artificial caps.
violates the artificial caps of 20, meaning that if the serial dictatorship were run under these artificial caps, some students would be unnecessarily rejected from their preferred schools. Formally, under preference profile $P^I$, the serial dictatorship under capacities of 30 Pareto dominates the serial dictatorship under capacities of 20, with 15 students being made strictly better off.

This example highlights the main shortcoming of artificial caps: in order to guard against the worst-case scenario of not filling some floors, artificial caps eliminates a large block of seats ex-ante, without regard for the submitted preferences of the students. A better mechanism would utilize the information contained in the student preferences to determine the capacities in effect at the final matching. This is the idea behind the dynamic quotas approach we introduce: first start with high capacities equal to 30 for each school. Run the algorithm and check whether, given the submitted preferences, all constraints are satisfied. If so, output this as the final matching. If not, lower the capacity of one school by one seat and repeat. Lowering the capacities gradually in this manner will produce a final matching that Pareto dominates artificial caps.

In general, one might be worried that allowing the capacities to depend on the submitted preferences in this manner would result in a loss of strategyproofness. In Section 5, we show that this is not the case: as long as the order in which the capacities are reduced is determined exogenously to the preferences, strategyproofness will be retained.

### 2.2 Applications

We now describe the details of several markets with distributional constraints. We note that while in some settings the use of artificial caps is openly discussed and publicly known (examples below),
in others, artificial caps may be used in such a way that it is unobserved to outsiders. That is, policymakers may decide to impose artificial caps in order to implicitly satisfy some floors, but to an outside analyst, it would appear as if the artificial caps were the “true” ceilings and there were no floors. This may happen in school districts which have control over the assignment to every school within their boundary. Since the idea behind artificial caps is an intuitive one that naturally occurs to policymakers, we believe that our mechanisms may be applicable to many markets that have up to now dealt with floors in such a suboptimal manner simply for lack of a better option. Once mechanisms are available that can handle floors, they may begin to explicitly appear in more markets.

Medical residency markets

A well-documented example of distributional concerns in matching markets is the so-called rural hospital problem often observed in matching newly graduated doctors to hospital residency programs. For example, Talbott (2007) notes that the United States as a whole has 280 doctors per 100,000 people, but the 18-county Mississippi Delta area has only 103 doctors per 100,000 people. Similar doctor shortages in rural areas are present in many countries, such as the United Kingdom and Australia, among others (Shallcross (2005); Nambiar and Bavas (2010)).

In Japan, in order to address this issue, the Japan Residency Matching Program (JRMP) imposes regional caps on the number of doctors who can be assigned to each of the country’s 47 prefectures. For example, in 2008, 860 positions were offered in Osaka (a popular urban area), but the JRMP decided to reduce this to 533 (a reduction of almost 40%) before running the deferred acceptance algorithm (similar reductions occurred in most urban areas; see Kamada and Kojima (2013)). The end goal of this policy is not to limit the number of doctors in urban areas per se, but to satisfy some implicit floors in rural areas. Our results suggest that the JRMP may improve outcomes by instead modeling the floor constraints explicitly and using dynamic quotas. Doing so will satisfy the actual distributional goals and make all doctors (weakly) better off, compared to the current approach of imposing regional caps and hoping that the resulting distribution of doctors turns out to be satisfactory.7

Military cadet matching

As a second example, consider military cadet matching. At the end of every year, over one thousand cadets at the United States Military Academy (USMA) submit preferences over Army branches (Aviation, Infantry, etc.) in which they would like to serve. After collecting cadet preferences,

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7 As of now, the JRMP does not model floor constraints explicitly, and Kamada and Kojima (2013) identify flaws in their mechanism when the regional caps are taken as the true objective. We should also note that while our formal model considers constraints at the institutional level, our mechanisms can be modified to accommodate caps at the regional level.
USMA then runs a matching algorithm to determine the assignment of cadets to branches (a process referred to as branching).  

An October 1, 2007 memorandum from the Army Deputy Chief of Staff to USMA entitled “Branch Allocation Methodology” describes the following three phase procedure. In Phase 1, the Army provides USMA with floors and ceilings for each branch, based on current staffing needs. In Phase 2, given these floors and ceilings, USMA calculates “demand pegs”, which are the ceilings that will be used at each branch when the matching algorithm is run in Phase 3. These demand pegs will be lower than the true ceilings provided in Phase 1, in order to ensure that all of the true ceilings and floors are satisfied; they are analogous to what we call artificial caps. Then, in Phase 3, the matching algorithm is run using the demand pegs from Phase 2. Similar to the markets above, conducting the branching using the demand pegs is a way for USMA to ensure that the floors given by the Army are satisfied, but it may do so inefficiently. The memo states that this is done because “there is no ex-ante closed form algorithm that optimizes program participation subject to manning requirements.” Providing such algorithms and studying their properties are precisely the goals of this paper.

School choice

In the examples cited above, all agents were of the same type and only aggregate constraints were imposed. In school choice, school districts care not only about the total number of students in each school, but also about the demographic composition within the schools. Some school districts implement diversity plans voluntarily (e.g., Cambridge, Massachusetts and Chicago, Illinois), while in other cities, it is court-ordered (in the case of Sheff v. O’Neill (1996), the Connecticut Supreme Court ordered that schools in Hartford, Connecticut be integrated). Historically, school districts often imposed diversity constraints based on race (for example, the 1965 Racial Imbalance Act in Massachusetts). In 2007, the United States Supreme Court limited the use of race as an explicit factor, but upheld that the state has a compelling interest in promoting diversity and that it is “an interest that a school district, in its discretion and expertise, may choose to pursue” (Parents Involved in Community Schools v. Seattle School District No. 1 (2007)).

In practice, the banning of race as an explicit factor has led school districts to pursue diversity by classifying students using other demographic characteristics (the basis of the classification will be immaterial for our purposes). For example, consider Jefferson County, Kentucky, which contains

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8See Sönmez and Switzer (2013) and Sönmez (2013) for further details.
9Military branching can also be thought of as a special case of a firm (the Army) that must assign employees to projects (the branches), with each project having a minimum staffing requirement. For example, some technology firms in Silicon Valley use centralized mechanisms to assign new interns to positions (with managers also submitting preferences over employees). In a similar vein, for many new medical residents, the first year after medical school is a transitional year in which they rotate as a clinician through various departments of a hospital. While the hospitals try to accommodate preferences as much as possible, each department has a minimum staffing requirement.
the city of Louisville and was also a party to the aforementioned court decision. After it was ruled that race could no longer be used as a determining factor, the school district partitioned the city geographically and required that between 15-50% of each school come from certain lower income areas of the city (Echenique and Yenmez (2012)).

Similarly, Cambridge, Massachusetts started its controlled choice plan in 1980, using race as the main factor. In 2001, Cambridge voluntarily shifted from classifying students by race to classifying them as either high or low socioeconomic status (SES) based whether or not they qualify for the federal Free and Reduced Lunch program. The school district then requires that each school be within 10% of the district-wide average for each SES class, which translates into floors and ceilings of approximately 25-45% for low SES students and 55-75% for high SES students at each school. Cambridge is a useful benchmark to keep in mind because they release some public data on their school choice process (including the number of students who list each school as a first choice). We will make use of this data when conducting our computer simulations in Section 6.

2.3 Related literature

Early papers to discuss distributional constraints in matching focused on the rural hospital problem and obtained mostly negative results. Papers such as Gale and Sotomayor (1985a,b), Roth (1984, 1986), Martinez et al. (2000), and Hatfield and Milgrom (2005) prove various versions of the “rural hospital theorem,” which says that if a doctor or a position at a hospital is unmatched at some stable matching, then they are unmatched at any stable matching. This suggests that the rural hospital problem is difficult to solve without imposing any additional structure on the market, and is what led the Japan Residency Matching Program (JRMP) to impose regional caps on the number of doctors who can be assigned to urban areas. Kamada and Kojima (2013) study this market in detail, and point out flaws in the current JRMP mechanism when the regional caps are taken as the true goals. Floor constraints are not present in their model (because the JRMP itself does not model them explicitly), but making the floor constraints explicit and using our dynamic quotas mechanisms is another possible approach that may improve the performance of the market.

Since it is an important goal of many school districts, diversity constraints have been discussed in the school choice literature, but most work thus far has only dealt with upper quotas/ceilings.}

\[10\] While school districts often state their diversity goals in terms of percentages for simplicity, in practice, the algorithms are often run using absolute numbers. For example, a school that was assigned two high SES students and one low SES student would satisfy Cambridge’s stated percentage goals, but this likely would not be an acceptable assignment because there are too few students overall. Thus, when running the algorithm, Cambridge converts the stated percentages into numbers of seats (see http://www3.cpsd.us/video/controlled_choice_video for a video describing the implementation of the Cambridge algorithm, aimed at parents). See the formal model in Section 3 for further discussion of this distinction.

\[11\] Afacan (2013) studies whether hospitals can manipulate their preferences to change the number of positions filled. Sönmez (1997) studies the complementary question of whether hospitals can manipulate their capacities to obtain a more preferred assignment of doctors.
Abdulkadiroğlu and Sönmez (2003), the seminal paper on school choice from a mechanism design perspective, show how type specific ceilings can be easily incorporated into standard matching mechanisms such as deferred acceptance and top trading cycles. Ceiling constraints do not fully capture diversity constraints, however, since they can still result in completely segregated schools. In addition, in a model with two types of students (majority and minority), Kojima (2012) points out that simple ceiling constraints can actually make all minority students (the supposed beneficiaries) worse off. Hafalir et al. (2013) correct this by proposing deferred acceptance with minority reserves, a mechanism further generalized by Kominers and Sönmez (2013), who introduce slot-specific priorities. Abdulkadiroğlu (2005), Erdil and Kumano (2012), and Echenique and Yenmez (2012) study various generalizations of school priorities over sets of students and how they can capture certain types of diversity goals.\(^\text{12}\)

Many distributional constraints are not captured by the ceilings and reserves modeled in the above papers. Our model instead treats the ceilings and floors as “hard” constraints, i.e., constraints that must be satisfied at any feasible matching. Such hard constraints complicate the problem considerably, and lead to the incompatibility of several important properties (non-wastefulness, elimination of justified envy, and strategyproofness) that could be achieved simultaneously in the previous models. Recent literature has begun to attempt to deal with hard floor constraints. In the context of object allocation, Budish et al. (2013) study what types of constraints admit expected assignments that can be implemented as lotteries over deterministic assignments. Ehlers et al. (2012) study a school choice model with floors and ceilings similar to that studied here, but due to the aforementioned impossibility results, they advocate for a “soft” interpretation of the constraints, i.e., they allow matchings that violate the floors and ceilings.\(^\text{13}\) While soft bounds may be an acceptable approach in some settings, there are many situations in which it is inadequate, such as medical markets suffering from the rural hospital problem, school districts with court-mandated desegregation guidelines, or the military, where minimum manning requirements must be filled. This paper provides mechanisms that can be used in these settings. Fragiadakis et al. (2013) also study a model with floor constraints. They propose strategyproof mechanisms that satisfy all floors, but their mechanisms run differently from those given in this paper and they must introduce non-standard notions of fairness. In addition, their model is restricted to the case of a single type, so they are unable to handle diversity constraints, which we view as an important contribution of the current paper.

This paper is also related to a large literature on incentives in mechanism design broadly and in matching mechanisms in particular. Dubins and Freedman (1981) and Roth (1982) were the

\(^{12}\)See also Westkamp (2013), who proposes similar mechanisms in the context of German university admissions, and Braun et al. (2013), who conduct an experimental analysis of these mechanisms.

\(^{13}\)They also provide an algorithm for hard constraints that eliminates justified envy among same types and is constrained non-wasteful, but this mechanism is not strategyproof.
first to prove that the deferred acceptance algorithm of Gale and Shapley (1962) is strategyproof for the proposing side in simple one-to-one matching models. Many papers have since generalized these strategyproofness results to allow schools to have multiple seats with more complex priority structures (Martinez et al. (2000); Abdulkadiroğlu (2005); Hatfield and Milgrom (2005); Hatfield and Kominers (2009, 2012); Hatfield and Kojima (2010)). At a formal level, the presence of the floors makes our model quite different from these papers. In particular, they often rely on the existence of student-optimal stable matchings, a condition which fails in our setting. Nevertheless, we are able to show that the new mechanism we propose will be strategyproof for students. This is especially important in school choice, where strategyproofness is often seen as a way to “level the playing field” for parents who are unable or do not have the resources to strategize effectively (see Ergin and Sönmez (2006), Chen and Sönmez (2006), and Pathak and Sönmez (2008, 2012), who discuss negative consequences of manipulable mechanisms).\footnote{Strategyproofness is not without a cost, as is shown in a recent strand of the school choice literature started by Abdulkadiroğlu et al. (2011) that finds that non-strategyproof mechanisms may sometimes outperform strategyproof ones on welfare grounds, at least in equilibrium (see also Featherstone and Niederle (2011), Troyan (2012), and Akyol (2013)). However, the equilibria of these mechanisms can be complex, and it may be difficult for many parents to calculate best responses.}

Finally, the problem of distributional constraints has also garnered interest in the computer science community, where many of the results are negative. For example, Biró et al. (2010) study college admissions in Hungary, in which colleges are allowed to declare minimum quotas for their programs, and Hamada et al. (2011) study hospital-resident matching with lower bounds. Both papers focus mainly on (computational) hardness results: the former shows that the problem of determining the existence of a stable matching is NP-complete, while the latter shows that the same is true of finding a matching that minimizes the number of blocking pairs. These papers provide another perspective which says that introducing floors into matching markets complicates the problem substantially, though they do not propose specific mechanisms nor study incentive or efficiency issues, as we do here.

3 Model and artificial caps

In this section, we describe our formal model of matching with distributional constraints. For concreteness, we use the language of school choice, with the interpretation that the distributional constraints correspond to the diversity goals of a school district. However, our mechanisms can also be applied to many other real-world matching markets, including those in the previous section.
3.1 Primitives

The model consists of a set $I = \{i_1, \ldots, i_n\}$ of $n$ students and a set $S = \{s_1, \ldots, s_m\}$ of $m$ schools. $\Theta = \{\theta_1, \ldots, \theta_r\}$ is a finite set of student types. Each student is of exactly one type, and the type of every student is public information. For concreteness, one may think of the types as different socioeconomic classes (as in Cambridge), but in general, they can correspond to any feature on which a school district wishes to impose distributional constraints. If $|\Theta| = 1$, then all students are of one type, and the floors and ceilings will correspond to aggregate constraints on the total number of students assigned to each school.\footnote{If we interpret the students as military cadets and the schools as branches of the military, this corresponds to constraints on the number of cadets assigned to each school.}

The function $\tau : I \rightarrow \Theta$ gives the type of each student. $I_\theta$ is the set of students of type $\theta$.

Each student $i$ has a strict preference relation $P_i$ over $S$, while each school $s$ has a strict priority relation $\succ_s$ over $I$. Vectors of such relations, one for each agent, are denoted $P_i = (P_i)_{i \in I}$ for the students and $\succ_S = (\succ_s)_{s \in S}$ for the schools. Let $\mathcal{P}$ denote the set of all individual preference relations, and $\mathcal{P}^n$ denote the set of all preference profiles $P_i$. The student preferences are their own private information. As is standard in the school choice literature, the school priorities are fixed and known to all students. In applications, priorities are often set by law, and depend on such things as the distance a student lives from a school, whether or not a student has a sibling attending the school, or whether a student speaks a certain language (for example, in certain “immersion schools” in Cambridge).

Each school $s$ has a type-specific floor $L_{s,\theta}$ (sometimes called a lower quota) and a type-specific ceiling $U_{s,\theta}$ (sometimes called an upper quota) for each type $\theta$. In addition, school $s$ has a total capacity of $Q_s$. We assume $0 \leq L_{s,\theta} \leq U_{s,\theta} \leq Q_s$ for all $(s, \theta)$. We collect the capacities for all schools in a vector $Q = (Q_s)_{s \in S}$ and the floors and ceilings in corresponding matrices $L = (L_{s,\theta})_{s \in S, \theta \in \Theta}$ and $U = (U_{s,\theta})_{s \in S, \theta \in \Theta}$.

A matching is a correspondence $\mu : I \cup S \rightarrow I \cup S$ that describes which students are assigned to which schools. Formally, $\mu$ must satisfy: (i) $\mu(i) \in S$ for all $i \in I$, (ii) $\mu(s) \subseteq I$ for all $s \in S$, and (iii) $\mu(i) = s$ if and only if $i \in \mu(s)$.

Let $\mathcal{M}$ denote the set of matchings. For any $\mu \in \mathcal{M}$,
we let \( \mu_\theta(s) = \mu(s) \cap I_\theta \) be the set of type \( \theta \) students assigned to school \( s \) under matching \( \mu \). Matching \( \mu \) is feasible if \( L_{s,\theta} \leq |\mu_\theta(s)| \leq U_{s,\theta} \) and \( |\mu(s)| \leq Q_s \) for all \((s, \theta)\). In words, a feasible matching is one that satisfies all of the type-specific floors and ceilings, as well as the capacities. Let \( \mathcal{M}_f \subseteq \mathcal{M} \) denote the set of feasible matchings. We assume throughout the paper that the set of feasible matchings is nonempty; this requires that the distributional constraints be consistent with the number of students of each type actually present in the market.\(^{18}\)

Remark 1. If \(|\Theta| = 1\), the distinction between capacities \(Q_s\) and type-specific ceilings \(U_{s,\theta}\) becomes irrelevant; that is, we can set \(U_{s,\theta} = Q_s\) for all \(s\).

A mechanism \( \psi : \mathcal{P}^n \to \mathcal{M} \) is a function that maps preference profiles to matchings. If the students submit \( P_I \in \mathcal{P}^n \), then \( \psi(P_I) \in \mathcal{M} \) is the resulting matching. We write \( \psi_s(P_I) \) for the school to which student \( i \) is assigned, and \( \psi_P(P_I) \) for the set of students assigned to school \( s \). We say that \( \psi \) is feasible if \( \psi(P_I) \in \mathcal{M}_f \) for all \( P_I \in \mathcal{P}^n \).

### 3.2 Important properties

Given two matchings \( \mu, \nu \in \mathcal{M}_f \), we say that \( \mu \) Pareto dominates \( \nu \) if \( \mu(i) \succ_R \nu(i) \) for all \( i \in I \) and \( \mu(i) \succ_R \nu(i) \) for some \( i \in I \).\(^{19}\) If \( \mu \in \mathcal{M}_f \) is not Pareto dominated by any other \( \nu \in \mathcal{M}_f \), then we say that \( \mu \) is Pareto efficient. Note that in our definition of efficiency, only the welfare of the students is considered. This is consistent with the school choice literature, in which school seats are viewed as objects to be consumed by the students (see Abdulkadiroğlu and Sönmez (2003)).

We say student \( i \) of type \( \theta \) claims an empty seat at school \( s \) if (i) \( s \in P_i \mu(i) \), (ii) \(|\mu(s)| < Q_s\) and \(|\mu_\theta(s)| < U_{s,\theta}\), and (iii) \(|\mu_\theta(\mu(i))| > L_{\mu(i),\theta}\). If no student claims an empty seat under matching \( \mu \), then \( \mu \) is non-wasteful. In words, non-wastefulness means that whenever a student prefers a school \( s \) to her current assignment, it is impossible to move her to \( s \) without violating feasibility.

A second property is elimination of justified envy, a fairness requirement commonly used in school choice settings.\(^{20}\) Given a matching \( \mu \), we say that student \( i \in \mu(s) \) justifiably envies student \( i' \in \mu(s') \) if (i) \( s' \in P_i \mu(i) \), (ii) \( i \succeq i' \), and (iii) there exists an alternative matching \( \nu \in \mathcal{M}_f \) such that \( \nu(i) = s' \), \( \nu(i') \neq s' \), and \( \nu(j) = \mu(j) \) for all \( j \neq i, i' \). If no student justifiably envies any other, then the matching eliminates justified envy. In words, student \( i \) justifiably envies \( i' \) if she prefers the school of student \( i' \), has higher priority than \( i' \) at this school, and \( i \) and \( i' \) can be reassigned without violating any constraints.\(^{21}\)

---

\(^{18}\)Formally, necessary conditions for non-emptiness are (i) \( \sum_{\theta \in \Theta} L_{s,\theta} \leq Q_s \), (ii) \( \sum_{s \in S} L_{s,\theta} \leq |I_\theta| \leq \sum_{s \in S} U_{s,\theta} \), and (iii) \(|I| \leq \sum_{s \in S} Q_s \).

\(^{19}\)We use \( R_i \) to denote the weak preference relation corresponding to \( P_i \), i.e., \( s R_i s' \) if and only if \( s P_i s' \) or \( s = s' \).

\(^{20}\)For example, this was an important criterion to administrators of the Boston school district when they were redesigning their school assignment mechanism. See Abdulkadiroğlu et al. (2005b).

\(^{21}\)In two-sided matching models without distributional constraints, non-wastefulness and elimination of justified envy are often combined into one definition called stability, which is usually then given a positive interpretation. We must separate the two definitions due to impossibility results caused by the introduction of the floors (discussed
The above properties have counterparts for mechanisms. Mechanism \( \psi \) is non-wasteful if \( \psi(P_I) \) is a non-wasteful matching for all \( P_I \in \mathcal{P}^n \), and \( \psi \) eliminates justified envy if \( \psi(P_I) \) is a matching that eliminates justified envy for all \( P_I \in \mathcal{P}^n \). We say that mechanism \( \psi \) Pareto dominates mechanism \( \varphi \) if

\[
\text{for all } P_I : \psi_i(P_I) R_i \varphi_i(P_I) \text{ for all } i \in I
\]

\[
\text{for some } P_I : \psi_i(P_I) P_i \varphi_i(P_I) \text{ for some } i \in I.
\]

The last important property relates to incentives for students to report truthfully in a mechanism. Mechanism \( \psi \) is strategyproof if \( \psi_i(P_I) R_i \psi_i(P_0^i, P_i) \) for all \( i \in I, P_I \in \mathcal{P}^n, \) and \( P_0^i \in \mathcal{P} \).

In words, a mechanism is strategyproof if no student can ever gain by misreporting her preferences, no matter what the other students report. Strategyproofness is viewed as an important property for many reasons.

- First, strategyproof mechanisms advance the so-called Wilson Doctrine (Wilson (1987)), which argues that to be successful, market designs should not be sensitive to specific assumptions on agent beliefs (see also Bergemann and Morris (2005)). Strategyproof mechanisms satisfy the Wilson Doctrine in its strongest sense, since truthful reporting is optimal for any beliefs agents may have.

- Second, from a practical perspective, school districts are interested in strategyproof mechanisms because they are strategically simple for parents to play. The school district can inform the parents that all they must do is submit their true preferences, and unsophisticated parents who are unable or do not have the resources to strategize effectively will not be disadvantaged. From this viewpoint, strategyproofness is a type of strategic fairness or way to “level the playing field”.\(^{22}\)

- Third, school districts often lack hard data on what makes schools desirable to parents. School choice mechanisms produce rich data on parent preferences, and, if the mechanism is strategyproof, it is reasonable to assume the data is truthful. This data can then be analyzed and used to improve schools in accordance with parent preferences.\(^{23}\)


\(^{23}\)See, for example, Elissa Gootman “Lafayette Among 5 High Schools to Close,” *New York Times*, December 14, 2006, which reports that demand data from the NYC assignment algorithm was an important reason for the closing of South Shore High School.
For these reasons, many cities have opted for school choice mechanisms that are strategyproof (among them, New York City, Boston, and New Orleans). Strategyproofness has been an important design consideration in other settings as well, such as hospital-resident matching (Roth (1991); Roth and Peranson (1999)).

3.3 Deferred acceptance (DA)

Consider the following algorithm, which is a generalization of the deferred acceptance algorithm proposed by Abdulkadiroğlu and Sönmez (2003).

Deferred acceptance (DA)

Step 1 Each student applies to the first school on her preference list. Each school $s$ considers all students who have applied to it, and tentatively accepts students as follows:

(i) Type-specific seats: for each type $\theta$, school $s$ accepts the $L_{s,\theta}$ highest-ranked type $\theta$ students according to $\succ_s$.

(ii) Open seats: for any students remaining in the applicant pool, school $s$ admits students one-by-one from the top of its priority order, unless either some type-specific ceiling $U_{s,\theta}$ would be violated or $Q_s - \sum_{\theta \in \Theta} L_{s,\theta}$ open seats have already been filled. All students not accepted are rejected.

Step $k$ Each student who was rejected in step $k - 1$ applies to her most preferred school that has not yet rejected her. Each school $s$ considers its new applicants in step $k$ jointly with the students tentatively admitted from step $k - 1$, and again tentatively accepts students in its applicant pool in the same manner as above. All students not accepted are rejected.

The key feature of DA is that at each step, the acceptances are only tentative. This means that if a student is rejected from her favorite school $s_1$ in round 1, when she applies to her second-choice school $s_2$ in round 2, she still has an opportunity to be admitted to $s_2$ if she has higher priority than one of $s_2$’s previously accepted students (that is, she does not lose out to the students accepted in round 1, because those acceptances were “deferred”).

In the version of the DA algorithm defined above, each school reserves $L_{s,\theta}$ seats exclusively for students of type $\theta$; the remaining $Q_s - \sum_{\theta \in \Theta} L_{s,\theta}$ seats are open seats, that can go to students of any type, subject to the ceiling constraints $U_{s,\theta}$. When the floors are set to 0 at all schools, the above algorithm is equivalent to that defined by Abdulkadiroğlu and Sönmez (2003). In these environments, DA is non-wasteful, eliminates justified envy, and is strategyproof, three properties
which have made it a successful mechanism in the field.\textsuperscript{24} Unfortunately, with floor constraints, DA may produce a matching that is not feasible. This is shown in the following example.

**Example 2.** There are 4 schools $S = \{s_1, s_2, s_3, s_4\}$ and three students $I = \{\ell_1, h_1, h_2\}$. There are two types $\Theta = \{\ell, h\}$, and student $\ell_1$ is of type $\ell$ (“low SES”) while students $h_1$ and $h_2$ are of type $h$ (“high SES”). The floors, ceilings, and capacities of the schools are given in the following table, as are the student preferences.

<table>
<thead>
<tr>
<th>Schools</th>
<th>Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{s,\ell}$</td>
<td>$L_{s,h}$</td>
</tr>
<tr>
<td>$s_1$</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
</tr>
<tr>
<td>$s_4$</td>
<td>0</td>
</tr>
</tbody>
</table>

Consider running DA on this market. The output is the matching $\mu$ defined by

$$
\mu = \begin{pmatrix}
  s_1 & s_2 & s_3 & s_4 \\
  h_1 & h_2 & \ell_1 & \emptyset
\end{pmatrix}.
$$

This matching is not feasible, however, since the floor constraint at school $s_4$, which must be assigned one type $h$ student, is not satisfied. The key issue is the following: even though school $s_4$ is “reserving” its seat for a type $h$ student, no type $h$ student applies to $s_4$ in the running of the algorithm, and so its floor remains unfilled.

### 3.4 Artificial caps

Because of the infeasibility of DA, many policymakers opt for the intuitive solution of imposing artificially lower ceilings and capacities, and then running DA. Imposing sufficiently strict artificial caps will guarantee that all schools receive enough applications to fill all floors. This artificial caps DA (ACDA) mechanism likely owes its popularity to the fact that it is feasible, simple to implement, and that it inherits several of the good properties of DA (see Theorem 2).\textsuperscript{25}

To define ACDA formally, let $(\bar{U}, \bar{Q})$ be a collection of artificial caps. Artificial caps are simply an alternative collection of type-specific ceilings $\bar{U} = (\bar{U}_{s,\theta})_{s \in S, \theta \in \Theta}$ and capacities $\bar{Q} = (\bar{Q}_s)_{s \in S}$, which need not equal the true ceilings and capacities $(U, Q)$. Note that the artificial caps do not contain any floors.

\textsuperscript{24}Strategyproofness of DA was first shown by Dubins and Freedman (1981) and Roth (1982) for one-to-one matching models. It has since been generalized to many-to-one matching in various ways (see, for example, Hatfield and Milgrom (2005) and Abdulkadiroğlu (2005)).

\textsuperscript{25}It is also possible to impose artificial caps and use other mechanisms besides deferred acceptance, such as the serial dictatorship or top trading cycles (Shapley and Scarf (1974)). We discuss this in detail Section 7.
Let \( \mathcal{M}(\bar{U}, \bar{Q}) = \{ \mu \in \mathcal{M} : |\mu_{\theta}(s)| \leq \bar{U}_{s,\theta} \text{ and } |\mu(s)| \leq \bar{Q}_s \text{ for all } (s, \theta) \} \). In words, the set \( \mathcal{M}(\bar{U}, \bar{Q}) \) is the set of matchings that respect the artificial caps \((\bar{U}, \bar{Q})\). Note that in general, the matchings in the set \( \mathcal{M}(\bar{U}, \bar{Q}) \) need not be feasible, as some floors may still be violated.

**Definition 1.** Artificial caps \((\bar{U}, \bar{Q})\) ensure a feasible match if \( \mathcal{M}(\bar{U}, \bar{Q}) \subseteq \mathcal{M}_f \).

In words, ensuring a feasible match simply means that any matching that satisfies the artificial ceilings and capacities also implicitly satisfies the (true) ceilings, capacities, and floors.

The first natural question to ask is whether such a feasibility ensuring \((\bar{U}, \bar{Q})\) exists. That this is true is shown in the theorem below.

**Theorem 1.** The set of vectors \((\bar{U}, \bar{Q})\) that ensure a feasible match is nonempty.

The proof chooses some feasible \( \mu \) and sets \( \bar{U}_{s,\theta} = |\mu_{\theta}(s)| \) and \( \bar{Q}_s = |\mu(s)| \) for all \((s, \theta)\), which corresponds to predetermining exactly the number of students of each type \( \theta \) who will be assigned to each school before students even submit their preferences. While this is one choice that will always work, there will in general be many choices of \((\bar{U}, \bar{Q})\) that ensure a feasible match.

We then formally define the artificial caps deferred acceptance algorithm (ACDA) as the deferred acceptance algorithm using some artificial caps \((\bar{U}, \bar{Q})\) that ensure a feasible matching.

**Properties of ACDA**

Recall our three desiderata: non-wastefulness, elimination of justified envy, and strategyproofness. While DA satisfies all three simultaneously without floors, in the presence of floors, an impossibility result obtains: matchings that eliminate justified envy may not even exist (Ehlers et al. (2012)). This is perhaps not surprising, since school districts often use floors to give an advantage to certain groups of students who would not be able to get in to a school based on their priority alone. At the same time, this observation leads to a natural alternative fairness criterion: we say a matching/mechanism eliminates justified envy among same types if no student justifiably envies another student of her same type. This seems to be a reasonable criterion, because any remaining priority violations are in some sense caused by the diversity constraints, which the school district finds inherently valuable. In addition, note that if all students are of the same type, elimination of justified envy is equivalent to elimination of justified envy among same types.

We then have the following theorem.

**Theorem 2.** ACDA eliminates justified envy among same types and is strategyproof. However, ACDA may be wasteful.

The strategyproofness and envy-freeness of ACDA are inherited from the fact that DA is strategyproof and eliminates justified envy among same types.
Example 2, continued. We now give an example of ACDA using the market from Example 2. Consider the following artificial caps:

\[
(\bar{U}, \bar{Q}) = \begin{pmatrix}
1 & 0 & \bar{1} \\
1 & 0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
\end{pmatrix}.
\]

Each row corresponds to a school, and the first column of \(\bar{U}\) gives the ceilings for \(\ell\) type students, while the second column gives the ceilings for type \(h\) students. It is indeed true that any matching that satisfies these artificial caps will be feasible. For the preferences \(P = (P_{\ell 1}, P_{h 1}, P_{h 2})\) given above, the outcome of ACDA is

\[
\mu = \begin{pmatrix}
s_1 & s_2 & s_3 & s_4 \\
\ell_1 & \emptyset & h_1 & h_2 \\
\end{pmatrix}.
\]

In contrast to standard DA under the true ceilings and capacities, the outcome of ACDA is feasible. However, consider the alternative preferences

\[
P'_{\ell 1} : s_3, s_2, s_1, s_4 \\
P'_{h 1} : s_1, s_2, s_3, s_4 \\
P'_{h 2} : s_4, s_1, s_3, s_2
\]

Under \(P' = (P'_{\ell 1}, P'_{h 1}, P'_{h 2})\), ACDA produces

\[
\mu' = \begin{pmatrix}
s_1 & s_2 & s_3 & s_4 \\
\emptyset & \ell_1 & h_1 & h_2 \\
\end{pmatrix}.
\]

This is Pareto dominated by the feasible matching \(\mu''\) where every student is assigned to her first choice:

\[
\mu'' = \begin{pmatrix}
s_1 & s_2 & s_3 & s_4 \\
h_1 & \emptyset & \ell_1 & h_2 \\
\end{pmatrix}.
\]

Note that \(\mu''\) would have resulted if we had run DA under the true \((U, Q)\). This suggests the following mechanism as an improvement over ACDA. Start by running DA under \((U, Q)\). If the preferences are such that the resulting match is feasible (e.g., \(P'\)), the algorithm finishes. If not (e.g., \(P\)), then lower the ceilings and/or capacities slightly and run DA again. Continue this procedure until a feasible matching is reached. Effectively, this mechanism uses the submitted preferences of the agents (e.g., \(P\) or \(P'\)) before deciding whether or not to impose ceilings and capacities that are lower than the true ceilings and capacities; this results in all students being made (weakly) better off. We formalize this idea in the next section.
4 Dynamic quotas deferred acceptance (DQDA)

In this section, we define a new dynamic quotas deferred acceptance (DQDA) mechanism and provide an example to illustrate how it works.

4.1 Definition of DQDA

To describe the algorithm, fix a sequence \( \eta = \{(U^1, Q^1), (U^2, Q^2), \ldots, (U^K, Q^K)\} \) of ceiling-capacity vectors such that \((U^{k+1}, Q^{k+1}) \leq (U^k, Q^k) \leq (U, Q)\) for all \( k \), and \((U^K, Q^K)\) ensures a feasible matching. We call \( \eta \) a reduction sequence. In words, \( \eta \) is simply a monotonically decreasing sequence of ceilings and capacities for all schools such that the last entry in the sequence ensures a feasible matching. Let \( DA^{(U^1, Q^1)} : \mathcal{P}^n \to \mathcal{M} \) denote the DA mechanism (as defined Section 3) using ceilings and capacities \((U', Q')\). Then, consider the following algorithm.

**Dynamic quotas deferred acceptance (DQDA)**

**Stage 1** Starting with \((U^1, Q^1)\), compute \( DA^{(U^1, Q^1)}(P_I) \), the outcome of DA under \((U^1, Q^1)\). If \( DA^{(U^1, Q^1)}(P_I) \) is a feasible matching, end the algorithm and output this matching. If not, proceed to stage 2.

In general,

**Stage k** Lower the ceilings and capacities to \((U^k, Q^k)\) and compute \( DA^{(U^k, Q^k)}(P_I) \), the outcome of DA under \((U^k, Q^k)\). If \( DA^{(U^k, Q^k)}(P_I) \) is a feasible matching, end the algorithm and output this matching. If not, proceed to stage \( k + 1 \).

The basic idea behind DQDA is to start with high ceilings and capacities, and check whether given the submitted preferences, the output of DA satisfies the floors as well. If so, the algorithm ends with the high ceilings. If not, then we proceed to the next stage, and run DA under lower ceilings. We continue gradually lowering the ceilings in this manner until we reach an assignment that satisfies all feasibility constraints. The algorithm is guaranteed to produce a feasible matching, since we have chosen \( \eta \) such that the final entry \((U^K, Q^K)\) ensures a feasible match. For some extreme preference profiles, the algorithm may run all the way to stage \( K \), in which case DQDA is equivalent to ACDA using caps \((U^K, Q^K)\). However, for many others, it may end earlier, thereby wasting less seats than imposing artificial caps of \((U^K, Q^K)\), and making the students better off. The key is that, unlike ACDA, which lowers the ceilings without regard for student preferences, DQDA only does so after taking the submitted preferences of the students into account, which results in fewer seats being eliminated unnecessarily.
4.2 How should \( \eta \) be constructed?

The DQDA algorithm as defined takes the reduction sequence \( \eta \) as an input, and different choices of \( \eta \) will lead to different DQDA mechanisms.\(^{26}\) While the exact choice of \( \eta \) will depend somewhat on the preferences of policymakers, we can give some guidelines for how \( \eta \) should be constructed. In particular, in order to achieve good incentive and efficiency properties, \( \eta \) should satisfy the following for all \( k \):

- For one \((s, \theta)\): \( U_{s, \theta}^{k+1} = U_{s, \theta}^k - 1 \) and \( Q_s^{k+1} = Q_s^k - 1 \)
- For all \((s', \theta') \neq (s, \theta)\): \( U_{s', \theta'}^{k+1} = U_{s', \theta'}^k \)
- For all \( s'' \neq s \): \( Q_{s''}^{k+1} = Q_{s''}^k \)

In words, this says that in moving from stage \( k \) to \( k + 1 \), we choose a school-type pair \((s, \theta)\) and lower the type \( \theta \) ceiling at \( s \) and the capacity of \( s \) by exactly one seat; the ceilings and capacities of the remaining schools are unchanged. We call any such \( \eta \) a single-seat reduction sequence. This construction is needed for both incentive and efficiency reasons (we discuss this in more detail after the formal characterization of DQDA in the next section).

Even with this restriction, there are still many possible choices for \( \eta \). As we show below, any choice will Pareto dominate ACDA, and in order to retain strategyproofness, all that is needed is that \( \eta \) be fixed ex-ante, i.e., the submitted student preferences cannot affect the order in which ceilings are reduced. Within these constraints, policymakers may actively choose the ceiling that is to be reduced at each stage in order to achieve some desirable policy goals; alternatively, at each stage \( k \) we can randomly choose some pair \((s, \theta)\), subject to feasibility constraints.\(^{27}\)

4.3 DQDA example

Before discussing the appealing properties of DQDA, we provide an example of how the algorithm works.

**Example 3.** Let \( S = \{s_1, s_2, s_3, s_4\} \), \( \Theta = \{\ell, h\} \), and \( I = \{\ell_1, h_1, h_2\} \), as in Example 2. Consider the school quotas/priorities and student preferences given in the following table. All floors are zero except for the type \( h \) floor at \( s_4 \).

---

\(^{26}\)That is, \( \eta \) is part of the definition of the mechanism, rather than a primitive of the model. As an analogy, consider the serial dictatorship mechanism, which first fixes some ordering of the students, and then allows the students to pick their favorite schools according to this ordering. The formal definition of the serial dictatorship takes the fixed student ordering as an input, similar to how DQDA takes \( \eta \) as an input, and different orderings lead to different serial dictatorship mechanisms.

\(^{27}\)Continuing with the analogy from footnote 26, the random serial dictatorship chooses a student ordering randomly and implements the serial dictatorship using this ordering. Similarly, in our case, we can randomly choose some \( \eta \) and implement DQDA.
The last object needed is a reduction sequence. Consider the following:

\[
\]

In words, this reduction sequence starts with \((U^1, Q^1) = (U, Q)\). At the beginning of stage 2, we lower the capacity and type \(h\) ceiling at \(s_1\) by 1, while at the beginning of stage 3, we lower the capacity and type \(h\) ceiling at \(s_2\) by 1. Note that \(\eta\) is a single-seat reduction sequence, and that the final entry, \((U^3, Q^3)\), does indeed ensure a feasible matching.

The output of stage \(k = 1\) is

\[
\mu^1 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ h_1 & h_2 & \emptyset \end{pmatrix}.
\]

Note that at the end of stage 1, the type \(h\) floor at \(s_4\) is not satisfied, and so we must move to stage 2. At the end of stage 2, the output is

\[
\mu^2 = \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ \emptyset & h_1 & h_2 & h_1 \end{pmatrix}.
\]

Matching \(\mu^2\) does satisfy all of the primitive floors and ceilings, and thus there is no need to move to stage 3. The final output is \(\mu^2\), which Pareto dominates what would have occurred under ACDA using \((\bar{U}, \bar{Q}) = (U^3, Q^3)\).

5 Analysis of DQDA: Single type case

This section and the next contain our main theoretical results. There are important distinctions between the single type case and the multiple type case. In addition, the intuition is easier to grasp when all students are of the same type. Given this, and the fact that the single type model is particularly relevant for markets which only care about the aggregate number of agents assigned
to each institution (e.g., military branching, the Japanese medical match, some school districts) we first focus specifically on this case. Section 6 discusses multiple types.

We therefore assume for this section only that $\Theta = \{\theta\}$. When there is only a single type, the distinction between the type-specific ceilings and capacities is irrelevant. To avoid redundant notation, we ignore the type-specific ceilings $U$ and write a reduction sequence as a sequence of only capacity vectors: $\eta = \{Q^1, Q^2, \ldots, Q^K\}$, where each $Q^k = (Q^k_s)_{s \in S}$ is a vector of school capacities. We still assume that $\eta$ is a single-seat reduction sequence, which in this case means that in moving from $k$ to $k + 1$, exactly one school reduces its capacity by exactly one seat, and the capacities of the remaining schools are unchanged.

5.1 Properties of DQDA

The following is the main result of this section.

**Theorem 3.** Let $\eta = \{Q^1, \ldots, Q^K\}$ be a single-seat reduction sequence. Then, the following hold:

1. DQDA Pareto dominates ACDA under $Q^K$.
2. DQDA eliminates justified envy.
3. DQDA is strategyproof.

**Discussion of Theorem 3**

The intuition behind parts (1) and (2) is that the final matching produced by DQDA is equivalent to DA under capacities $Q^k$ for some $k \leq K$. Since $Q^k_s \geq Q^K_s$ for all $s$, there are more seats available at all schools, which makes all students better off (part (1)). Further, since within-stage DA eliminates justified envy, the final DQDA matching eliminates justified envy as well (part (2)).

Parts (1) and (2) can be shown by analyzing DA within each stage $k$ separately. Analyzing the incentive properties of DQDA is more complicated, because they cannot be understood by looking at each stage independently. The submitted preferences themselves have the potential to change what stage is the final stage of the algorithm, and with it the capacities in place at the final matching. This feature may make it seem that agents have the potential to manipulate (for example, by submitting preferences such that the algorithm ends earlier, resulting in more total seats available).

To understand why DQDA is strategyproof, it is helpful to consider the types of potential manipulations a student $i$ can make. Let $k$ be the final stage of DQDA if $i$ submits her true preferences $P_i$. There are three classes of manipulations $P_i'$: contractions ($P_i'$ causes the algorithm to end in some $k' < k$), extensions ($P_i'$ causes the algorithm to end in some stage $k' > k$), and manipulations which do not change the final stage ($P_i'$ causes the algorithm to end in stage $k' = k$). The last category is the easiest to rule out: if both $P_i$ and $P_i'$ cause DQDA to end in stage $k$, then,
since DQDA is equivalent to standard DA in stage $k$, strategyproofness of DA implies that any such manipulation is not profitable. Next, consider extensions. If $P'_i$ causes the mechanism to continue beyond stage $k$, this means that there are less seats available for all students. This intuitively makes every student worse off, and so extensions will never be profitable.

The most difficult types of manipulations to rule out are contractions. It may seem at first that a contraction might be profitable for $i$ if continuing the algorithm would eliminate a seat that $i$ could have received by submitting false preferences and ending the algorithm early, under higher ceilings. Fortunately, this seemingly intuitive reasoning turns out to be incorrect. To understand why, consider a student $i$ with preferences $P_i: s_1, s_2, s_3, \ldots$. Let all students other than $i$ apply first, and assume after this is done, there is only one floor seat left to be filled, at school $s_2$. Now, the algorithm ends the next time any student applies to $s_2$. One option available to student $i$ is to lie and list school $s_2$ as her true first choice, thereby ending the algorithm immediately (a contraction) with her receiving $s_2$. If $i$ instead submits her true preferences and first applies to $s_1$, she may initiate a chain of rejections that ends with some other student applying to $s_2$, in which case $i$ receives $s_1$, her true first choice. The key, though, is even if $i$ is eventually rejected from $s_1$ (for example, if $\eta$ specifies that the capacity of $s_1$ is to be reduced), she will then simply apply to $s_2$ and the algorithm ends. The seat at $s_2$ will always be available until someone applies to it, at which point the algorithm ends and all assignments are made permanent. Thus, there is no harm in $i$ reporting her true top choices, because seats at lower-ranked schools are always available to her. By applying to her true top choices, she may cause a chain of rejections that lead to some other student filling these seats, leaving her with a more preferred assignment.

5.2 Optimality of DQDA

To summarize, Theorem 3 shows that DQDA outperforms ACDA on all dimensions: it satisfies the same incentive and fairness properties, while improving on ACDA with respect to efficiency in a Pareto sense. Thus, in our view, market designers would be better served by using a dynamic quotas mechanism. The next natural question is whether DQDA is itself an optimal mechanism. The following result says that this is indeed the case, if we also require strategyproofness.

**Theorem 4.** Let $\eta = \{Q^1, \ldots, Q^K\}$ be a single-seat reduction sequence that satisfies $Q^1 = Q$. If $\psi$ is a feasible mechanism that Pareto dominates DQDA under $\eta$, then $\psi$ is not strategyproof.

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28The order in which students are allowed to apply is irrelevant, a result first shown without distributional constraints by McVitie and Wilson (1971).

29In the appendix, we show that if $\eta$ is a single-seat reduction sequence, then DQDA is equivalent to the following algorithm: in stage $k$, rather than starting from the empty matching, start with the matching $DA^{Q^{k-1}}(P_f)$ from the end of the previous stage. Then, lower the capacities to $Q^k$. This causes exactly one student to be rejected, who then applies to her next most preferred school, which then rejects a student, and so forth. This rejection chain stops when no further student is rejected under the current capacities $Q^k$. Then lower the capacities to $Q^{k+1}$, etc., until a feasible matching is reached.
Another way to phrase Theorem 4 is to say that DQDA is on the Pareto frontier of strategyproof mechanisms. Thus, any market which imposes aggregate constraints on the total number of agents assigned to each institution (e.g., branch assignment in the Army, school districts that desire at least a certain number of students in each school for cost efficiency reasons) cannot improve on DQDA without sacrificing incentives. Given the importance of strategyproofness to many institutions, this theorem gives a sense in which DQDA is an optimal choice of mechanism.

6 Analysis of DQDA: Multiple type case

In this section, we return to the general model and discuss the properties of DQDA when students can be of multiple types. With multiple types, each school once again has type-specific ceilings and floors $U_{s,\theta}$ and $L_{s,\theta}$, and a distinct capacity $Q_s$. Reduction sequences are again written as $\eta = \{(U^1, Q^1), \ldots, (U^K, Q^K)\}$.

6.1 Properties of DQDA with multiple types

We start by providing the analogue of Theorem 3 with multiple types.

**Theorem 5.** Let $\eta = \{(U^1, Q^1), \ldots, (U^K, Q^K)\}$ be a single-seat reduction sequence. Then, the following hold:

1. DQDA Pareto dominates ACDA under $(U^K, Q^K)$.
2. DQDA eliminates justified envy among same types.
3. DQDA is strategyproof.

Theorem 5 is similar to Theorem 3, but the definition of a single-seat reduction sequence is more complex in the multiple type case. Also, in the single type case, DQDA eliminates all justified envy, but here, it only eliminates justified envy among same types.$^{30}$

Additionally, the intuition is more complicated in the multiple type case. For part (1), DQDA still ends under some ceilings and capacities $(U^K, Q^K) \geq (U^K, Q^K)$, and it seems that higher ceilings and capacities should make the students better off. However, the full argument is more complex with multiple types. Formally, we need for each $s$ that the set of students chosen from any given

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$^{30}$Recall that with multiple types, there is an impossibility result that says that matchings that eliminate justified envy (across types) may not even exist (see Section 3), and so the most we can hope to achieve is elimination of justified envy among same types. If, in addition, we require non-wastefulness, another impossibility result obtains: matchings that simultaneously eliminate justified envy among same types and are non-wasteful may also fail to exist (see Ehlers et al. (2012) and Fragiadakis et al. (2013)). Thus, one of these properties must be weakened. Markets that use ACDA are opting to keep elimination of justified envy among same types and weaken non-wastefulness. However, as Theorem 5 shows, ACDA weakens non-wastefulness more than necessary, as DQDA Pareto dominates ACDA while still satisfying elimination of justified envy among same types and strategyproofness. This same discussion carries over to the single type case, where elimination of justified envy among same types is equivalent to elimination of justified envy.
set of applicants is weakly smaller (in the set inclusion sense) in later stages compared to earlier stages. With only a single type, this follows immediately, because all students have equal access to all seats, and so less seats is always worse from the perspective of every student. With multiple types, this is not necessarily true. Consider two students, $i$ and $j$, such that $\tau(i) = \theta$ and $\tau(j) = \theta'$. Lowering the $(s, \theta)$ ceiling alone (without also lowering the capacity of school $s$) might be beneficial to $j$, because it might cause $s$ to reject $i$, which then opens up a seat for $j$. This would imply that $j$ is better off in later stages under lower ceilings and capacities, and hence DQDA might not Pareto dominate ACDA. The assumption that $\eta$ is a single-seat reduction sequence ensures that this does not happen.

A similar issue arises with strategyproofness: lowering the $(s, \theta)$ ceiling alone could make extensions profitable for $j$. This is because an extension may cause $i$ to be rejected, which gives $j$, who is of a different type than $i$, access to a seat at school $s$ that she did not have previously. The single-seat reduction sequence restriction ensures that even if $i$ is rejected, this seat cannot be reassigned to $j$ (or any student), and so any such manipulation is not profitable.

Remark 2. In the appendix, we define a more general model that uses the notion of a school choice function, which is an arbitrary function that assigns to each set of potential applicants the subset of students admitted by the school (the set-up is similar to Abdulkadiroğlu (2005) or Hatfield and Milgrom (2005)). We next define a generalized DQDA algorithm where $\eta$ is a sequence of choice functions, which allows our model to be applied to settings where schools may have more general preferences over sets of students. While DQDA can be defined quite generally, to ensure good properties such as strategyproofness, additional structure on the school choice functions is needed. First, we require that within each stage, every school’s choice function satisfies substitutability and the law of aggregate demand as defined by Hatfield and Milgrom (2005). Next, we must impose conditions on how choice functions are connected across stages. Monotonicity says that, fixing the set of applicants at a school, the set of rejected students expands as we move to later stages. This purpose of this condition is similar to substitutability: both guarantee that as the algorithm progresses, a school will never want to admit a student it has previously rejected. The last condition, minimality, guarantees that in moving from stage $k$ to stage $k + 1$, at most one student will be rejected. We then show that single-seat reduction sequences as defined here induce choice functions that satisfy these conditions, and use this to prove strategyproofness.

Recall that in the single type case, Theorem 4 shows that DQDA is on the Pareto frontier of strategyproof mechanisms. The proof of Theorem 4 does not extend to the multiple type case. Whether DQDA is on the Pareto frontier of strategyproof mechanisms with multiple types is an open question.\textsuperscript{31}

\textsuperscript{31}We discuss this in further detail in Appendix D. We show via an example that it is possible to find sequences $\eta$ such that DQDA is Pareto dominated by an alternative mechanism $\psi$ that is strategyproof. However, if we use
6.2 Endogenous-reduction DQDA (EDQDA)

Even if it turns out that DQDA is on the efficient frontier of strategyproof mechanisms with multiple types, strategyproofness itself is still a limitation; that is, it may be possible to improve efficiency under weaker incentive constraints. For example, papers such as Abdulkadiroğlu et al. (2011) and Troyan (2012) show that the (manipulable) Boston mechanism can result in efficiency gains over DA in some settings. However, these results rely on students playing complicated best-responses, and so it is unclear if these efficiency gains will actually be realized in equilibrium. On other hand, if it were possible to find a mechanism that increased efficiency while still making truthful reporting “almost” a dominant strategy, we could be confident that the efficiency gains would actually be realized. In this section, we provide such a mechanism.

To motivate the new mechanism, we first discuss two features of the reduction sequence \( \eta \) that were necessary for strategyproofness, but that intuitively limit the efficiency of DQDA. The first is that, whenever we lower a type-specific ceiling \( U_{s,\theta} \), we must also lower the capacity \( Q_s \) of school \( s \). This means that when we reject a type \( \theta \) student from \( s \), we cannot assign her seat to a different type of student, even if all of that type’s floors are already satisfied, which is wasteful. The second problematic feature from an efficiency standpoint is the fixed nature of \( \eta \). This is problematic because in moving from \( k \) to \( k+1 \), we may lower the quota of a type that has already filled all floors. For example, consider a market with two types, \( \theta_1 \) and \( \theta_2 \), and say that at the end of stage \( k \), all \( \theta_1 \) floors are satisfied, but some \( \theta_2 \) floors are still unfilled. Intuitively, it seems we should reduce a type \( \theta_2 \) ceiling, but this will not occur if \( \eta \) directs a type \( \theta_1 \) ceiling to be lowered in stage \( k+1 \).

Given these observations, we introduce a new mechanism called endogenous-reduction DQDA (EDQDA). It runs similarly to DQDA, with corrections to alleviate the inefficiencies of the previous paragraph. To define it, we use a slightly different definition of a reduction sequence.

Let \( \rho = \{(s^1, \theta^1), \ldots, (s^K, \theta^K)\} \) be a sequence of school-type pairs, where each \( (s^k, \theta^k) \in S \times \Theta \). \( \rho \) is a baseline order for reducing the ceilings, but, unlike for DQDA, we will skip over an entry if all floors for the corresponding type have already been met. In addition, we will only reduce the type-specific ceilings, and not the capacities. The same entry may appear multiple times in \( \rho \).

Endogenous-reduction DQDA (EDQDA)

Set \( U^1 = U \).

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32Note also that it is possible to describe any single-seat reduction sequence \( \eta \) from the definition of DQDA by listing the school-type pair whose capacity and ceiling are reduced at each step.
Stage 1 Starting with the empty matching, run DA under \((U^1, Q)\), and set \(\mu^1 = DA(U^1, Q)(P_I)\).

(a) If \(\mu^1\) is a feasible matching, end the algorithm and output this matching.

(b) Otherwise, let \(\Theta = \{\theta \in \Theta : |\mu^1_\theta(s)| < L_{s,\theta} \text{ for some } s \in S\}\) be the set of types for which at least one floor constraint is not yet satisfied, and let \(Y = \{(s, \theta) \in S \times \Theta : |\mu^1_\theta(s)| > L_{s,\theta}\}\) be the set schools that have an excess of these types of students. Let \((s, \theta)\) be the element of \(Y\) that occurs earliest in \(\rho\). Set \(U^2_{s,\theta} = U^1_{s,\theta} - 1\) and \(U^2_{s',\theta'} = U^1_{s',\theta'}\) for all other \((s', \theta')\), and delete the earliest occurrence of \((s, \theta)\) from \(\rho\). Proceed to stage 2.

In general,

Stage \(k\) Starting with the empty matching, run DA under \((U^k, Q)\) and set \(\mu^k = DA(U^k, Q)(P_I)\).

(a) If \(\mu^k\) is a feasible matching, end the algorithm and output this matching.

(b) Otherwise, let \(\Theta = \{\theta \in \Theta : |\mu^k_\theta(s)| < L_{s,\theta} \text{ for some } s \in S\}\) be the set of types for which at least one floor constraint is not yet satisfied, and let \(Y = \{(s, \theta) \in S \times \Theta : |\mu^k_\theta(s)| > L_{s,\theta}\}\) be the set schools that have an excess of these types of students. Let \((s, \theta)\) be the element of \(Y\) that occurs earliest in \(\rho\). Set \(U^{k+1}_{s,\theta} = U^k_{s,\theta} - 1\) and \(U^{k+1}_{s',\theta'} = U^k_{s',\theta'}\) for all other \((s', \theta')\), and delete the earliest occurrence of \((s, \theta)\) from \(\rho\). Proceed to stage \(k + 1\).

EDQDA functions similarly to DQDA, except instead of following the reduction sequence in order from start to finish, we find the earliest entry \((s^k, \theta^k)\) for which \(s^k\) has an excess of type \(\theta^k\) students and not all type \(\theta^k\) floors have been met. We then reduce the \(\theta^k\) ceiling at \(s^k\) by one and leave everything else fixed. We thus may skip entries if the types corresponding to those entries have already met all floor constraints, which does not occur in DQDA. In addition, we only lower the type-specific ceilings, and not the capacities, so if a type \(\theta\) ceiling is lowered in order to satisfy a floor elsewhere, that student’s seat is not “wasted”, and can be taken by a student of a different type. To ensure that EDQDA produces a feasible matching, \(\rho\) must be chosen in such a way that \((U^K, Q)\) ensures a feasible match, where \(U^K\) is defined as \(U^K_{s,\theta} = U_{s,\theta} - \sum_{k=1}^K 1\{ (s^k, \theta^k) = (s, \theta) \}\), where \(1\{\cdot\}\) is an indicator function that takes on a value of 1 if the \(k^{th}\) entry of \(\rho\) is \((s, \theta)\).

These modifications intuitively make EDQDA a more efficient mechanism. While it turns out that EDQDA will not be more efficient in a Pareto sense, the simulations performed below show that on average, the students will prefer EDQDA to both ACDA and DQDA. The cost of these welfare gains is that EDQDA is no longer strategyproof. However, EDQDA may still be a successful mechanism in practice, provided that the potential manipulations are not too easy to enact. To formalize this idea, the matching literature has developed several notions of approximate incentive compatibility. In the next section, we use one of these notions to argue that EDQDA satisfies good large market incentive properties. This, together with the simulation results that show that
EDQDA is on average preferred by the students, suggests it can be a successful mechanism in applications.

Remark 3. In addition, note that once again, the final matching produced by EDQDA is equivalent to DA under some ceilings and capacities \((U', Q')\), and so EDQDA will eliminate justified envy among same types (the argument is equivalent to the one used to prove Theorem 5, part (2)).

6.3 Large markets

In this section, we show formally that EDQDA has good large market incentive properties. There are many ways to formalize the notion of a large market limit. We choose to use the concept of \textit{strategyproofness in the large} (SPL) proposed by Azevedo and Budish (2013).\textsuperscript{33} We choose this particular formalization because whether a mechanism is SPL or not turns out to be a good predictor of whether it is a successful mechanism in practical applications. Azevedo and Budish (2013) show that non-SPL mechanisms (e.g., pay-as-bid auctions for Treasury bills, priority matching algorithms in hospital-residency markets) tend to perform poorly in the field and are eventually abandoned, while their SPL counterparts (uniform price auctions, DA) are successful and in continued use, even though they are only SPL, and not fully strategyproof.\textsuperscript{34} This suggests that SPL is a practically useful second-best to strategyproofness, and so weakening incentives to SPL while at the same time increasing efficiency may improve performance, at least in large enough markets.\textsuperscript{35}

To show that our mechanism is SPL, we must expand the formal model. We consider a sequence of markets, indexed by \(n \in \mathbb{N}\), where \(n\) is the number of agents, and \(I^n = \{i_1, \ldots, i_n\}\) is the set of agents in market \(n\). \(\Theta\) is a finite set of \textit{quota types}, and each student is of exactly one type in \(\Theta\). The set \(\Theta\) is fixed for all \(n\), but the number of students of each type \(\theta, |I^n_\theta|\), grows according to some fixed sequence. The set of schools \(S = \{s_1, \ldots, s_m\}\) is also fixed for all \(n\), but the capacity of the schools may increase with \(n\). Specifically, for each market \(n \in \mathbb{N}\), school \(s\) has a capacity of \(Q^n_s\), and type-specific floors and ceilings of \(L^n_{s, \theta}\) and \(U^n_{s, \theta}\). As in the original model, we collect

\textsuperscript{33}For other large market incentive compatibility notions that are closely related to SPL, see Immorlica and Mahdian (2005), Kojima and Pathak (2009), and Kojima et al. (2013), who study the large market properties of DA, or Cle and Kojima (2010) and Kojima and Manea (2010), who do the same for the probabilistic serial mechanism. One of the main advantages of SPL is that it is not tailored to a specific mechanism, and so can be applied more broadly.

\textsuperscript{34}While DA is strategyproof in one-sided markets such as school choice where the schools are not strategic players, it is not strategyproof in markets where both sides are strategic players (e.g., doctors and hospitals). Nevertheless, DA \textit{is} SPL in these markets (for both sides), and has indeed been a successful mechanism in real-world applications.

\textsuperscript{35}There is strong support for the idea that incentives play an important role in determining the success or failure of the mechanisms mentioned above. See, for example, Friedman (1991), who argues against pay-as-bid auctions and in favor of uniform price auctions on incentive grounds, and Roth (1991), who makes a similar argument for DA over priority match mechanisms in U.K. medical residency markets. Note also that the proposed improvements are only SPL, and not fully strategyproof.
these quotas into matrices $L^n, U^n$, and $Q^n$. We assume throughout that $(L^n, U^n, Q^n)$ is such that at least one feasible matching exists for every market $n$.

The definition of strategyproofness in the large that we use is a cardinal concept, and so we introduce a finite set of payoff (utility) types $T$. Corresponding to each $t_i \in T$ is a von Neumann-Morgenstern expected utility function $u_{t_i} : \Delta S \to [0, 1]$, where $\Delta S$ is the set of lotteries over schools. Preferences are private, in that an agent’s payoff depends only on her type payoff $t_i$ and outcome (lottery). Each utility type $t_i$ also has an associated ordinal preference relation over schools, which we denote $P_{t_i}$.\footnote{The ordinal preference relation can be obtained by comparing utility values over degenerate lotteries that place probability 1 on each school.}

A second difference from the original model is how we construct the school priorities. Each school has a finite number of priority classes $Z_s = \{1, \ldots, |Z_s|\}$. Each student $i$ is assigned to one priority class at each school. Each $s$ has a strict ranking of priority classes; without loss of generality, we assume that each school ranks $1 >_s 2 >_s \cdots >_s Z_s$. Each school ranks all students in a higher priority class above all students in a lower, and ties within a priority class will be broken using a random lottery (see below). For each $i, z_i \in Z = \times_{s=1}^m Z_s$ is an $m$-dimensional vector that denotes $i$’s priority class at each school. One practical interpretation of priority classes is that each class corresponds to a certain zone, with students living within a certain radius of a school receiving higher priority for their neighborhood school than those living farther away. Consistent with this interpretation, we will sometimes refer to $Z$ as a set of zones for concreteness and simplicity, but we emphasize that the priority classes can be based on other factors.\footnote{Formally, it is necessary to divide the students into a fixed number of priority classes to ensure semi-anonymity of our algorithms (in the sense of Kalai (2004)), which is needed for the large market results to hold. See Azevedo and Budish (2013), who also must impose similar conditions to show that standard DA is SPL.}

To summarize, let $\Lambda = \Theta \times Z \times T$. A student’s overall type is then an element $\lambda_i \in \Lambda$. Since $\Theta, Z$ and $T$ are all finite, $\Lambda$ is finite as well.

For each market $n$, define $n(\theta, z)$ as the number of students of quota-zone type $(\theta, z)$. We assume that $n(\theta, z) \to \infty$ for all $(\theta, z)$, so that the number of agents of each type $(\theta, z)$ grows large according to some fixed sequence.

**Definition 2.** A (direct) mechanism $\{n_n\}_{n \in \mathbb{N}, \Lambda}$ is a sequence of allocation functions $\psi^n : \Lambda^n \to \Delta(S^n)$ such that every allocation in the support of $\psi^n(\lambda)$ is feasible for all $n$ and all $\lambda \in \Lambda^n$.\footnote{Formally, it is necessary to divide the students into a fixed number of priority classes to ensure semi-anonymity of our algorithms (in the sense of Kalai (2004)), which is needed for the large market results to hold. See Azevedo and Budish (2013), who also must impose similar conditions to show that standard DA is SPL.}

Note that a mechanism as defined here produces a random allocation. For notational purposes, the inputs are vectors of types $\lambda \in \Lambda^n$, but each student $i$ is restricted to reporting $\theta_i$ and $z_i$ truthfully; the only private information is her payoff type $t_i$.

Given a distribution of payoff types $\pi \in \Delta T$, define for a student $i$ of quota-zone type $(\theta_i, z_i)$
the following quantity:

\[ \phi^n_{(\theta, z)}(t') = \sum_{\lambda \in \Lambda^n} \psi^\lambda_i(\lambda_i, \lambda_{-i}) \Pr(\lambda_{-i} | t_{-i} \sim \text{iid}(\pi)). \]

where \( \lambda'_i = (\theta_i, z_i, t'_i) \) and \( \Pr(\lambda_{-i} | t_{-i} \sim \text{iid}(\pi)) \) gives the probability that the realized type profile of the \( n-1 \) other agents is \( \lambda_{-i} = (\theta_{-i}, z_{-i}, t_{-i}) \), given that the payoff types are drawn iid from some distribution \( \pi \) (recall that \( \theta_{-i} \) and \( z_{-i} \) are fixed). In words, \( \phi^n_{(\theta, z)} \) gives the outcome an agent of type \( (\theta_i, z_i) \) receives when she reports her payoff type as \( t'_i \) and the payoff type reports of the other students are drawn according to \( \pi \).

We are now ready to formally define strategyproofness in the large. Given a finite set \( X \), let \( \Delta X \) denote the set of full-support probability distributions over \( X \).

**Definition 3.** (Azevedo and Budish (2013)) Mechanism \( \{ (\psi^n)_{n \in \mathbb{N}}, \Lambda \} \) is **strategyproof in the large (SPL)** if for any \( \varepsilon > 0 \) and any \( \pi \in \Delta T \), there exists an \( n_0 \) such that for all \( n \geq n_0 \), \( (\theta, z) \in \Theta \times Z \), and all \( t_i, t'_i \in T \):

\[ u_i(\phi^n_{(\theta, z)}(t_i, \pi)) \geq u_i(\phi^n_{(\theta, z)}(t'_i, \pi)) - \varepsilon. \]

Last, we must define the EDQDA mechanism in this setting. Given some collection of reduction sequences \( \{ \rho^n \}_{n \in \mathbb{N}} \) defined as above, the mechanism in market \( n \) proceeds as follows. First, draw a vector of lottery numbers \( \ell \in [0, 1]^n \) uniformly at random, where \( \ell_i \) denotes the lottery number of student \( i \). Then, create a strict priority relation for each school \( s_i \), as follows:

\[ i \succ^n_s j \iff z_{i,s} > z_{j,s} \text{ or } [z_{i,s} = z_{j,s} \text{ and } \ell_i > \ell_j], \]

where \( z_{i,s} \) is student \( i \)'s priority class (zone) at \( s \). Let \( \mu^n(\lambda, \ell) \) be the matching produced by the EDQDA algorithm (as defined the previous section) using type-specific floors \( L^n \), type-specific ceilings \( U^n \), capacities \( Q^n \), priorities \( \succ^n_s \), reduction sequence \( \rho^n \), and (ordinal) preferences \( (P_{ti})_{i \in I^n} \). Then, define \( E^n \) as:

\[ E^n(\lambda) = \int_{\ell \in [0,1]^n} \mu^n(\lambda, \ell) d\ell. \]

The main theorem of this section is the following.

**Theorem 6.** The mechanism \( \{ (E^n)_{n \in \mathbb{N}}, \Lambda \} \) is strategyproof in the large.

Intuitively, in a large enough market, it is unlikely that student \( i \)'s report will have an effect on the final stage of the algorithm, and thus it is unlikely that she will be able to profitably manipulate. At a formal level, we prove the result by showing that EDQDA satisfies an envy-freeness condition identified by Azevedo and Budish (2013) as sufficient for strategyproofness in the large.
6.4 Simulations

We have thus far argued from a theoretical perspective that our new mechanisms should lead to improved performance of matching markets with floor constraints, as they increase efficiency while still satisfying good incentive and fairness properties. However, the theoretical results do not say anything about the number of students who are made better off by the use of our mechanisms. To answer this question, we turn to computer simulations.

The purpose of these simulations is two-fold: first, to approximate an actual school choice market and obtain a sense of the magnitude of the potential gains from our mechanisms; second, to conduct comparative statics to better understand in what types of markets our mechanisms will have the biggest impact. With these dual goals in mind, we use the details of kindergarten assignment in Cambridge, MA (for which limited data is publicly available) as an “anchor” to set the number of students, schools, and student types, but also structure the simulations with enough flexibility to allow us to conduct comparative statics with respect to preference correlation, quotas, and capacities.\footnote{38}

Simulation parameters

There are \( n = 750 \) students, \( m = 12 \) schools, and two possible types \( \Theta = \{\ell, h\} \). There are 250 students of type \( \ell \) (“low SES”) and 500 students of type \( h \) (“high SES”).

Student preferences are determined as follows. Student \( i \)’s utility for school \( s \) is \( u_i(s) = \alpha v^c(s) + (1 - \alpha) v^p_i(s) \), where \( v^c(s) \) is a common utility component that is the same across students, and \( v^p_i(s) \) is \( i \)’s private, idiosyncratic utility for school \( s \). The common component \( v^c(s) \) and all private components \( v^p_i(s) \) are drawn iid uniformly from \([0,1]\).\footnote{39} Ordinal preferences are then created from these cardinal preferences. By varying \( \alpha \), we can study how mechanism performance varies as a function of preference correlation. A value of \( \alpha = 0 \) corresponds to uncorrelated preferences, while \( \alpha = 1 \) is completely common preferences. To get a sense of the degree of preference correlation in a real market, we can look at the Cambridge data, which lists for each school the number of students who rank it as their first choice. The value of \( \alpha \) corresponding to the data is \( \alpha = 0.13 \).\footnote{40}

School priority vectors are drawn uniformly at random, independently across schools. For the school floors, ceilings, and capacities, we consider two cases: low flexibility and high flexibility.\footnote{38}The Cambridge data can be accessed at http://www3.cpsd.us/department/frc/FRC.\footnote{39}This method of drawing preferences is common in the matching literature; see, for example, Hafalir et al. (2013) and Míralles (2009). Using other distributions (e.g., normal) leads to similar results.\footnote{40}This value was obtained by finding the \( \alpha \) that minimized the distance between 1000 simulated distributions of first choices and the empirical distribution from the Cambridge data. While indicative of student preferences, care should be taken in interpreting this number, because Cambridge uses a non-strategyproof mechanism (immediate acceptance), and so it is unknown if the reported preferences are truthful (which is one of the common arguments made in support of strategyproof mechanisms). However, because our proposed mechanisms are strategyproof, by varying \( \alpha \) we are able to get a sense of how our mechanisms would perform if implemented, even if the true \( \alpha \) were different from that implied by the current data.
Table 1 gives the type-specific floors, ceilings, and capacities for the two cases. We treat all schools symmetrically, and so the numbers in the table correspond to the floors, ceilings, and capacities for each school. In the high flexibility case, the floors are lower and the ceilings are higher (compared to the low flexibility case), meaning there is a wider range of possible final assignments in the high flexibility case. Given the primitive floors, ceilings, and capacities \((L, U, Q)\), the artificial capacities \((\bar{U}, \bar{Q})\) are chosen as the highest symmetric values that ensure a feasible matching.

<table>
<thead>
<tr>
<th></th>
<th>Floors</th>
<th>Ceilings</th>
<th>Capacities</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Low flexibility</strong></td>
<td>Primitive</td>
<td>(14,36)</td>
<td>(39,69)</td>
</tr>
<tr>
<td></td>
<td>Artificial</td>
<td>-</td>
<td>(21,42)</td>
</tr>
<tr>
<td><strong>High flexibility</strong></td>
<td>Primitive</td>
<td>(14,14)</td>
<td>(76,76)</td>
</tr>
<tr>
<td></td>
<td>Artificial</td>
<td>-</td>
<td>(21,44)</td>
</tr>
</tbody>
</table>

Table 1: The floors, ceilings, and capacities at each school for the low and high flexibility cases. For entries \((x, y)\), \(x\) corresponds to the low type \(\ell\) and \(y\) corresponds to the high type \(h\).

We last must discuss how we construct \(\eta\) (for DQDA) and \(\rho\) (for EDQDA). As noted before, there are many possible ways to do this. Since it is not obvious ex-ante which should be chosen, at each stage we randomly choose which quota to reduce, subject to the constraint that all \((s, \theta)\) must be reduced once before any is reduced for a second time. For \(\eta\), we also reduce the capacity of school \(s\) and each stage, while for \(\rho\), we do not. For more details on the specification of \(\eta\) and \(\rho\), see the appendix.

**Simulation results**

For each set of parameters we consider, we run 150 iterations. To compare our mechanisms, the outcome metric we use is the average rank distribution over these 150 iterations. The rank distribution plots, for each \(x\), the number of students who receive their \(x^{th}\) or better choice. It is one of the common metrics publicly reported by school districts when evaluating their school choice mechanisms (for example, on its website, the Cambridge school district says that “85% of students receive one of their top 3 choices,” which is a simplified reporting of the rank distribution).\(^41\)

In Figure 4, we plot the rank distributions for our mechanisms under different choices of parameters. To read the figures, take the top left panel as an example: it says that under ACDA, about 620 students on average get their first choice, about 725 get their first or second choice, etc., while under EDQDA, about 700 students get their first choice, 745 get their first or second choice, etc. For clarity, we only plot the beginning of the rank distributions, because at higher ranks \(x\), essentially all students are getting their \(x^{th}\) or better choice under all mechanisms.

\(^{41}\)Motivated by the fact that authorities are often concerned with rank distributions, Featherstone (2011) investigates mechanisms that optimize the rank distribution in the context of object allocation without priorities.
Figure 4: The number of students assigned to their $x^{th}$ or better choice for various parameter values, averaged over 150 iterations. Higher plots correspond to better mechanisms (on average) for the students. For clarity, we only plot the beginning of the distributions, but the dominance relation $EDQDA > DQDA > ACDA$ holds for all values of $x$. 
In the figure, we plot the rank distributions for six parameterizations: three values of $\alpha (\alpha = 0, 0.13, 0.26)$ and two levels of flexibility. In the appendix, we report different results for other parameter values, but the three main takeaways can be obtained from Figure 4.

1. Comparing ACDA, DQDA, and EDQDA: In every case, the average rank distribution for ACDA is stochastically dominated by DQDA, which in turn is itself stochastically dominated by EDQDA. This confirms the Pareto dominance of DQDA over ACDA (Theorem 5). While there is no analogous Pareto dominance of EDQDA over DQDA, the simulations show that it is the case that the students will on average be better off under EDQDA compared to both DQDA and ACDA. The results suggest that there are significant gains to be had from our mechanisms, with on average over 20% more students receiving their first choice under EDQDA compared to ACDA for some parameterizations (see the appendix).

2. Comparative statics with respect to preference correlation: Within each column, the flexibility is fixed, but the preference correlation increases as we move from the top row to the bottom. As can be seen in the figure, the gains from our mechanism are larger when the preference correlation is smaller. Intuitively, this is because when the preference correlation is small, it is more likely that all floors will be filled in the early stages of the dynamic quotas mechanisms. As the preference correlation becomes larger, this becomes less likely, and dynamic quotas is more likely to end in later stages, making it closer to artificial caps. However, even at high correlations, there are still gains to be had from our mechanisms (additional results in the appendix).

3. Comparative statics with respect to flexibility: Within each row, the preference correlation $\alpha$ is fixed, but the flexibility increases as we move from the left column to the right. As can be seen in the figure, the gains from our mechanism are larger when there is more flexibility. Intuitively, when there is less flexibility (i.e., the floors and ceilings are “close”), dynamic quotas becomes more similar to artificial caps. In the extreme case when all ceilings are equal to all floors and there is no flexibility, DQDA is equivalent to ACDA (and to EDQDA). As the flexibility increases, there is more room for our dynamic quotas mechanisms to respond to the submitted preferences of the agents, and hence the gains obtained from using a dynamic quotas mechanism increase.

In summary, while our mechanisms will produce efficiency gains for any parametrization, the potential gains are increasing in flexibility and decreasing in preference correlation. However, we still recommend the use of DQDA or EDQDA even in markets with low flexibility or high preference correlation, because they will make the students better off on average, without sacrificing fairness or incentive properties.
7 Extensions

7.1 General mechanisms with dynamic quotas

For most of the paper, we have focused specifically on deferred acceptance. However, the main ideas behind artificial caps and dynamic quotas can in fact be applied much more broadly to any mechanism that has inputs for ceiling constraints but not for floors, including immediate acceptance, the serial dictatorship, and top trading cycles.

Formally, we define a class of mechanisms called upper quota mechanisms. Upper quota mechanisms are indexed by vectors of ceilings and capacities \((U', Q')\), which we refer to jointly as "upper quotas". Recall that \(\mathcal{M}(U', Q') = \{\mu \in \mathcal{M} : |\mu_\theta(s)| \leq U'_{s, \theta} \text{ and } |\mu(s)| \leq Q'_s \text{ for all } (s, \theta)\}\) is the set of all matchings that respect \((U', Q')\). An upper quota mechanism is a function \(\psi(U', Q') : P^n \rightarrow \mathcal{M}\) such that \(\psi(U', Q')(P_I) \in \mathcal{M}(U', Q')\) for all \(P_I \in P^n\). Note that upper quota mechanisms always satisfy the given ceilings and capacities \((U', Q')\), but need not satisfy any floors.

A collection of mechanisms, one for each \((U', Q')\), is denoted \(\Psi := \{\psi(U', Q')\}_{(U', Q')}.\) We refer to \(\Psi\) as a class of upper quota mechanisms. As an example, \(\Psi\) could be the class of DA mechanisms as defined in Section 3: \(\Psi = \{DA(U', Q')\}_{(U', Q')}\).

**Definition 4.** Let \((U', Q')\) and \((U'', Q'')\) be such that \((U', Q') \leq (U'', Q'')\) and \(\sum_{\theta \in \Theta}(U''_{s, \theta} - U'_{s, \theta}) \leq Q''_s - Q'_s\) for all \(s \in S\). \(\Psi\) is resource monotonic if, for all such \((U', Q')\) and \((U'', Q'')\), \(\psi(U', Q')\) Pareto dominates \(\psi(U'', Q'').\)

Resource monotonicity means that raising the ceilings and capacity of a school makes all students better off, provided that the type-specific ceilings are not raised more than the capacity.\(^{42}\)

**Artificial caps**

Recall that \((\bar{U}, \bar{Q})\) ensures a feasible match if \(\mathcal{M}(\bar{U}, \bar{Q}) \subseteq \mathcal{M}_f\). If \((\bar{U}, \bar{Q})\) ensures a feasible match, \(\psi(U, Q) : P^n \rightarrow \mathcal{M}\) is by definition a feasible mechanism. We call such a mechanism an artificial caps mechanism.

**Remark 4.** While artificial caps mechanisms and upper quota mechanisms are closely related, there is an important distinction: an artificial caps mechanism is a particular type of upper quota mechanism, where the quotas \((\bar{U}, \bar{Q})\) are chosen to be strict enough that any matching produced by the

\(^{42}\)Resource monotonicity has been used as an important axiom in many other allocation settings as well (see, for example, Ehlers and Klaus (2004), Kesten (2006), and Thomson (2005)). The previous works do not have type-specific ceilings, and prior notions of resource monotonicity say that raising just the capacity of a school makes all agents better off (in a Pareto sense). This is implied by Definition 4, but for our purposes, we want to allow the type-specific ceilings to be raised as well. However, we must do this in a “controlled” manner: raising the type-specific ceilings by more than the number of capacity seats may make students of other types whose ceilings were not raised worse off (see the discussion after Theorem 5).
mechanism is feasible. Upper quota mechanisms, on the other hand, may or may not satisfy the floors (depending on the submitted preferences).

We now generalize dynamic quotas to any class of mechanisms $\Psi$.

**Dynamic quotas $\Psi$ ($DQ^\Psi$)**

Fix a sequence of ceiling-capacity vectors $\eta = \{(U^1, Q^1), (U^2, Q^2), \ldots, (U^K, Q^K)\}$ such that: (i) $(U^{k+1}, Q^{k+1}) \leq (U^k, Q^k) \leq (U, Q)$ for all $k$, and (ii) $(U^K, Q^K)$ ensures a feasible match. The algorithm then proceeds in a series of stages.

**Stage 1** Calculate $\psi(U^1, Q^1)(P_I)$. If $\psi(U^1, Q^1)(P_I)$ is a feasible matching, end the algorithm and output this matching. If not, proceed to stage 2.

In general,

**Stage $k$** Calculate $\psi(U^k, Q^k)(P_I)$. If $\psi(U^k, Q^k)(P_I)$ is a feasible matching, end the algorithm and output this matching. If not, proceed to stage $k + 1$.

We define dynamic quotas $\Psi$ as the function $DQ^\Psi : \mathcal{P}^n \rightarrow \mathcal{M}$ that produces, for each input, the matching at the end of the above algorithm.

Dynamic quotas DA is a special case of $DQ^\Psi$ when $\Psi$ is the class of deferred acceptance mechanisms, but dynamic quotas can be applied to many other choices of $\Psi$. We discuss common classes of mechanisms used in practice below.

**Theorem 7.** Fix $P_I \in \mathcal{P}^n$. Then, $DQ^\Psi(P_I) = \psi(U^k, Q^k)(P_I)$ for some $k \leq K$. If, in addition, $\Psi$ is resource monotonic and $\eta$ is a single-seat reduction sequence, then the dynamic quotas mechanism $DQ^\Psi$ Pareto dominates the artificial caps mechanism $\psi(U^K, Q^K)$.

The above theorem can be thought of as a generalization of the result that DQDA Pareto dominates ACDA to other classes of mechanisms $\Psi$.

**Applying dynamic quotas to other common mechanisms**

We now define three other mechanisms commonly used in the field that can be adapted to our dynamic quotas framework: the serial dictatorship, immediate acceptance (also known as “the Boston mechanism”), and top trading cycles.
Mechanism 1: Serial dictatorship under \((U', Q')\)

Fix an ordering of the students \(\succ^{SD}\) (without loss of generality, we let \(i_1 \succ^{SD} \ldots \succ^{SD} i_n\)).

**Step 1** Consider student \(i_1\). Assign \(i_1\) to her most preferred school \(s\) according to \(P_{i_1}\) that has not yet reached its capacity \(Q'_s\) or type-specific ceiling \(U'_{s, \tau(i_1)}\).

In general,

**Step \(k\)** Consider student \(i_k\). Assign \(i_k\) to her most preferred school \(s\) according to \(P_{i_k}\) that has not yet reached its capacity \(Q'_s\) or type-specific ceiling \(U'_{s, \tau(i_k)}\).

We define \(SD(U', Q') : \mathcal{P}^n \rightarrow \mathcal{M}\) as the function that produces, for each input, the matching at the end of the above algorithm. We let \(SD = \{SD(U', Q')\}_{(U', Q')}\) denote the class of serial dictatorship mechanisms.

The serial dictatorship fixes some ordering of students and allows the students to pick their favorite school in this order. The serial dictatorship is a commonly used mechanism in assignment problems where the agents do not have priorities over the objects to be assigned.

Mechanism 2: Immediate acceptance under \((U', Q')\)

**Step 1** Each student applies to the first school on her preference list. Each school \(s\) admits students one-by-one according to \(\succ_s\), unless admitting a student would violate either the capacity \(Q'_s\) or some type-specific ceiling \(U'_{s, \beta}\). Reduce the capacity of each \(s\) by the total number of students admitted, and reduce each type-specific ceiling by the number of students of that type admitted.

In general,

**Step \(k\)** Each student not already accepted in some previous step applies to her \(k^{th}\) choice school.

Each school \(s\) admits students one-by-one according to \(\succ_s\), unless admitting a student would violate either the updated capacity or updated type-specific ceilings from the previous step. Reduce the capacity of each \(s\) by the total number of students admitted, and reduce each type-specific ceiling by the number of students of that type admitted.

We define \(IA(U', Q') : \mathcal{P}^n \rightarrow \mathcal{M}\) as the function that produces, for each input, the matching at the end of the above algorithm. We let \(IA = \{IA(U', Q')\}_{(U', Q')}\) denote the class of immediate acceptance algorithms.

The immediate acceptance algorithm runs similarly to the deferred acceptance algorithm with one crucial difference: at the end of each step, the acceptances are permanent (“immediate”), rather than tentative. The immediate acceptance algorithm is popular in many school choice settings.
Variants of it are currently used in Cambridge, MA and Minneapolis, MN, and the algorithm has been used previously in Seattle, WA, and Boston, MA. The main drawback of immediate acceptance is that it is not strategyproof (nor is it even strategyproof in large markets and is anecdotally “easy” to manipulate; see Kojima and Pathak (2009) or Azevedo and Budish (2013)). Nevertheless, it is still a commonly used mechanism in the field, and so understanding its properties with regard to dynamic quotas is important.

**Mechanism 3: Top trading cycles under \((U', Q')\)**

**Step 1** Each student \(i\) points to her most preferred school which has a seat available for her type (i.e., both \(Q'_s > 0\) and \(U'_{s,r(i)} > 0\)). Each school \(s\) points to the student with highest priority according to \(>_{s}\). There is at least one cycle. Every student in a cycle is assigned a seat at the school she is pointing to and is removed. For the schools that are assigned a student, their capacity and relevant type-specific ceiling are reduced by one. If a school has no more capacity, it is removed from the market. If there is at least one student remaining, move to the next step.

In general,

**Step \(k\)** Each remaining student \(i\) points to her most preferred school which has a seat available for her type. Each school \(s\) points to the student with highest priority according to \(>_{s}\). There is at least one cycle. Every student in a cycle is assigned a seat at the school she is pointing to and is removed. For the schools that are assigned a student, their capacity and relevant type-specific ceiling are reduced by one. If a school has no more capacity, it is removed from the market. If there is at least one student remaining, move to the next step.

We define \(\text{TTC}^{(U', Q')} : \mathcal{P}^n \to \mathcal{M}\) as the function that produces, for each input, the matching at the end of the above algorithm. We let \(\text{TTC} = \{\text{TTC}^{(U', Q')}\}_{(U', Q')}\) be the class of TTC mechanisms.

TTC mechanisms were first proposed for school choice problems by Abdulkadiroğlu and Sönmez (2003), who show that TTC is strategyproof and Pareto efficient. The main drawback of TTC is that it will not eliminate justified envy (nor will it eliminate justified envy among same types), because TTC allows students to trade their priorities. Versions of TTC are currently in use in school districts in San Francisco and New Orleans.

The serial dictatorship, immediate acceptance, and top trading cycles are all upper quota mechanisms used in practice. As such, they are all natural candidates for use in the presence of floor constraints as well. By imposing sufficiently strict artificial caps, it is possible to ensure a feasible matching by running any of these mechanisms. However, doing so will result in the same inefficiencies as with deferred acceptance, because artificial caps still eliminates seats ex-ante, ignoring important information contained in the student preferences. We thus argue that any
market that uses artificial caps SD/IA/TTC would be better served by switching to dynamic quotas SD/IA/TTC. Both SD and IA are indeed resource monotonic, while TTC is not.\textsuperscript{43} Theorem 7 then implies that dynamic quotas SD/IA will Pareto dominate artificial caps SD/IA. Since TTC is not resource monotonic, the Pareto dominance of dynamic quotas over artificial caps will not hold, but it will still be true that the final matching implemented under dynamic quotas will have higher ceilings than artificial caps, which intuitively will make the students better off on average (just as simulations showed that students were better off on average under EDQDA than ACDA, even though EDQDA does not Pareto dominate ACDA). We thus suggest that policymakers may want to consider dynamic quotas over artificial caps, even if resource monotonicity does not hold formally.

7.2 Outside options

In the main model, we made the assumption that all students find all schools acceptable, and vice-versa. In this section, we allow students to have an outside option, denoted $O$. We let the set of schools now be $S \cup \{O\}$, and students have a strict preference relation $P_i$ over this set. Formally, $O$ can be treated as a school just like any other, only with infinite capacity. We say a mechanism $\psi : \mathcal{P}^n \to M$ is \textit{individually rational} if $\psi_i(P_I) \in O$ for all $i \in I$.

\textbf{Theorem 8.} Assume that $L_{s,\theta} > 0$ for some $(s, \theta)$. There is no mechanism $\psi$ that is both feasible and individually rational.

The above result says that it is impossible to guarantee an outcome that satisfies all floor constraints if students can take an outside option. In practice, how far a matching is from satisfying the floors will depend on the number of agents who actually have a feasible outside option. This is not an issue in some settings, such as military branching, where all cadets have signed contracts to serve in the military. In school choice, all students are legally required to be enrolled in some school. If a student is not able to be assigned any school on her submitted preference list, districts will assign them to some other school, generally the school closest to their home with an available seat (in New York City, this process is called “administrative assignment”). Some students may then choose to attend a private school, but many may not have such a viable outside option.\textsuperscript{44}

While it is impossible to guarantee a matching that is feasible in the formal sense, the stricter the artificial caps imposed, the closer the resulting matching will be to filling the floors. However,

\textsuperscript{43}The serial dictatorship is a special case of deferred acceptance when all schools use the same priority relation equal to $\succ^{SD}$, and so resource monotonicity of $SD$ follows from resource monotonicity of $DA$. For $IA$, we prove resource monotonicity in Appendix F. Kesten (2006) shows that $TTC$ is not resource monotonic.

\textsuperscript{44}For example, in 2003, before the redesign of the high school assignment mechanism in New York City, over 30,000 of the 90,000 students were assigned to a school they did not submit a preference for, with the vast majority attending their assigned school. See Abdulkadiroğlu et al. (2005a).
artificial caps may once again go “too far”, and eliminate more seats than necessary. This can end up pushing too many students to leave the school district and take their outside option.

The next theorem shows that dynamic quotas will again outperform artificial caps, even allowing for outside options. With an outside option, dynamic quotas works similarly to before. The only difference is, if for some preferences $P_I$ the algorithm runs all the way to stage $K$, then the final matching output is $\psi^{(U^K,Q^K)}(P_I)$, whether this matching is feasible or not. Then, we have the following results.

**Theorem 9.** Assume that $\Psi$ is resource monotonic, every $\psi^{(U',Q')} \in \Psi$ is individually rational, and let $\eta = \{(U^1,Q^1),\ldots,(U^K,Q^K)\}$ be a single-seat reduction sequence. Then, the following hold:

1. The dynamic quotas mechanism $DQ^\Psi$ Pareto dominates the artificial caps mechanism $\psi^{(U^K,Q^K)}$.

2. The number of students assigned to schools in $S$ is weakly greater under $DQ^\Psi$ than under the artificial caps mechanism $\psi^{(U^K,Q^K)}$ (equivalently, the number of students assigned to the outside option $O$ is weakly less under $DQ^\Psi$ than under the artificial caps mechanism $\psi^{(U^K,Q^K)}$).

3. The number of floor seats left unfilled is weakly less under $DQ^\Psi$ than under the artificial caps mechanism $\psi^{(U^K,Q^K)}$.

Any school district that imposes some type of upper quotas/artificial caps runs the risk of pushing some students to take an outside option. This will be true of any mechanism. How high this cost is, and whether it is outweighed by the benefits of diverse student bodies, are factors that must be determined by each school district based on its own needs and goals. Given that many school districts do indeed believe there are sufficient benefits to diversity, the results above suggest that dynamic quotas is a better approach to achieving it than artificial caps: all students (weakly) prefer dynamic quotas, more students stay in the public school district, and the diversity goals are not harmed.

**Remark 5.** The results above may be in particular relevant for the Japan Residency Matching Program, where doctors often submit short preference lists. Rather than imposing artificial caps from the outset and running DA, the JRMP might be better served by running a DQDA type mechanism. In the worst case, the matching produced by DQDA will be the same as under artificial caps, but DQDA may in fact end earlier, under higher capacities for all hospitals.

### 8 Conclusion

This paper shows that a common approach used in many matching markets with distributional constraints may result in avoidable inefficiencies. We propose new mechanisms based on a concept of dynamic quotas that recover these inefficiencies by allocating seats more flexibly, based on the
submitted preferences of the students, while still satisfying all constraints. We show that it is possible to improve upon the existing approaches in a Pareto sense without compromising fairness or incentive properties, suggesting that the use of our mechanisms should improve the performance of matching markets in the field. In addition, we analyze the aforementioned incentive/efficiency trade off further, and show how relaxing incentives to large-market strategyproofness allows us to use mechanisms that increase efficiency even more.

The main motivation for most of our formal modeling decisions was diversity goals in school choice, though our mechanisms can be applied in many other settings as well, including military cadet branching, hospital-resident matching, or other markets where distributional constraints are important. We take no position on the merits of imposing distributional constraints; many school districts that impose such constraints do so because they believe that there are social benefits to diverse educational environments. We instead follow the paradigm of market design research advocated by Roth (2002) and take the constraints as empirical realities. We then attempt to provide practical mechanisms that optimize student welfare within this framework. We do argue, however, that markets that impose artificial caps as a way to satisfy some “implicit” floor constraints (such as the Japan Residency Matching Program or other markets in which the true distributional constraints are not publicly stated) should consider modeling their goals more explicitly and switching to a dynamic quotas mechanism similar to those provided in this paper. Doing so will improve agent welfare while at the same time ensuring that the true distributional goals are met.

References


### A  A more general framework

In this section, we define a model that allows for more general school choice functions and feasibility constraints. We then introduce some conditions on choice functions and preliminary theorems that will be useful in the proof of strategyproofness (found in the next section), though they also may be of independent interest. To simplify the flow of the argument, the proofs of some lemmas will be found in Appendix C.

#### A.1  Primitives of the general model

The primitives of the general model once again consist of a set of schools $S = \{s_1, \ldots, s_m\}$ and a set of students $I = \{i_1, \ldots, i_n\}$. Each student has a strict preference relation $P_i$ on the set of schools $S$. The priorities of the schools are now described differently. For any set $X$, let $2^X$ denote the power set of $X$. For each school $s$ we define a **choice function** $\text{Ch}_s : 2^I \rightarrow 2^I$, where $\text{Ch}_s(I')$ denotes the set of students admitted to school $s$ when its choice set is $I'$, i.e., $\text{Ch}_s(I') \subseteq I'$ is school $s$’s highest priority subset of students. This set up is similar to the model of Hatfield and
In the baseline model, $Ch_s(I)$ would be the choice function as defined in the description of DA in Section 3. Corresponding to each choice function is a rejection function $Rej_s(I') = I' \setminus Ch_s(I')$. Let $Ch := \{Ch_{s_1}, \ldots, Ch_{s_m}\}$ denote a vector of choice functions, one for each school.

The following two conditions on choice functions were identified by Hatfield and Milgrom (2005) as key for strategyproofness of DA in a model without floor constraints.

**Definition 5.** Choice function $Ch_s$ is **substitutable** if $I_0 \sqsupseteq I_{00}$ implies $Rej_s(I_0) \sqsupseteq Rej_s(I_{00})$.

**Definition 6.** Choice function $Ch_s$ satisfies the **law of aggregate demand** if $I_0 \sqsupseteq I_{00}$ implies $|Ch_s(I_0)| \leq |Ch_s(I_{00})|$.

These properties will be needed for our analysis for similar reasons as in Hatfield and Milgrom (2005). We thus assume that all choice functions defined from here forward satisfy both substitutability and the law of aggregate demand.

Let $I(s) = \{I' \subseteq I : Ch_s(I') \text{ for some } I'' \subseteq I\}$. In words, $I(s)$ is a set consisting of all possible assignments for school $s$, obtained by considering every potential set of applicants $I''$ that $s$ may have the opportunity to choose from. Beyond the individual school choice functions, the school district can also impose additional feasibility constraints. In this more general setting, we assume for each school $s$ the school district defines a priori a subset $I_f(s) \subseteq I(s)$ of feasible assignments. The set of feasible matchings $M_f$ is then defined as follows:

$$M_f = \{\mu \in M : \mu(s) \in I_f(s) \text{ for all } s \in S\}.$$ 

Note that the definition of $I_f(s)$ and IRS (footnote 45) imply that if $\mu \in M_f$, then $Ch_s(\mu(s)) = \mu(s)$ for all $s$. As in the main text, we assume that the set of feasible matchings is nonempty (if this were not the case, then the feasibility constraints are not consistent with the primitives of the model, and so must be altered).

### A.2 Generalized DQDA

We now define a generalized version of the DQDA algorithm that takes as an input a reduction sequence of choice functions. Given a vector of choice functions $Ch' = \{Ch'_{s_1}, \ldots, Ch'_{s_m}\}$, let

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45 Aygün and Sönmez (2013) point out a technical ambiguity in the model of Hatfield and Milgrom (2005), noting that to ensure the choice functions are derived from a well-defined underlying priority relation $\succ_s$ over sets of students, one must assume a condition called irrelevance of rejected students (IRS), which, in our setting, says that if $Ch_s(I'') \subseteq I' \subseteq I''$, then $Ch_s(I') = Ch_s(I'')$. We assume below that all of our choice functions satisfy substitutability and the law of aggregate demand, which Aygün and Sönmez (2013) show implies IRS, and so we are justified in working directly with the choice functions, rather than the underlying priority relation.

46 Note that by IRS (footnote 45), $I' \in I(s)$ implies that $I' = Ch_s(I')$.

47 In the main text, $I_f(s)$ would consist of all assignments that satisfy a school’s type-specific floor and ceiling constraints and capacities. In the standard school choice model (e.g., Abdulkadiroğlu and Sönmez (2003)), $I_f(s) = I(s)$. 

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Generalized dynamic quotas deferred acceptance (GDQDA) Let $\eta = \{Ch^1, \ldots, Ch^K\}$ be a sequence of choice function vectors such that $DA_{Ch^K}(P_I) \in M_f$ for all $P_I \in P^n$.\textsuperscript{48}

Stage 1 Starting with the empty matching, compute $DA_{Ch^1}(P_I)$. If $DA_{Ch^1}(P_I) \in M_f$, end the algorithm and output this matching. If not, proceed to stage 2.

In general,

Stage $k$ Starting with the empty matching, compute $DA_{Ch^k}(P_I)$. If $DA_{Ch^k}(P_I) \in M_f$, end the algorithm and output this matching. If not, proceed to stage $k + 1$.

Though at this point very little structure has been imposed on $\eta$, GDQDA will produce a feasible matching for any input preferences $P_I$ (in the extreme case, it will produce $DA_{Ch^K}(P_I)$, which by definition is feasible). However, without additional structure, the GDQDA mechanism may not have good incentive or efficiency properties. As mentioned above, two important properties are that within each stage, $Ch^k_s(\cdot)$ satisfies both substitutability and the law of aggregate demand for all $k$ and $s$.\textsuperscript{49} The next two conditions impose structure on how the choice functions evolve across stages.

**Definition 7.** Reduction sequence $\eta = \{Ch^1, \ldots, Ch^K\}$ is **monotonic** if $Rej^k_s(I') \subseteq Rej'^{k''}_s(I')$ for all $s \in S$, $I' \subseteq I$, and $k'' \geq k'$.

Remark 6. If $\eta$ is monotonic and $Ch^k_s$ is substitutable for all $s$ and $k$, we say $\eta$ is **monotonically substitutable**. Note that if $\eta$ is monotonically substitutable, then $I' \subseteq I''$ and $k' \leq k'' \implies Rej^k_s(I') \subseteq Rej'^{k''}_s(I'')$.\textsuperscript{50}

**Definition 8.** Reduction sequence $\eta$ is **minimal** if the following hold for all $k$: (i) for exactly one $s$, $0 \leq |Ch^k_s(I')| - |Ch^{k+1}_s(I')| \leq 1$ for all $I' \subseteq I$ and (ii) for all remaining $s' \neq s$, $Ch^{k+1}_{s'}(\cdot) = Ch^k_s(\cdot)$.

Minimality guarantees that when moving from stage $k$ to $k + 1$, fixing the set of applications, at most one student will be rejected. Lemma 2 below shows that in the baseline model in the main text, any single-seat reduction sequence $\eta$ induces a sequence of choice functions that is monotonic and minimal and satisfies substitutability and the law of aggregate demand for all $k$ and all $s$.

\textsuperscript{48}Note that such a $Ch^K$ can always be found by choosing any $\mu \in M_f$ and setting $Ch^K_s(I') = I' \cap \mu(s)$ for all $I' \subseteq I$ and all $s \in S$. Of course, it will also in general be possible to ensure a feasible match with much less restrictive choice functions.

\textsuperscript{49}The choice of $Ch^K$ defined in footnote 48 does indeed satisfy both substitutability and the law of aggregate demand.

\textsuperscript{50}$Rej^k_s(I') \subseteq Rej'^k_s(I'') \subseteq Rej'^{k''}_s(I'')$, where the first inclusion is by substitutability and the second is by monotonicity.
A.3 Alternate descriptions of the GDQDA algorithm

We now provide an alternative algorithm that (under the above conditions) will be equivalent to generalized DQDA that we call **persistent DQDA (PDQDA)**. In the main proof of strategyproofness, we will move back and forth between GDQDA and PDQDA. Briefly, stage $k$ of PDQDA will start with the tentative matching from stage $k-1$ and reduce the choice functions by moving them forward to $Ch^k$, causing a rejection chain (in contrast to GDQDA, which begins each stage from the empty matching). Formally, given a sequence $\eta = \{Ch^1, \ldots, Ch^K\}$, we define the following algorithm.

**Persistent DQDA**

**Stage 1**

1. **Step 0** Set $A^1_s(0) = \emptyset$ for all $s \in S$.
2. **Step 1** Choose some student $i^1$ who applies to her favorite school, $s^1$. Let $A^1_{s^1}(1) = \{i^1\}$ and $A^1_s(1) = A^1_s(0)$ for all other $s \in S$. Each school $s \in S$ tentatively accepts the students in $Ch^1_{s^1}(A^1_{s^1}(1))$ and rejects the rest.
3. **Step t** Choose a student $i^t$ that is not tentatively accepted by any school, and let her apply to her most preferred school $s^t$ that has not yet rejected her. Let $A^1_{s^t}(t) = A^1_{s^t}(t-1) \cup \{i^t\}$ and $A^1_s(t) = A^1_s(t-1)$ for all $s \neq s^t$. Each school $s \in S$ tentatively accepts the students in $Ch^1_{s^t}(A^1_{s^t}(t))$, and rejects all other students.

Stage 1 terminates when every student is either tentatively accepted by some school $s \in S$ or has applied to all schools and been rejected.\(^{51}\) This happens in a finite number of steps $T^1$. Let the resulting matching be $\nu^1$, where $\nu^1(s) = Ch^1_s(A^1_s(T^1))$ for all $s \in S$. If $\nu^1 \in \mathcal{M}_f$, end the algorithm and output matching $\nu^1$. If not, proceed to stage 2.

In general,

**Stage $k$**

1. **Step 0** Set $A^k_s(0) = A^{k-1}_s(T^{k-1})$ for all $s \in S$, and let each school tentatively accept $Ch^k_s(A^k_s(0))$ and reject all remaining students.
2. **Step 1** Choose a student $i^1$ that is not tentatively accepted by any school, and let her apply to her most preferred school $s^1$ that has not yet rejected her (in this stage, or any previous stages). Let $A^k_{s^1}(1) = A^k_{s^1}(0) \cup \{i^1\}$ and $A^k_s(1) = A^k_s(0)$ for all other $s \in S$. Each school $s \in S$ tentatively accepts the students in $Ch^k_{s^1}(A^k_{s^1}(1))$ and rejects the rest.

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\(^{51}\)Note that in this definition, we allow the possibility that a student applies to and is rejected from every school, in which case she is unmatched at the end of the algorithm. Under suitable choice functions that ensure feasible matchings, this will not be an issue.
Step \( t \)  Choose a student \( i^t \) that is not tentatively accepted by any school, and let her apply to her most preferred school \( s^t \) that has not yet rejected her (in this stage, or any previous stages). Let \( A^k_s(t) = A^k_s(t-1) \cup \{i^t\} \) and \( A^k_s(t) = A^k_s(t-1) \) for all other \( s \in S \). Each school \( s \in S \) tentatively accepts the students in \( \text{Ch}_s^k(A_s^k(t)) \) and rejects the rest.

Stage \( k \) terminates when every student is tentatively accepted by some school \( s \in S \) or has applied to all schools and been rejected. This happens in a finite number of steps \( T^k \). Let the resulting matching be defined by \( \nu^k(s) = \text{Ch}^k_s(\text{A}^k_s(T^k)) \) for all \( s \in S \). If \( \nu^k \in M_f \), end the algorithm and output matching \( \nu^k \). If not, proceed to stage \( k + 1 \).

The above description of the algorithm makes use of the cumulative offer process of Hatfield and Milgrom (2005) (see also Hatfield and Kojima (2009)). As the cumulative offer process progresses, schools continually accumulate applications from students, and at each point, hold on to their most preferred set students among all of those who have cumulatively applied to it. Students who are not currently held by any school make new applications to their most preferred school that has not yet rejected them. Note that, as it stands, there is no assumption of consistency imposed on the algorithm, since some student could be assigned to more than one school. However, we show that under our conditions on \( \eta \), this will not be an issue: at the end of each stage of PDQDA, each student will be assigned to at most one school, and the assignment will be equivalent to that at the end of stage \( k \) of GDQDA.

**Theorem 10.** Let \( \nu^k \) be the assignment at the end of stage \( k \) of the PDQDA algorithm, and \( \mu^k \) be the assignment at the end of stage \( k \) of the GDQDA algorithm. Assume that \( \eta \) is monotonic and that \( \text{Ch}_s^k(\cdot) \) satisfies substitutability and the law of aggregate demand for all \( k \) and \( s \). Then \( \nu^k = \mu^k \) for all \( k \).

**Proof.** As in the definition of the algorithm above, let \( A^k_s(t) \) denote the cumulative offer set of school \( s \) at step \( t \) of stage \( k \) under PDQDA. For each stage \( k \), let \( T^k \) denote the final step of stage \( k \). Therefore, at the end of stage \( k \) of PDQDA we have \( \nu^k(s) = \text{Ch}^k_s(A^k_s(T^k)) \).

Now consider GDQDA. Within a stage \( k \), we are simply running DA under choice functions \( \text{Ch}^k \). We can define an analogous within-stage cumulative offer process. Let \( B^k_s(t) \) denote the cumulative offer set of school \( s \) at step \( t \) of the (within-stage) cumulative offer process. Because GDQDA starts each stage \( k \) from the empty matching, \( B^k_s(0) = \emptyset \) for all \( s \) and \( k \). Let \( \hat{T}^k \) denote the last step of stage \( k \). Hatfield and Milgrom (2005) show that the matching produced by this cumulative offer process is equivalent to deferred acceptance, i.e. \( \mu^k(s) = \text{Ch}^k_s(B^k_s(\hat{T}^k)) \) for all \( s \in S \) and all \( k \).

The following lemma, which is proved in Appendix C, is key to the argument.
Lemma 1. For all $k$ and all $s$, $\mathcal{A}_s^k(T^k) = \mathcal{B}_s^k(\hat{T}^k)$.

Note that in general, $T^k \neq \hat{T}^k$, but the cumulative offer sets are still the same. With this lemma (proved in Appendix C), the result follows easily: $\nu^k(s) = \text{Ch}_s^k(\mathcal{A}_s^k(T^k)) = \text{Ch}_s^k(\mathcal{B}(\hat{T}^k)) = \mu^k(s)$, where the first and third equalities are by definition, and the second is by Lemma 1. Since this holds for all $k$ and all $s$, the proof is complete. $\blacksquare$

Theorem 11. Let $\eta = \{\text{Ch}_1, \ldots, \text{Ch}_K\}$ be monotonic and assume that $\text{Ch}_s^k(\cdot)$ satisfies substitutability and the law of aggregate demand for all $k$ and all $s$. Then, GDQDA under $\eta$ Pareto dominates DA under $\text{Ch}^k$.

Proof. The final matching produced by GDQDA is equivalent to the matching produced by DA under choice functions $\text{Ch}^k$ for some $k' \leq K$. The outcome of DA in stage $k$ is equivalent to the outcome of the cumulative offer process in stage $k$ (Hatfield and Milgrom (2005); Hatfield and Kojima (2010)). By monotonicity, $\text{Ch}_s^k(I') \subseteq \text{Ch}_s^{k'}(I')$ for all $I' \subseteq I$. The result then follows from Lemma 1 of Kamada and Kojima (2013). $\blacksquare$

B  Proofs omitted from the main text

Once again, proofs of any lemmas not given here can be found in Appendix C.

Theorem 1

Consider any feasible matching $\mu \in \mathcal{M}_f$, and define $\bar{Q}_s = |\mu(s)|$ and $\bar{U}_{s,\theta} = |\mu_\theta(s)|$ for all $(s, \theta)$. We show that $(\bar{U}, \bar{Q})$ ensures a feasible match. Note that $\sum_{s' \in S} \bar{U}_{s', \theta} = |I_\theta|$. Then, consider any other matching $\nu \in \mathcal{M}$ that respects $(\bar{U}, \bar{Q})$. $\nu$ respects all true ceilings and capacities $(U, Q)$ by definition. We must show that $\nu$ also respects all floors. Assume not, i.e., assume there is some pair $(s, \theta)$ such that $|\nu_\theta(s)| < L_{s,\theta}$. Then, $\sum_{s' \in S} |\nu_\theta(s')| < \sum_{s' \in S} \bar{U}_{s', \theta} = |I_\theta|$.\footnote{The first inequality follows from the fact that $\sum_{s' \in S \setminus \{s\}} |\nu_\theta(s')| \leq \sum_{s' \in S \setminus \{s\}} \bar{U}_{s', \theta}$ (because $\nu$ respects $(\bar{U}, \bar{Q})$) and that $|\nu_\theta(s)| < L_{s,\theta} \leq \bar{U}_{s, \theta}$.} But this implies that under $\nu$, there exists some student $i$ of type $\theta$ who is not assigned to any school, and so $\nu \notin \mathcal{M}$, which is a contradiction.

Theorem 2

In the proof of Theorem 5 below, we show that for any stage $k$ of DQDA, the within-stage choice functions of the schools satisfy substitutability and the law of aggregate demand. Since ACDA is equivalent to DA using the stage $K$ choice functions, strategyproofness follows from Hatfield and Milgrom (2005), who show that substitutability and the law of aggregate demand are sufficient for strategyproofness.
To show that ACDA eliminates justified envy among same types, note that ACDA is equivalent to DA under \((U^K, Q^K)\). Denote the matching produced by DA under \((U^K, Q^K)\) as \(\mu\). Assume that some student \(i\) envies another student \(j\) of her same type: \(\mu(j)P \mu(i)\) and \(\tau(i) = \tau(j) = \theta\). Let step \(t\) be the step of the DA algorithm at which \(i\) is rejected from \(\mu(j)\). In step \(t\), \(i\) is rejected because the type \(\theta\) specific seats are filled with \(L_{\mu(j),\theta}\) students of type \(\theta\) ranked higher than \(i\) according to \(\succ_{\mu(j)}\), and the open seats are filled with either: (i) \(U^K_{\mu(j),\theta} - L_{\mu(j),\theta}\) students of type \(\theta\) ranked higher than \(i\) according to \(\succ_{\mu(j)}\) or (ii) \(Q^K_{\mu(j)} - \sum_{\theta \in \Theta} L_{\mu(j),\theta}\) students of any type ranked higher than \(i\) according to \(\succ_{\mu(j)}\). As the algorithm progresses, a student accepted in step \(t\) can only be rejected from the type \(\theta\) specific seats only if a higher-ranked student of type \(\theta\) applies, and the same is true of the students at the open seats. Thus, at the end of the algorithm, all students assigned to \(\mu(j)\) through either the type \(\theta\) specific seats or the open seats must be ranked higher than \(i\). Since \(\tau(j) = \theta\) as well, this implies that \(j \succ_{\mu(j)} i\), i.e., \(i\) does not justifiably envy \(j\).

**Theorems 3 and 5**

Theorem 3 is a special case of Theorem 5, and so we prove the latter. With slight abuse of notation, let \(\eta = \{Ch^1, \ldots, Ch^K\}\) be the sequence of choice functions induced by \(\{(U^1, Q^1), \ldots, (U^K, Q^K)\}\), which, recall, is assumed to be a single-seat reduction sequence. We first note the following lemma, which will be useful in the proofs of both parts (1) and (3). For the proof, see Appendix C.

**Lemma 2.** The reduction sequence \(\eta = \{Ch^1, \ldots, Ch^K\}\) is monotonic and minimal, and \(Ch^k(\cdot)\) satisfies substitutability and the law of aggregate demand for all \(k\) and \(s\).

We now prove each part of Theorem 5.

**Part (1)**

Follows from Lemma 2 and Theorem 11.

**Part (2)**

The final matching produced by DQDA is equivalent to DA under \(Ch^k\) for some \(k \leq K\). That DA under \(Ch^k\) eliminates justified envy among same types can be shown in the same manner as the proof of Theorem 2, where we showed that DA under \(Ch^K\) eliminates justified envy among same types.

\(^{53}\)Of course, there may be students of other types \(\theta' \neq \theta\) whom \(i\) envies and has higher priority over, since these students could be assigned through the type \(\theta'\) specific seats.
Part (3)

Fix the reports of the other students at $P_{-i}$, and let $i$’s true preferences be $P_i$. We will show that there is no report $P_i'$ that will give $i$ a better assignment than reporting the truth.

We will make use of the fact that the DA algorithm within each stage $k$ is equivalent to the cumulative offer process of Hatfield and Milgrom (2005) (see the discussion in Appendix A). It is well-known (see, e.g., McVitie and Wilson (1971), Dubins and Freedman (1981), Hatfield and Milgrom (2005), or Hatfield and Kojima (2010)) that in the cumulative offer process, the order in which students are allowed to apply does not matter, as any such order will lead to the same final matching. We consider running the algorithm using an ordering in which student $i$ applies last in each stage. For each stage $k$, let $\hat{A}_s^k$ denote the cumulative set of applicants that school $s$ receives in the cumulative offer process under Ch$_s^k$ on all students other than $i$. Then, let $i$ enter the market, which causes a rejection chain, which records the action of the algorithm:

<table>
<thead>
<tr>
<th>Step</th>
<th>Action</th>
<th>Cumulative offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>None</td>
<td>$A_s^k(0) = \hat{A}_s^k$ for all $\hat{s}$</td>
</tr>
<tr>
<td>1</td>
<td>Student $i$ applies to school $s$</td>
<td>$A_s^k(1) = A_s^k(0) \cup {i}$, $A_s^k(1) = A_s^k(0)$ for all $\hat{s} \neq s$</td>
</tr>
<tr>
<td>2</td>
<td>$s$ rejects $i'$</td>
<td>$A_s^k(2) = A_s^k(1)$ for all $\hat{s}$</td>
</tr>
<tr>
<td>3</td>
<td>$i'$ applies to $s'$</td>
<td>$A_{s'}^k(3) = A_{s'}^k(2) \cup {i'}$, $A_{s'}^k(3) = A_{s'}^k(2)$ for all $\hat{s} \neq s'$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The rejection chain in stage $k$ ends at the first step $T_k$ for which some student $i'$ applies to some school $s'$ and $s'$ does not reject a student. At this point, the applicant pools for each school are $A_s^k(T_k)$, and the final matching $\mu^k$ is defined by $\mu^k(s) = \text{Ch}_s^k(A_s^k(T_k))$. Note that by substitutability, if $i \in \text{Rej}_s^k(A_s^k(t))$ for some step $t$, then $i \in \text{Rej}_s^k(A_s^k(t'))$ for all $t' \geq t$. In particular, $i \in \text{Rej}_s^k(A_s^k(T_k))$, which implies that under $\mu^k$, each student is assigned to exactly one school. Also, note that the law of aggregate demand guarantees at each rejection step, at most one student is rejected.

For any school $s$ and applicant pool $A \subseteq I$, define $\delta_s(A) = \sum_{\theta \in \Theta} \max\{L_{s,\theta} - |A \cap I_\theta|, 0\}$. In words, $\delta_s(A)$ is the number of floor seats unfilled at $s$ when its applicant pool is $A$.\(^{54}\)

Let $\Delta^k = \sum_{s \in S} \delta_s(\hat{A}_s^k)$. In words, $\Delta^k$ is the total number of floor seats unfilled at all schools in stage $k$ after all students but $i$ have applied.

\(^{54}\)Recall that at every stage $k$, each school $s$ always reserves $L_{s,\theta}$ seats for students of type $\theta$, and so if $|A \cap I_\theta| \leq L_{s,\theta}$, then all type $\theta$ students will be chosen, while if $|A \cap I_\theta| > L_{s,\theta}$, at least $L_{s,\theta}$ students of type $\theta$ will be chosen.
Lemma 3. The following hold for all $k$:

(i) If $\Delta^k = 0$, then $DA^{Ch^k}(P'_i, P_{-i}) \in \mathcal{M}_f$ for all $P'_i \in \mathcal{P}$.
(ii) If $\Delta^k > 1$, then $DA^{Ch^k}(P'_i, P_{-i}) \notin \mathcal{M}_f$ for all $P'_i \in \mathcal{P}$.
(iii) $\Delta^k \geq \Delta^{k+1} \geq \Delta^k - 1$.

If $\Delta^1 = 0$, then, all floor seats have been filled before $i$ enters the market in stage 1, and the DQDA algorithm ends in stage 1 for any report $P'_i$ of agent $i$. Therefore, from the perspective of agent $i$, the mechanism is equivalent $DA^{Ch^1}$ which is known to be strategyproof, and so she has no profitable manipulation. So, assume that $\Delta^1 > 1$. Lemma 3, part (iii) then implies that there is some critical stage $k^*$ for which $\Delta^{k^*} = 1$ and $\Delta^{k'} > 1$ for all $k' < k^*$. Thus, we can ignore all stages $k < k^*$, since by Lemma 3, part (ii), the algorithm will not end in stage $k'$ for any report of student $i$.

Thus, consider beginning DQDA in stage $k^*$. From here forward we work with the PDQDA algorithm (which, by Theorem 10 is equivalent to DQDA). Starting with offer sets for the schools equal to $\hat{\mathcal{A}}^*_{s}$, let $i$ enter the market with some reported preferences $P'_i$. Her entering again causes a rejection chain:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Cumulative offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k^*$</td>
<td>0</td>
<td>None</td>
<td>$\mathcal{A}^{k^<em>}_{s}(0) = \hat{\mathcal{A}}^{k^</em>}_{s}$ for all $\hat{s}$</td>
</tr>
<tr>
<td>1</td>
<td>Student $i$ applies to school $s$</td>
<td>$\mathcal{A}^{k^<em>}_{s}(1) = \mathcal{A}^{k^</em>}_{s}(0) \cup {i}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$s$ rejects $i'$</td>
<td>$\mathcal{A}^{k^<em>}_{s}(1) = \mathcal{A}^{k^</em>}_{s}(0)$ for all $\hat{s} \neq s$</td>
<td></td>
</tr>
<tr>
<td>$\mathcal{R} = $</td>
<td>2</td>
<td>$i'$ applies to $s'$</td>
<td>$\mathcal{A}^{k^<em>}_{s}(2) = \mathcal{A}^{k^</em>}_{s}(1)$ for all $\hat{s}$</td>
</tr>
<tr>
<td></td>
<td>$\mathcal{A}^{k^<em>}_{s}(3) = \mathcal{A}^{k^</em>}_{s}(2) \cup {i'}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\mathcal{A}^{k^<em>}_{s}(3) = \mathcal{A}^{k^</em>}_{s}(2)$ for all $\hat{s} \neq s'$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$k^* + 1$</td>
<td>0</td>
<td>Choice functions become $Ch^{k^*+1}$</td>
<td>$\mathcal{A}^{k^<em>+1}_{s}(0) = \mathcal{A}^{k^</em>}_{s}(3)$ for all $\hat{s}$</td>
</tr>
<tr>
<td>1</td>
<td>School $s''$ rejects student $i''$</td>
<td>$\mathcal{A}^{k^<em>+1}_{s}(1) = \mathcal{A}^{k^</em>+1}_{s}(0)$ for all $\hat{s}$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\vdots$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The rejection chain simply records the action of the algorithm, but note that now, we include quota reductions as part of the rejection chain (stage $k^* + 1$, step 0 above).

There are several features to note about rejection chains. First, at stage $k$, step $t$ of the rejection chain, the set of applicants being tentatively held by school $s$ is $Ch^k_s(\mathcal{A}^*_s(t))$. Second, note that monotonicity and substitutability guarantee that if at some stage and step $(k, t)$ we have $i \in Rej^k_s(\mathcal{A}^k_s(t))$, then $i \in Rej^k_{s'}(\mathcal{A}^k_{s'}(t'))$ for all $(k', t')$ such that either (i) $k' > k$ or (ii) $k' = k$ and
$t' \geq t$. This, together with the law of aggregate demand and minimality, guarantee that at each step, there is at most one student who is not currently being held by any school, and it is this student who will make the next application according to her preferences.\textsuperscript{55} Third, within a stage, the school rejecting a student at some step is the same as the last school to receive an application. However, across stages, this may not be true. For example, at stage $k^*$, step 3, $s'$ receives an application. After the choice functions are reduced to $\text{Ch}^{k^*+1}$, school $s''$ rejects a student, but it may be that $s'' \neq s'$.

At stage $k^*$, $\Delta k^* = 1$, which means that $\delta_y(\tilde{A}^{k^*}_y) = 1$ for some school $y$ and $\delta_s(\tilde{A}^{k^*}_s) = 0$ for all $s \neq y$. By definition of $\delta_s(\cdot)$, it must be that $|\tilde{A}^{k^*_s} \cap I_\phi| = L_{y,\phi} - 1$ for some type $\phi \in \Theta$, and $|\tilde{A}^{k^*_s} \cap I_\theta| \geq L_{s,\theta}$ for all $(s, \theta) \neq (y, \phi)$. In words, this means that every school has enough students in its choice set to fill all type-specific floors except for school $y$, which is one student short of filling its type $\phi$ floor.

We next note the following important fact about rejection chains:

The algorithm ends the next time a type $\phi$ student applies to $y$ \hfill (1)

This is an “if and only if” statement: the algorithm ends the next time a type $\phi$ student applies to $y$, and cannot end earlier. When (1) occurs in some step $t$ of some stage $k \geq k^*$, no student is rejected from $y$, and stage $k$ ends. At this point, all schools have filled all floors, and so the algorithm ends and all tentative assignments are made permanent.

The next part of the proof is inspired by the Scenario Lemma from Dubins and Freedman (1981), who were the first to prove strategyproofness of DA in a simple one-to-one matching model. Define a scenario $S_i$ as a sequence of applications for agent $i$, i.e., a partial rank ordering over $S$. So, a scenario could be $S_i = \{s, u, v\}$, which means that $i$ first applies to $s$, then to $u$, then to $v$. The list need not include all schools. Each scenario induces a rejection chain as above. We use $R(S_i)$ to denote the rejection chain corresponding to scenario $S_i$. The rejection begins with $i$ applying to the first school in $S_i$, and then records all subsequent applications, rejections, and quota reductions. The rejection chain for any scenario $S_i$ ends in one of two ways:

(1) some student $j$ of type $\phi$ applies to school $y$

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Cumulative offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$t$</td>
<td>$j$ applies to $y$</td>
<td>$A^k_y(t) = A^k_y(t-1) \cup {j}$</td>
</tr>
</tbody>
</table>

(2) $i$ is rejected by the last school in $S_i$:

<table>
<thead>
<tr>
<th>Stage</th>
<th>Step</th>
<th>Action</th>
<th>Cumulative offer sets</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k$</td>
<td>$t$</td>
<td>$i$ is rejected by $v$</td>
<td>$A^k_s(t) = A^k_s(t-1)$ for all $s$</td>
</tr>
</tbody>
</table>

\textsuperscript{55}The law of aggregate demand guarantees this within a stage, while minimality guarantees it across stages.
The following lemma is key to the remainder of the proof.

**Lemma 4. (Scenario Lemma with Quota Reductions)** Consider two scenarios \( S_i \) and \( \hat{S}_i \) such that every school in \( \hat{S}_i \) is also named in \( S_i \) (order is immaterial), and assume that in \( R(S_i) \), student \( i \) applies to every school in \( S_i \). Then, every line in \( R(\hat{S}_i) \) also occurs at some point in \( R(S_i) \).

With this lemma in hand, we can continue the proof. Suppose, without loss of generality, that \( i \)'s true preferences are \( P_i : s_1, s_2, \ldots, s_m \). Say that if \( i \) submits her true preferences, she receives school \( s_h \), and suppose that there is some scenario \( \hat{S}_i = \{u, \ldots, v\} \), where \( i \) gets some school \( v \) such that \( vP_is_h \). Then, rejection chain \( R(\hat{S}_i) \) must end with some student \( j \) of type \( \phi \) applying to \( y \), while \( i \) is assigned to \( v \).

**Case (i):** \( sP_iv \) for all \( s \in \hat{S}_i \setminus \{v\} \).

Compare \( \hat{S}_i \) to a scenario \( S_i = \{s_1, \ldots, s_{h-1}\} \). By assumption, \( \hat{S}_i \subseteq S_i \), and in \( R(S_i) \), \( i \) makes every application in \( S_i \).

By the Scenario Lemma with Quota Reductions, every application in \( R(S_i) \) is also made in \( R(\hat{S}_i) \). In particular, \( j \) must also apply to \( y \) in \( R(S_i) \), which contradicts the fact that \( R(S_i) \) ends with \( i \) being rejected by \( s_{h-1} \).

**Case (ii):** \( vP_is \) for at least one \( s \in \hat{S}_i \).

Delete all schools \( s \in \hat{S}_i \) such that \( vP_is \) to create a smaller scenario \( \hat{S}_i \subseteq \hat{S}_i \). By case (i), \( R(\hat{S}_i) \) must end with \( i \) rejected by \( v \). Since \( \hat{S}_i \subseteq \hat{S}_i \), the Scenario Lemma with Quota Reductions implies that \( i \) must also be rejected by \( v \) in \( R(\hat{S}_i) \), which is a contradiction.

**Theorem 4**

Let \( DQDA \) denote the DQDA mechanism under some reduction sequence \( \eta = \{Q^1, \ldots, Q^K\} \) such that \( Q^1 = Q \), and let \( \psi \) be a mechanism that dominates \( DQDA \). Let \( P_i \) be a preference profile such that \( \psi_i(P_i)DQDA_i(P_i) \) for some \( i \in I \), and \( \psi_i(P_i)R_iDQDA_i(P_i) \) for all \( i \in I \). Let \( i \) be a student that is made strictly better off under \( \psi \). Define \( DQDA_i(P_i) = s_i \) and \( \psi_i(P_i) = \hat{s}_i \) (implying \( \hat{s}_iP_is_i \)).

Let the students other than \( i \) submit profile \( P_{-i} \), and define \( k^* \) as the critical stage of DQDA such that DQDA will not end before stage \( k^* \) for any preferences of agent \( i \) (as in Appendix A), and let \( \mu^{k^*} \) be the tentative matching after every student but \( i \) has entered the market. There are two cases.

**Case (i):** \( k^* = 0 \)

In this case, the algorithm ends in stage 1 for any report of agent \( i \). Let \( S = \{s \in S : |\mu^1(s)| < Q_s\} \) and \( R \) be the rejection chain initiated by \( i \) entering the market. Note that the rejection chain ends the first time an application is made to a school in \( S \).

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56 This follows because \( i \) receives \( s_h \) when he submits his true preferences.
Claim 1. Let $P'_i$ be some report for agent $i$, and define $P'_i = (P'_i, P_{-i})$.$^{57}$ Then, we have $\psi_s(P'_i) = DQDA_s(P'_i)$ for all $s \in S$.

Proof. First we show that $\psi_s(P'_i) \subseteq DQDA_s(P'_i)$. If not, then there exists some $j \in \psi_s(P'_i)$, but $j \notin DQDA_s(P'_i)$. However, under DQDA, no student was rejected from $s$, which means $DQDA_j(P'_i)P'_j = \psi_j(P'_i)$, contradicting that $\psi(P'_i)$ dominates $DQDA(P'_i)$. Second, we show that $DQDA_s(P'_i) \subseteq \psi_s(P'_i)$. If not, then there is some student $j \in DQDA_s(P'_i)$, but $j \notin \psi_s(P'_i)$. By domination, $j$ must have been rejected from $\psi_j(P'_i)$ under DQDA because $\psi_j(P'_i)$ was filled to capacity. Let $\psi_j(P'_i) = s_j$. In order to assign $j$ to $s_j$ under $\psi$, there must be some student $j' \in DQDA_j(P'_i)$, but $\psi_j(P'_i) \neq DQDA_{j'}(P'_i)$.$^{58}$ By domination, $\psi_{j'}(P'_i)P'_{j'} \subseteq DQDA_{j'}(P'_i)$. By the same argument, there must be some $j'' \in DQDA_{j''}(P'_i)$, but $\psi_{j''}(P'_i)P'_{j''} \subseteq DQDA_{j''}(P'_i)$, where $s_{j''} = DQDA_{j''}(P'_i)$. Continuing this line of reasoning, we reach a contradiction. Thus, $\psi_s(P'_i) = DQDA_s(P'_i)$ for all $s \in S$. $\blacksquare$

By the above claim, $s_i \notin S$. Fix some $y \in S$, and note that $\hat{s}_i, P_iy$. Consider an alternative report for $i$, $P'_i$, which ranks $\hat{s}_i$ first, $y$ second, and everything else remains the same. Define $P'_i = (P'_i, P_{-i})$.

Then, $DQDA_i(P'_i) = y$. By Claim 1, $\psi_i(P'_i) = y$. But now, $\psi_i(P'_i) = \hat{s}_iP'_iy = \psi_i(P'_i)$, and so $\psi$ is manipulable.

Case (ii): $k^* \geq 1$

We begin with the following claim. Define

$$S(P_i) = \{s \in S : |DQDA_s(P_i)| = L_s \text{ and } s \text{ never rejects a student in the running of } DQDA(P_i)\}.$$  

Claim 2. If a matching $\nu$ dominates $DQDA(P_i)$ under preferences $P_i$, then $\nu(s) = DQDA_s(P_i)$ for all $s \in S(P_i)$.

Proof. Define $I(P_i) = \{i \in I : DQDA_i(P_i) \notin S(P_i)\}$. Since no student is ever rejected from a school in $S(P_i)$, for all $j \in I(P_i)$ and all $s \in S(P_i)$ we have $DQDA_j(P_i)P_j = \psi_j(P'_i)$, but $\psi_j(P'_i) \neq DQDA_j(P'_i)$. Since $\nu$ dominates $DQDA(P_i)$, $\nu(j) \notin S(P_i)$ for all $j \in I(P_i)$. Since $|\nu(s)| \geq L_s$ for all $s \in S$ by feasibility, it follows that $\nu(s) = DQDA_s(P_i)$ for all $s \in S(P_i)$.$^{59}$

$^{57}$We allow for the possibility that $P'_i = P_i$, in which case $P'_i = P_i$. Also, note that for $j \neq i$, $P'_j = P_j$.

$^{58}$If this were not the case, then $|\psi_j(P'_i)| > |DQDA_j(P'_i)| = Q_s$, i.e., $\psi$ is not a feasible mechanism.

$^{59}$More specifically, note that $|I \setminus I(P_i)| = \sum_{s \in S(P_i)} L_s$. Since no students in $I(P_i)$ can be assigned to a school in $S(P_i)$, and the schools in $S(P_i)$ must all fill their lower quotas, $\psi_j(P'_i) \in S(P_i)$ for all $j \in I \setminus I(P_i)$. This shows that $|\psi_s(P_i)| = |DQDA_s(P_i)|$ for all $s \in S(P_i)$. To see that $\psi_s(P_i) = DQDA_s(P_i)$, we first show that $\psi_s(P_i) \subseteq DQDA_s(P_i)$. Assume not. Then, there exists some $j \in \psi_s(P_i)$, but $j \notin DQDA_s(P_i)$. Since $s$ did not reject any students under DQDA, it follows that $DQDA_j(P_i)P_j = \psi_j(P'_i)$, and so $j$ is worse off under $\psi$, which contradicts that $\psi$ dominates DQDA. Finally, we show $DQDA_s(P_i) \subseteq \psi_s(P_i)$. If not, then there exists some $j \in DQDA_s(P_i)$, but $j \notin \psi_s(P_i)$. Since $|DQDA_s(P_i)| = |\psi_s(P_i)| = L_s$, there is some other $j'$ such that $j' \in \psi_s(P_i)$, but $j' \notin DQDA_s(P_i)$. We can then apply the same argument as in the first case to show that $j'$ is worse off under $\psi$, a contradiction.
Similar to the proof of strategyproofness, we can calculate $DQDA(P_i)$ by starting the algorithm in stage $k^*$. Denote $y$ as the unique school such that $|\hat{\mu}(y)| = L_y - 1$. Allow $i$ to enter the market using preferences $P_i$, and let $\mathcal{R}$ denote the rejection chain that occurs. The rejection chain (and the algorithm) end as soon as $y$ gets its next application. Note that it must be that this application is from some student $j \neq i$, which also implies that $\hat{\mu}_i P_s y$.

Construct an alternative preference profile $P_i' = (P_i', P_{-i})$, where $P_i'$ is defined such that $\hat{s}_i$ is first, $y$ is second, and the relative ordering of all other schools is the same as in $P_i$:

$$P_i' : \hat{s}_i, y, \ldots$$

Claim 3. $DQDA_i(P_i') = y$ and $y \in S(P_i')$.

Proof. As in the proof of strategyproofness (Theorem 5), we can calculate $DQDA(P_i)$ starting with $\hat{\mu}(y)$ in stage $k^*$ and allowing $i$ to apply last. (The tentative matching before $i$ enters, $\hat{\mu}(y)$, is the same under $P_i'$, since only $i$'s preferences have changed). Let $\mathcal{R}'$ denote the rejection chain that occurs upon $i$ entering the market. Once again, the algorithm ends with the next application to $y$, which implies that $|DQDA_y(P_i')| = L_y$ and that $y$ never rejects a student, i.e., $y \in S(P_i')$.

To show that $DQDA_i(P_i') = y$, note that strategyproofness of DQDA gives $DQDA_i(P_i') \neq \hat{s}_i$. Thus, $i$ must be rejected from $\hat{s}_i$ at some step $t$ of rejection chain $\mathcal{R}'$. Step $t + 1$ must then be $i$ applying to $y$. Since $y$ has an empty seat, $i$ is accepted, and the algorithm ends. ■

Claim 4. $\psi_i(P_i') = y$

Proof. This follows immediately from Claim 2, since by assumption the matching $\psi(P_i')$ dominates matching $DQDA(P_i)$ under preferences $P_i'$. ■

Now, consider the market where the true preferences are $P_i'$. The definition of $\hat{s}_i$ and Claim 4 imply $\hat{s}_i = \psi_i(P_1 P_i') \psi_i(P_i') = y$, which means that $\psi$ is not strategyproof.

Theorem 6

To simplify notation, for a student of type $\lambda_i = (\theta_i, z_i, t_i)$, we define $g_i = (\theta_i, z_i)$, and refer to $g_i$ as student $i$'s group. Recall that students cannot misreport their group and, as described in the text, the number of students in each group, $n_g$, grows according to some fixed sequence such that $n_g \to \infty$ for all groups $g$.

We now state a no-envy condition that will be sufficient for a mechanism to be SPL, and then show that EDQDA does indeed satisfy this condition.

Definition 9. (Azevedo and Budish (2013)) A mechanism $\{(\psi^n)_{n \in \mathbb{N}}, \Lambda\}$ is envy-free but for tie-breaking (EFTB) if for each $n$ there exists a function $x^n : (\Lambda \times [0, 1])^n \to \Delta(S^n)$ that is

60Otherwise, $DQDA_i(P_i) = y$. Then, by Claim 2, $\psi_i(P_i) = y$, which contradicts that $\psi_i(P_i) P_i DQDA_i(P_i)$.
symmetric over its coordinates and such that

$$\psi^n(\lambda) = \int_{\ell \in [0,1]^n} x^n(\lambda, \ell) d\ell$$

and if $\ell_i \geq \ell_j$ and $i$ and $j$ belong to the same group $g$, then $u_{\ell_i}(x^n_i(\lambda, \ell)) \geq u_{\ell_i}(x^n_j(\lambda, \ell))$.

Given this definition, we have the following lemma.

**Lemma 5.** The mechanism $\{(E^n)_{n \in \mathbb{N}}, \Lambda\}$ is envy-free but for tie-breaking.

** Proof.** To show this, we must exhibit a function $x^n$ that satisfies the properties of Definition 9. Such a function is given immediately by $\mu^n(\lambda, \ell)$, as defined in Section 6.3. Recall that by definition we have $E^n(\lambda) = \int_{\ell \in [0,1]^n} \mu^n(\lambda, \ell) d\ell$

for all $n \in \mathbb{N}$. The function $\mu^n$ is clearly symmetric over its coordinates by construction. Thus, the last thing we need to show is that an agent $i$ in group $g$ never envies another agent $j$ in the same group $g$ with a lower lottery number. Assume that $i$ and $j$ belong to the same group, but $\ell_i > \ell_j$. Recall that $n_s$ is the (post-lottery) priority relation for school $s$. The fact that $i$ and $j$ belong to the same group and $\ell_i > \ell_j$ imply that the priority relations are such that $i > n_s j$ for all $s \in S$. Now, $\mu^n(\lambda, \ell)$ is equivalent to the matching produced by standard DA under some quotas $(U^{n,k}, Q^n)$, priorities $(>^n_s)_{s \in S}$, and (ordinal) preferences $(P_i)_{i \in I^n}$. This matching is eliminates all justified envy for same types with respect to the strict priorities $(>^n_s)_{s \in S}$, which implies that $\mu^n_i(\lambda, \ell) R_i \mu^n_j(\lambda, \ell)$, and so $u_{\ell_i}(\mu^n_i(\lambda, \ell)) \geq u_{\ell_i}(\mu^n_j(\lambda, \ell))$, which proves the lemma. $\blacksquare$

Given this lemma, the theorem follows from Proposition 1 of Azevedo and Budish (2013), which states that if a mechanism satisfies EFTB, it is SPL.\(^{61}\)

**Theorem 7**

Since $(U^K, Q^K)$ ensures a feasible match, we know that $DQ^\Psi$ ends no later than stage $K$ for every $P_I$; for some $P_I$, it may end in some stage $k < K$ if $\psi_i^{(U^K, Q^K)}(P_I)$ is a feasible match. For the second part, $\eta$ being a single-seat reduction sequence implies $\sum_{\theta \in \Theta} (U^{k,\theta}_s - U^{K,\theta}_s) \leq Q^k_s - Q^K_s$ for all $s \in S$. Then, by resource monotonicity, $DQ^\Psi_i(P_I) = \psi_i^{(U^K, Q^K)}(P_I) R_i \psi_i^{(U^K, Q^K)}(P_I)$ for all $i \in I$. Since this holds for all $P_I$, the result follows.

\(^{61}\)Our model is slightly different, since the number of students in each group $g$ grows deterministically, rather than stochastically. In Appendix C of their paper, Azevedo and Budish (2013) note that their proofs hold when the number of students in each group grows deterministically and the utility types are drawn iid within each group, which is the case for our model.
Theorem 8

Consider a preference profile $P_I$ such that $P_i$ ranks $O$ first for every $i \in I$. Individual rationality requires that $\psi_i(P_I) = O$ for all $i \in I$. But now, $\psi_s(P_I) = \emptyset$, and so $\psi(P_I)$ is not feasible.

Theorem 9

As in the proof of Theorem 7, $DQ^\psi(P_I) = \psi^{(U^k,Q^k)}(P_I)$ for some $k \leq K$. The same argument there shows part (1). For part (2), assume that $DQ^\psi$ ends in some stage $k < K$ (if not, the result is obvious). By part (1), $DQ^\psi_i(P_I)R_i\psi_i^{(U^k,Q^k)}(P_I)R_iO$, which means that $DQ^\psi_i(P_I) = O \implies \psi_i^{(U^k,Q^k)}(P_I) = O$, which delivers the result. For part (3), if $DQ^\psi$ ends in stage $K$, then $DQ^\psi(P_I) = \psi^{(U^k,Q^k)}(P_I)$, and so both mechanisms fill the same number of floor seats. If $DQ^\psi$ ends in some stage $k < K$, then the resulting matching $\psi^{(U^k,Q^k)}(P_I)$ must be feasible, i.e., there are zero floor seats that are left unfilled, and so the result holds.

C Proofs of lemmas

In this section, we collect the proofs of various lemmas used in the previous two appendices.

Proof of Lemma 1

We first start with two sub-lemmas.

Lemma 6. If $k' \geq k$, then $B^{k'}_s(\hat{T}^k) \subseteq B^k_s(\hat{T}^{k'})$ for all $s$.

Proof. By monotonicity, $Ch^k_s(I') \subseteq Ch^{k'}_s(I')$ for all $s \in S$ and all $I' \subseteq I$. The statement then follows from Lemma 1, part (2) of Kamada and Kojima (2013). $\blacksquare$

The next sub-lemma makes use of the following definition of stability. Note that this is only a technical definition used to prove the results below, and is not related to the justified envy definitions used in the main text. Let $Ch' := \{Ch'_1, \ldots, Ch'_m\}$ be a vector of choice functions (which again need not be equal to Ch).

Definition 10. A matching $\mu$ is stable with respect to $Ch'$ if:

(i) $\mu(s) = Ch'_s(\mu(s))$ for all $s \in S$

(ii) there exist no pair $(i, s)$ such that $sP_i\mu(i)$ and $i \in Ch'_s(\mu(s) \cup \{i\})$.

Part (i) says that a school does not unilaterally reject any student assigned to it. The corresponding “individual rationality” property holds for students automatically, because we assume that all students find all schools acceptable. Given this definition, we have the following lemma.

$\blacksquare$See also the “Capacity Lemma” of Konishi and Unver (2006) for a related result.
Lemma 7. $\nu^k$ is a matching and is stable with respect to $Ch^k$. 

Proof. We must first show that $\nu^k$ as defined is a matching, i.e., that at the end of the cumulative offer process, no student is assigned to more than one school. Within a stage $k$, substitutability implies $Rej_s^k(A_s^k(t-1)) \subseteq Rej_s^k(A_s^k(t))$ for all $t = 1, \ldots, T^k$, and across stages, monotonicity ensures that $Rej_s^{k-1}(A_s^{k-1}(T^{k-1})) \subseteq Rej_s^k(A_s^k(0))$ for all $s$. Thus, if $i$ is rejected at some step $t$ of some stage $k$, then $i$ is rejected in all later steps and stages, implying that no student is assigned to more than one school. In addition, by irrelevance of rejected students (see footnote 45), $Ch_s^k(A_s^k(T^k)) = \nu^k(s) = Ch_s^k(\nu^k(s))$ and $i \notin Ch_s^k(\nu^k(s) \cup \{i\})$ for all $i$ such that $sP_i \nu^k(i)$, since $i$ must have been rejected from $s$. Therefore, $\nu^k$ is stable with respect to $Ch^k$. 

We now use induction to show

$$A_s^k(T^k) = B_s^k(\hat{t}^k)$$

for all $s \in S$.\footnote{It generally will not be the case that $T^k = \hat{t}^k$.} Since in stage 1, both algorithms are just standard DA using $Ch^1$, (2) holds for $k = 1$. Assume the inductive hypothesis that (2) holds for $1, \ldots, k - 1$. We show that this implies it holds for $k$ as well.

First, note that since stage $k$ of GDQDA is simply the DA algorithm under $Ch^k$, we know that $\mu^k$ is the student-optimal stable match with respect to $Ch^k$ (Hatfield and Milgrom (2005)). Further, $\nu^k$ is some stable match with respect to $Ch^k$, which means that $\mu^k(i)R_i \nu^k(i)$ for all $i \in I$. This implies that $B_s^k(\hat{t}^k) \subseteq A_s^k(T^k)$ for all $s \in S$.\footnote{If this were not the case, then there exists some $s$ such that $A_s^k(T^k) \not\subseteq B_s^k(\hat{t}^k)$, and let $t'$ be the first step of stage $k$ of PDQDA such that $A_s^k(t') \not\subseteq B_s^k(\hat{t}^k)$ for all $t < t'$ and all $s \in S$, but $A_s^k(t') \not\subseteq B_s^k(\hat{t}^k)$. Let $i$ be the student who applies to $s$ at step $t'$. This means that $i$ is rejected from $\mu^k(i)$ (the school she is matched to under GDQDA) in some step of stage $k$ of PDQDA; let the earliest of these steps be $t''$, so that $i \in Rej^k_{\mu^k(i)}(A_{\mu^k(i)}(t''))$. Further, note that $t'' < t'$, which, by the definition of $t'$, implies that $A_{\mu^k(i)}(t'') \subseteq B_{\mu^k(i)}(\hat{t}^k)$. Substitutability of the choice functions within stage $k$ then implies that that $Rej^k_{\mu^k(i)}(A_{\mu^k(i)}(t'')) \subseteq Rej^k_{\mu^k(i)}(B_{\mu^k(i)}(\hat{t}^k))$, which means $i \in Rej^k_{\mu^k(i)}(B_{\mu^k(i)}(\hat{t}^k))$, which contradicts the fact that $i$ is assigned to school $\mu^k(i)$ under GDQDA in stage $k$.}

$$\mu^k(i)R_i \nu^k(i).$$\footnote{Such a $t'$ exists because $A_{\mu^k(i)}(0) = A_{\mu^k(i)}^{-1}(T^k - 1) = B_{\mu^k(i)}^{-1}(\hat{t}^k - 1) \subseteq B_{\mu^k(i)}(\hat{t}^k)$ for all $s \in S$, where the first equality is by definition, the second is by the inductive hypothesis, and the set inclusion is by Lemma 6.}

\footnote{Note that $i$ must be rejected at some step $t''$ of stage $k$ (and not in an earlier stage). To see this, assume that $i$ was rejected from $\mu^k(i)$ in some earlier stage $k'$ of PDQDA. This implies that $\mu^k(i)P_i \nu^k(i)$. Since $k' < k$, Lemma 1, part (1) of Kamada and Kojima (2013) implies that $\mu^k(i)P_i \mu^k(i)$. Combining these two inequalities, we conclude that $\mu^k(i)P_i \nu^k(i)$, which contradicts the inductive hypothesis.}
Proof of Lemma 2

Substitutability

Consider a school \( s \), stage \( k \), and set of students \( I' \) such that \( i \in \text{Rej}_s^k(A') \), and another set of students \( A'' \) such that \( A' \subseteq A'' \). Let \( \tau(i) = \theta \). When the set of applicants is \( A' \), student \( i \) is rejected because the type \( \theta \) specific seats are filled with \( L_{s,\theta} \) higher ranked type \( \theta \) students, and the open seats are filled with either (i) \( U_{s,0}^k - L_{s,0} \) higher ranked type \( \theta \) students or (ii) \( Q_s^k - \sum_{\theta \in \Theta} L_{s,\theta} \) higher ranked students of any type. In either case, since all students in open seats are filled with either (i) \( U_{s,0}^k - L_{s,0} \), and set of students \( \Theta \), the type \( \theta \) specific seats will once again be filled with \( L_{s,\theta} \) higher ranked type \( \theta \) students, and either condition (i) or (ii) will also still hold. So, \( i \in \text{Rej}_s^k(A'') \) as well.

Monotonicity

If \( s' \) is not the school whose quotas are reduced in moving from stage \( k \) to \( k+1 \), then \( \text{Rej}_s^k(A) = \text{Rej}_{s'}^{k+1}(A) \) trivially. So, let \( s \) be the school whose capacity is reduced in moving from stage \( k \) to \( k+1 \): \( Q^k_s = Q^k_s - 1 \) and \( U_{s,\theta}^{k+1} = U_{s,\theta}^k - 1 \), while \( U_{s,\theta'}^{k+1} = U_{s,\theta'}^k \) for all other \( \theta' \neq \theta \).

We want to show that \( \text{Rej}_s^k(A) \subseteq \text{Rej}_s^{k+1}(A) \) for all \( k \) and all \( A \subseteq I \).

To do so, we show the contrapositive:

\[
i \in \text{Ch}_s^{k+1}(A) \implies i \in \text{Ch}_s^k(A).
\]

Assume not, and let \( i \) be the highest ranked student according to \( \succ_s \) such that \( i \in \text{Ch}_s^{k+1}(A) \), but \( i \notin \text{Ch}_s^k(A) \) (equivalently, \( i \in \text{Rej}_s^k(A) \)). Let \( \tau(i) = \theta' \), which may or may not be equal to \( \theta \). If \( i \) is admitted through the type \( \theta' \) specific seats in stage \( k+1 \), then she is one of the \( L_{s,\theta'} \) highest ranked type \( \theta' \) students in \( A \), and so she will be admitted in stage \( k \) as well. So, \( i \) must be admitted through an open seat in stage \( k+1 \). Therefore, when \( i \)'s application is considered in stage \( k+1 \), the following both hold: (a) at most \( U_{s,\theta'}^{k+1} - L_{s,\theta'} - 1 \) higher ranked type \( \theta' \) students have been accepted to the open seats and (b) at most \( Q_s^{k+1} - \sum_{\theta \in \Theta} L_{s,\theta} - 1 \) students in total have been accepted to the open seats. Define \( J = \{ j \in \text{Rej}_s^{k+1}(A) : j \succ_s i \} \). Note that (b) implies that for all \( j \in J \), the type-specific ceiling \( U_{s,\tau(j)}^{k+1} \) is reached before \( j \)'s application is considered in stage \( k+1 \).

Since \( Q_s^k = Q_s^{k+1} + 1 \), for \( i \) to be rejected in stage \( k \), there must be two \( j_1, j_2 \in J \) such that \( j_1, j_2 \in \text{Ch}_s^k(A) \); without loss of generality, let \( j_1 \succ_s j_2 \succ_s i \). Since \( j_1 \in \text{Ch}_s^k(A), \) it must be that \( \tau(j_1) = \theta \).

Then, after \( j_1 \) is admitted, the \( U_{s,\theta}^k \) ceiling is now binding (since \( U_{s,\theta}^k = U_{s,\theta}^{k+1} + 1 \) and all \( j \succ_s j_1 \) who are admitted in stage \( k+1 \) are also admitted in stage \( k \) by the assumption that \( i \) is the highest-ranked such student for which this is not the case). So, when \( j_2 \)'s application is considered in stage \( k+1 \), the following both hold: (a) at most \( U_{s,\theta'}^{k+1} - L_{s,\theta'} - 1 \) higher ranked type \( \theta' \) students have been accepted to the open seats and (b) at most \( Q_s^{k+1} - \sum_{\theta \in \Theta} L_{s,\theta} - 1 \) students in total have been accepted to the open seats. Define \( J = \{ j \in \text{Rej}_s^{k+1}(A) : j \succ_s i \} \). Note that (b) implies that for all \( j \in J \), the type-specific ceiling \( U_{s,\tau(j)}^{k+1} \) is reached before \( j \)'s application is considered in stage \( k+1 \).

\[^{68}\text{An inductive argument then implies that } \text{Rej}_s^k(A) \subseteq \text{Rej}_s^{k'}(A) \text{ for all } k' \geq k.\]

\[^{67}\text{If } \tau(j_1) \neq \theta, \text{ then } U_{s,\tau(j_1)}^k = U_{s,\tau(j_1)}^{k+1}. \text{ Then, since all } j \succ_s j_1 \text{ such that } j \text{ is accepted in stage } k+1 \text{ are also accepted in stage } k \text{ (by the assumption that } i \text{ is the highest-ranked student for which this is not the case), the ceiling for type } \tau(j_1) \text{ students will already have been reached in stage } k \text{ when } j_1 \text{'s application is considered, and } j_1 \text{ will be rejected.}\]
considered, she will be rejected, which is a contradiction.\footnote{If \(\tau(j_2) = \theta\), this follows from the previous sentence. If \(\tau(j_2) \neq \theta\), it follows from the same argument as in footnote 68.}

\textbf{Minimality}

As for monotonicity, we only need consider the school \(s\) whose quotas are reduced in moving from \(k\) to \(k + 1\). Let \(s\) be the school such that \(Q^{k+1}_s = Q^k_s - 1\) and \(U^{k+1}_{s,\theta} = U^k_{s,\theta} - 1\) for some \(\theta\), while \(U^{k+1}_{s,\theta'} = U^k_{s,\theta'}\) for all \(\theta' \neq \theta\). Consider a set of applicants \(\mathcal{A} \subseteq I\), with \(\text{Ch}^k_s(\mathcal{A})\) the set that is admitted in stage \(k\). There are four cases.

\textbf{Case (i):} \(|\text{Ch}^k_s(\mathcal{A})| < Q^k_s\) and \(|\text{Ch}^k_s(\mathcal{A}) \cap I_\theta| < U^k_{s,\theta}\)

In this case, \(\text{Ch}^{k+1}_s(\mathcal{A}) = \text{Ch}^k_s(\mathcal{A})\), which implies \(|\text{Ch}^k_s(\mathcal{A})| - |\text{Ch}^{k+1}_s(\mathcal{A})| = 0\).

\textbf{Case (ii):} \(|\text{Ch}^k_s(\mathcal{A})| = Q^k_s\) and \(|\text{Ch}^k_s(\mathcal{A}) \cap I_\theta| < U^k_{s,\theta}\)

Let \(i'\) be the lowest-ranked student admitted through the open seats in stage \(k\). Then, \(\text{Ch}^{k+1}_s(\mathcal{A}) = \text{Ch}^k_s(\mathcal{A}) \setminus \{i'\}\), which implies \(|\text{Ch}^k_s(\mathcal{A})| - |\text{Ch}^{k+1}_s(\mathcal{A})| = 1\).

\textbf{Case (iii):} \(|\text{Ch}^k_s(\mathcal{A})| < Q^k_s\) and \(|\text{Ch}^k_s(\mathcal{A}) \cap I_\theta| = U^k_{s,\theta}\)

Let \(i'\) be the lowest-ranked type \(\theta\) student admitted through the open seats in stage \(k\). Then, \(\text{Ch}^{k+1}_s(\mathcal{A}) = \text{Ch}^k_s(\mathcal{A}) \setminus \{i'\}\), which implies \(|\text{Ch}^k_s(\mathcal{A})| - |\text{Ch}^{k+1}_s(\mathcal{A})| = 1\).

\textbf{Case (iv):} \(|\text{Ch}^k_s(\mathcal{A})| = Q^k_s\) and \(|\text{Ch}^k_s(\mathcal{A}) \cap I_\theta| = U^k_{s,\theta}\)

Let \(i'\) be the lowest-ranked type \(\theta\) student admitted through the open seats in stage \(k\). Then, \(\text{Ch}^{k+1}_s(\mathcal{A}) = \text{Ch}^k_s(\mathcal{A}) \setminus \{i'\}\), which implies \(|\text{Ch}^k_s(\mathcal{A})| - |\text{Ch}^{k+1}_s(\mathcal{A})| = 1\).

\textbf{Law of aggregate demand}

Within a stage, school \(s\) admits students one-by-one until either some type-specific ceiling \(U^k_{s,\theta}\) or overall capacity \(Q^k_s\) has been reached. Therefore, more students in the applicant pool weakly increases the number of students admitted, and the law of aggregate demand is satisfied.

\textbf{Proof of Lemma 3}

We will use the following facts about \(\delta_s\):

(a) \(\mathcal{A} \subseteq \mathcal{A}^' \implies \delta_s(\mathcal{A}^') \leq \delta_s(\mathcal{A})\)

(b) if \(\delta_s(\mathcal{A} \cup \{i\}) < \delta_s(\mathcal{A}) \implies \text{Rej}^k_s(\mathcal{A} \cup \{i\}) = \text{Rej}^k_s(\mathcal{A})\) for all \(k\).

Fact (a) follows immediately from the definition of \(\delta\). Fact (b) follows because \(\delta_s(\mathcal{A} \cup \{i\}) < \delta_s(\mathcal{A})\) implies that student \(i\) is of some type \(\theta\) such that \(|\mathcal{A} \cap I_\theta| < L_{s,\theta}\). But this means that when the applicant pool at school \(s\) is \(\mathcal{A} \cup \{i\}\), student \(i\) is accepted through one of the type \(\theta\) seats. This does not affect the students accepted by the type \(\theta'\) seats for \(\theta' \neq \theta\), nor the open seats, and thus \(\text{Ch}^k_s(\mathcal{A} \cup \{i\}) = \text{Ch}^k_s(\mathcal{A})\), or equivalently, \(\text{Rej}^k_s(\mathcal{A} \cup \{i\}) = \text{Rej}^k_s(\mathcal{A})\).

Part (i): Since \(\Delta^k = 0\), the set \(\tilde{\mathcal{A}}^k_s\) contains at least \(L_{s,\theta}\) students of type \(\theta\) for all schools \(s\). Let \(\mathcal{A}^k_s(T^k)\) be the cumulative set of applicants to school \(s\) at the end of stage \(k\). Since \(\tilde{\mathcal{A}}^k_s \subseteq \mathcal{A}^k_s(T^k)\)
for any submitted preferences of student \( i \), \( \text{Ch}^k(A^k_s(T^k)) \) is a feasible assignment for school \( s \), and the algorithm ends in stage \( k \).

Part (ii): We can have \( \Delta^k > 1 \) in two ways: either \( \delta_s(\hat{A}^k_s) > 1 \) for some school, or \( \delta_s(\hat{A}^k_s) \) for multiple schools. First, consider \( \delta_s(\hat{A}^k_s) > 1 \) for some school \( s \). This means that there are at least 2 floor seats at \( s \) that are not yet filled because not enough students have applied to \( s \). When student \( i \) enters the market in stage \( k \), he causes a rejection chain that ends the first time a school gets an application from a student \( i' \) and does not reject an additional student. Therefore, at the end of stage \( k \), at most one of the unfilled floor seats at \( s \) can be filled, and so the assignment of school \( s \) will still not be feasible.

The case where \( \delta_s(\hat{A}^k_s) \geq 1 \) for multiple schools is argued similarly.

Part (iii): By Theorem 10, \( \hat{A}^{k+1} \) can be computed by starting with \( \hat{A}^k \) and then reducing the choice functions to \( \text{Ch}^{k+1} \). Doing so causes a rejection chain that ends the first time a student \( i' \) applies to a school \( s' \) and \( s' \) does not reject a student. Since \( \hat{A}^k \subseteq \hat{A}^{k+1} \), we have \( \delta_s(\hat{A}^{k+1}) \leq \delta_s(\hat{A}^k) \), which implies that \( \Delta^k \geq \Delta^{k+1} \). To see that \( \Delta^{k+1} \geq \Delta^k - 1 \), note that in the rejection chain, the first time a student applies to a school and fills a floor, no further student is rejected (by fact (b)) and the rejection chain ends. Thus, at the end of the rejection chain, at most one floor seat that wasn’t filled under \( \tilde{\mu}^k \) can be filled under \( \tilde{\mu}^{k+1} \), and so \( \Delta^{k+1} \geq \Delta^k - 1 \).

**Proof of Lemma 4**

We use the notation \((k, t)\) to denote the line corresponding to step \( k \), stage \( t \) of a rejection chain. Let \( A^k_s(t) \) denote the cumulative offer set of school \( s \) at line \((k, t)\) of \( \mathcal{R}(S_i) \), and \( \hat{A}^k_s(t) \) denote the corresponding set at line \((k, t)\) of \( \mathcal{R}(\hat{S}_i) \). Similarly, let \( t^k \) denote the final step of stage \( k \) under scenario \( S_i \), and \( \hat{t}^k \) denote the final step of stage \( k \) under scenario \( \hat{S}_i \). Last, let \( k_{end} \) denote the last stage of \( \mathcal{R}(S_i) \) and \( \hat{k}_{end} \) denote the last stage of \( \mathcal{R}(\hat{S}_i) \).

We prove the result by induction on the line index \((k, t)\). Line \((k^*, 0)\) of \( \mathcal{R}(\hat{S}_i) \) is “\( i \) applies to \( s' \)”, and this step occurs in \( \mathcal{R}(S_i) \) by assumption. So, make the inductive assumption that all lines up to \((k, t - 1)\) of \( \mathcal{R}(\hat{S}_i) \) also occur in \( \mathcal{R}(S_i) \). Then, consider the next line in \( \mathcal{R}(\hat{S}_i) \). There are three cases:

**Case (i): The next line \((k, t)\) is an application line.**

Line \((k, t)\) then reads “\( i' \) applies to \( s'' \)”. There are two cases. If \( i' = i \), then this application also occurs in \( \mathcal{R}(S_i) \) by assumption. If \( i' \neq i \), then let \( u \) be the school immediately before \( s' \) on the preference list of \( i' \). Because \((k, t)\) is an application line, \((k, t - 1)\) must be a rejection line in which student \( i' \) is rejected by \( u \). Since, by the inductive hypothesis, line \((k, t - 1)\) occurs somewhere in \( \mathcal{R}(S_i) \), student \( i' \) must be rejected from \( u \) at some point in \( \mathcal{R}(S_i) \), and will then, according to his preferences, apply to \( s' \) in the following line.

**Case (ii): The next line \((k, t)\) is a rejection line.**


Line \((k, t)\) then reads “\(s’\) rejects \(i’\)”. Thus, student \(i’\) must have already applied to \(i’\), either before \(i\) entered the market, or somewhere in rejection chain \(R(\hat{S}_i)\). The choice function at \(s’\) when \(i’\) is rejected is \(Ch_{s’}^{\hat{i}'}\), and the set of cumulative applicants is \(\hat{A}_{s’}^k(t)\), which implies that \(i’ \in \text{Rej}_{s’}^k(\hat{A}_{s’}^k(t))\).

By the inductive hypothesis, all students in \(\hat{A}_{s’}^k(t)\) also apply to \(s’\) under scenario \(S_i\); in other words, \(\hat{A}_{s’}^k(t) \subseteq A_{s’}^{k_{end}}(T_{k_{end}}^k)\). Further, the inductive hypothesis plus monotonic substitutability imply that \(\text{Rej}_{s’}^k(\hat{A}_{s’}^k(t)) \subseteq \text{Rej}_{s’}^{k_{end}}(\hat{A}_{s’}^k(t)) \subseteq \text{Rej}_{k_{end}}(A_{s’}^{k_{end}}(T_{k_{end}}^k))\). So, \(i’\) must be rejected from \(s’\) at some point under scenario \(S_i\), i.e., line \((k, t)\) must occur in \(R(S_i)\).

**Case (iii): The next line \((k + 1, 0)\) is a choice function reduction line.**

In this case, \((k + 1, 0)\) reads “The choice functions become \(Ch^{k+1}\)”. Assume to the contrary that this reduction does not occur under \(S_i\). Thus, \(R(S_i)\) ends in stage \(k\) under \(Ch^k\) (by the inductive hypothesis, \(R(S_i)\) reaches at least stage \(k\)). By Theorem 10, an alternative way to compute the outcome at the end of stage \(k\) under either scenario is to start with the empty matching and run the DA algorithm under \(\hat{S}_i\) and \(S_i\). Recall that \(\hat{A}_s^k\) is the cumulative set of applicants to school \(s\) before \(i\) enters the market (as before, since the preferences of all agents \(-i\) do not change, this is the same under either case). Note that \(\delta_s(\hat{A}_s^k) = 0\) for all \(s \neq y\), and \(\delta_y(\hat{A}_y^k) = 1\), where \(y\) has one type \(\phi\) floor seat left to be filled (if the latter did not hold, all school choice sets would be feasible even before \(i\) enters, and the mechanism would not continue to stage \(k + 1\) under scenario \(\hat{S}_i\)). Just as above, we can write a rejection chain for each scenario corresponding to running the DA algorithm **within stage** \(k\). Let \(R^k(S_i)\) and \(R^k(\hat{S}_i)\) denote these two rejection chains.

Since \(R(S_i)\) ends in stage \(k\), \(R^k(S_i)\) must end in one of two ways:

1. \(y\) gets an application from some student \(j\) of type \(\phi\):

   Stage | Step | Action | Offer sets
   ---- | ---- | ----- | --------
   \(k\) | \(t\) | \(j\) applies to \(y\) | \(A_y^k(t) = A_y^k(t-1) \cup \{j\}\) \(A_s^k(t) = A_s^k(t-1)\) for all \(s \neq s\)

2. \(i\) is rejected by the last school in \(S_i\):

   Stage | Step | Action | Offer sets
   ---- | ---- | ----- | --------
   \(k\) | \(t\) | \(i\) is rejected by \(s\) | \(A_s^k(t) = A_s^k(t-1)\) \(A_s^k(t) = A_s^k(t-1)\) for all \(s \neq s\)

On the other hand, \(R(\hat{S}_i)\) does not end in stage \(k\). This means that \(R^k(\hat{S}_i)\) must end with

Stage | Step | Action | Offer sets
---- | ---- | ----- | --------
\(k\) | \(t\) | Student \(i'\) applies to school \(v\) | \(\hat{A}_v^k(t) = \hat{A}_v^k(t-1) \cup \{i'\}\) \(\hat{A}_s^k(t) = \hat{A}_s^k(t-1)\) for all \(s \neq s\)

---

\(^{70}\) The inductive hypothesis implies that \(k_{end} \geq k\), and so the first set inclusion follows by monotonicity. The second follows by substitutability and \(\hat{A}_{s'}^k(t) \subseteq A_{s'}^{k_{end}}(T_{k_{end}}^k)\).
where either $v \neq y$ or $i'$ is not of type $\phi$.

Now, by the Scenario Lemma of Dubins and Freedman (1981), this last step of $R^k(\hat{S}_i)$ must also occur in $R^k(S_i)$. But, this means that this is the last step of $R^k(S_i)$, contradicting the above.

### D On the optimality of DQDA with multiple types

Theorem 4 shows that in the single type case, any DQDA algorithm that starts at the true capacities ($Q^1 = Q$) is on the Pareto frontier of strategyproof mechanisms. With multiple types, we show here that this is not the case.

Consider the following market.

<table>
<thead>
<tr>
<th></th>
<th>$L_{s,\ell}$</th>
<th>$L_{s,h}$</th>
<th>$U_{s,\ell}$</th>
<th>$U_{s,h}$</th>
<th>$Q_s$</th>
<th>$\succeq_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>$h_1 \succeq_s h_2 \succeq_s \ell_1$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$h_1 \succeq_s \ell_1 \succeq_s h_2$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>$\ell_1 \succeq_s h_2 \succeq_s h_1$</td>
</tr>
</tbody>
</table>

Let $\eta = \{(U^1, Q^1), (U^2, Q^2)\}$, where

$$(U^1, Q^1) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad (U^2, Q^2) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$ 

Note that $(U^1, Q^1) = (U, Q)$ and that $(U^2, Q^2)$ ensures a feasible match. Consider the preferences $P_t = (P_{h_1}, P_{h_2}, P_{\ell_1})$, where

$P_{h_1} : s_1, s_3, s_2$

$P_{h_2} : s_2, s_3, s_1$

$P_{\ell_1} : s_1, s_2, s_3$

and $P'_t = (P_{h_1}, P_{h_2}, P'_{\ell_1})$ where $P'_{\ell_1} = s_1, s_3, s_2$.

The output of the DQDA under these two profiles is

$$DQDA(P_t) = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & \{h_1, h_2\} & \{h_1\} \end{pmatrix}, \quad DQDA(P'_t) = \begin{pmatrix} s_1 & s_2 & s_3 \\ \emptyset & \{h_2\} & \{h_1, \ell_1\} \end{pmatrix}. $$

Consider an alternative mechanism $\psi$ defined by

$$\psi(P_t) = \psi(P'_t) = \begin{pmatrix} s_1 & s_2 & s_3 \\ \ell_1 & h_2 & h_1 \end{pmatrix}$$

and $\psi(P''_t) = DQDA(P''_t)$ for all $P''_t \neq P_t, P'_t$. Since, at $P_t$ and $P'_t$, $\psi$ makes $\ell_1$ better off, and does not change the assignments of the other students, $\psi$ dominates DQDA. In addition, $\psi$
is strategyproof. This can be seen as follows: first, because $\psi_i(P'') = DQDA_i(P''_i)$ for all $P''_i$ for $i = h_1, h_2$, strategyproofness of DQDA implies that students $h_1$ and $h_2$ have no profitable manipulations under $\psi$. Further, under $\psi$, student $\ell_1$ always receives her first choice school with respect to her reported preferences, and thus, no manipulation is profitable.

Thus, in this market, $\psi$ Pareto dominates DQDA under $\eta$ and is strategyproof. However, this is only one choice of $\eta$; since $\eta$ is not a primitive of the model, but rather is part of the description of a mechanism, we are free to choose $\eta$. So, consider an alternative reduction sequence $\hat{\eta} = \{(\hat{U}^1, \hat{Q}^1), (\hat{U}^2, \hat{Q}^2)\}$, where

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$  

In this case, DQDA produces

$$DQDA(P_I) = \begin{pmatrix} s_1 & s_2 & s_3 \\ h_1 & \ell_1 & h_2 \end{pmatrix}.$$  

Note that $\psi$ does not Pareto dominate $DQDA$, because student $h_1$ prefers her assignment under $DQDA$. In fact, no mechanism Pareto dominates $DQDA$. This is true because student $h_1$ cannot be made better off, while making either student $h_2$ or student $\ell_1$ better off while still ensuring feasibility would require making student $h_1$ worse off. Thus, in this market, DQDA under $\hat{\eta}$ is on the Pareto frontier of strategyproof mechanisms. Whether such an $\hat{\eta}$ can be found in the multiple type case for markets of any size is an open question.

E  EDQDA is not strategyproof

The proof is by example. Consider a market of 4 schools $S = \{s_1, s_2, s_3, s_4\}$ and three students $I = \{h_1, h_2, \ell_1\}$, with type space $\Theta = \{h, \ell\}$. Students $h_1$ and $h_2$ are of type $h$ and $\ell_1$ is of type $\ell$. (This is the same as the examples used in the main text.) The preferences of the students and priorities of the schools are as follows.

<table>
<thead>
<tr>
<th>$L_{s,\ell}$</th>
<th>$L_{s,h}$</th>
<th>$U_{s,\ell}$</th>
<th>$U_{s,h}$</th>
<th>$Q_s$</th>
<th>$\succ_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$h_1 \succ_{s_1} h_2 \succ_{s_1} \ell_1$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$h_1 \succ_{s_2} \ell_1 \succ_{s_2} h_2$</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>$\ell_1 \succ_{s_3} h_2 \succ_{s_3} h_1$</td>
</tr>
<tr>
<td>$s_4$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>$h_1 \succ_{s_4} h_2 \succ_{s_4} \ell_1$</td>
</tr>
</tbody>
</table>

Let $\rho = \{(s_2, h), (s_3, h)\}$. First consider the following preferences:
The output in stage 1 of EDQDA is

\[
\mu^1 = \left( \begin{array}{cccc}
s_1 & s_2 & s_3 & s_4 \\
\emptyset & h_1 & \ell_1 & h_2 \\
\end{array} \right)
\]

Since \( \mu^1 \) satisfies all of the true capacities and quotas, the algorithm ends here and the final matching is \( \mu^1 \). Now, consider an alternative preference profile where \( P_{h_1} \) and \( P_{h_2} \) are unchanged, but student \( \ell_1 \) submits preference \( P_{\ell_1}' : s_1, s_2, s_3, s_4 \). At the end of stage 1, the matching is

\[
\nu^1 = \left( \begin{array}{cccc}
s_1 & s_2 & s_3 & s_4 \\
\ell_1 & h_1 & \ell_2 & \emptyset \\
\end{array} \right).
\]

Note that the type \( h \) lower quota at \( s_4 \), \( L_{s_4,h} = 1 \), is not satisfied. Thus, according to \( \rho \), we lower the type \( h \) ceiling at \( s_2 \), and run DA again. The stage 2 output is

\[
\nu^2 = \left( \begin{array}{cccc}
s_1 & s_2 & s_3 & s_4 \\
h_1 & \ell_1 & \ell_2 & \emptyset \\
\end{array} \right).
\]

Once again, the lower quota at \( s_4 \) is not satisfied, so we move to stage 3, lowering the \( h \) ceiling at \( s_3 \). The output is

\[
\nu^3 = \left( \begin{array}{cccc}
s_1 & s_2 & s_3 & s_4 \\
h_1 & \ell_1 & \emptyset & h_2 \\
\end{array} \right).
\]

Since all quotas and capacities are satisfied, the algorithm ends here, and the final matching under the false preference is \( \nu^3 \). Note that student \( \ell_1 \) truly prefers \( s_2 \) to \( s_3 \), and so is better off submitting the false report \( P_{\ell_1}' \) instead of her true preferences \( P_{\ell_1} \).

The reason that the mechanism above was not strategyproof was that by lying, student \( \ell_1 \) prevents a lower quota for type \( h \) from being filled in round 1, leading to a seat for type \( h \) to be cut at \( s_2 \). Then, in round 2, since student \( h_1 \) can no longer be admitted to \( s_2 \), student \( \ell_1 \) can, and is thus made better off.

Note that in this case, the reduction sequence does not satisfy our assumption of monotonicity. To see this, let \( I' = \{ \ell_1, h_1 \} \), and note

\[
\text{Rej}^1_{s_2}(I') = \{ \ell_1 \} \not\subseteq \{ h_1 \} = \text{Rej}^2_{s_2}(I').
\]

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Consider two vectors \((U', Q')\) and \((U'', Q'')\) as in Definition 4, and let \(IA^{(U', Q')}\) and \(IA^{(U'', Q'')}\) denote the immediate acceptance algorithm under each. Assume to the contrary that \(IA^{(U'', Q'')}\) did not Pareto dominate \(IA^{(U', Q')}\). Consider a student \(i\) of type \(\theta\) such that \(IA^{(U', Q')}_{\theta}(P_1) = s'\), \(IA^{(U'', Q'')}_{\theta}(P_1) = s''\), and \(s'P,is''\) for some preference profile \(P_1\). Since \(IA^{(U'', Q'')}\) does not Pareto dominate \(IA^{(U', Q')}\), at least one such student and preference profile must exist. Say that under \(P_1\), student \(i\) ranks \(s'\) as her \(k\)th choice. Further, let \(i\) be a student who, in the running of \(IA^{(U'', Q'')}\), is rejected from the corresponding \(s'\) earliest.\(^{71}\)

Let \(J'(s')\) denote the set of students who, under \(IA^{(U', Q')}\), apply to \(s'\) in some step \(k' < k\) or who apply in step \(k\), but have higher priority at \(s'\) than \(i\), and let \(J'_A(s') \subseteq J'(s')\) be the subset of these students who are accepted by \(s'\). Further, let

\[
\Theta' = \{\theta \in \Theta : U'_{s', \theta} \text{ is binding when } i\text{'s application is considered}\},
\]

and let \(\bar{\Theta}' = \Theta \setminus \Theta'\). Note that for any \(\bar{\theta} \in \bar{\Theta}'\), no type \(\bar{\theta}\) students are rejected from \(s'\) before \(i\)’s application is considered under \(IA^{(U', Q')}\), because both \(U'_{s', \bar{\theta}}\) and \(Q'_{s'}\) are slack when \(i\)’s application is considered. This means that \(|J'_A(s')| = \sum_{\bar{\theta} \in \bar{\Theta}} U'_{s', \bar{\theta}} + \sum_{\bar{\theta} \in \bar{\Theta}} |J'(s') \cap I_{\bar{\theta}}| < Q'_{s'}\), where the second summation comes from the fact that all students with types in \(\Theta'\) who are considered before \(i\) are accepted, because the capacity constraint is slack (since \(i\) is accepted) and the type-specific ceiling is slack (by definition of \(\bar{\Theta}'\)).

Now consider running immediate acceptance under \((U'', Q'')\). Analogously define the sets \(J''(s')\) and \(J'_A(s')\). Now, we have \(J''(s') \subseteq J'(s')\).\(^{72}\) Note that \(|J''_A(s')| \leq \sum_{\bar{\theta} \in \bar{\Theta}} U''_{s', \bar{\theta}} + \sum_{\bar{\theta} \in \bar{\Theta}} |J'(s') \cap I_{\bar{\theta}}| < Q''_{s'}\).\(^{73}\) Thus, when \(i\)’s application is considered under \(IA^{(U'', Q'')}\), school \(s'\) still has an open seat. In addition, the type-specific ceiling \(U''_{s', \theta}\) is not binding because \(U''_{s', \bar{\theta}} \geq U'_{s', \bar{\theta}} > |J''(s') \cap I_{\bar{\theta}}|\), where the last inequality follows from \(J''(s') \subseteq J'(s')\) and \(|J'(s') \cap I_{\bar{\theta}}| < U'_{s', \bar{\theta}}\). So, \(i\) is not rejected from \(s'\) under \(IA^{(U'', Q'')}\), which is a contradiction.

\(^{71}\)That is, there is no student \(j\) who is rejected from her match \(IA^{(U', Q')}_j(P_j)\) in an earlier step of \(IA^{(U'', Q'')}\). There may be multiple such students rejected in stage \(k\), in which case, we choose some student \(i\) who has the highest priority at her corresponding \(s'\) of those students who applied to \(s'\) in stage \(k\) under \(IA^{(U', Q')}\).

\(^{72}\)If this were not the case, then some student \(j\) applies to \(s'\) before \(i\) is considered at \(s'\) under \(IA^{(U'', Q'')}\), but does not do so under \(IA^{(U', Q')}\). But this contradicts the fact that \(i\) is the earliest student rejected from her match \(IA^{(U', Q')}_{\theta}(P_1)\) in the running of \(IA^{(U', Q')}\).

\(^{73}\)The first inequality follows from the fact that for types in \(\Theta'\), at most \(U''_{s', \theta}\) students can be assigned to \(s\), and for types in \(\bar{\Theta}'\), at most \(|J'(s') \cap I_{\bar{\theta}}|\) have applied to \(s'\). The second inequality can be obtained by adding the inequalities \(\sum_{\bar{\theta} \in \bar{\Theta}} (U''_{s', \bar{\theta}} - U'_{s', \bar{\theta}}) \leq Q''_{s'} - Q'_{s'}\) (from the definition of \((U', Q')\) and \((U'', Q'')\)) and \(\sum_{\bar{\theta} \in \bar{\Theta}} |J'(s') \cap I_{\bar{\theta}}| < Q''_{s'} - \sum_{\bar{\theta} \in \bar{\Theta}} U''_{s', \bar{\theta}}\) (from the previous paragraph).
G Simulation appendix

In this appendix, we provide additional details on how \( \eta \) and \( \rho \) were chosen to run the simulations described in Section 6, and provide additional simulation results.

Constructing \( \eta \) and \( \rho \)

In this section, we describe in detail how we choose \( \eta \) and \( \rho \) to run the simulations in Section 6. Essentially, at each stage, we randomly choose one school-type pair \((s, \theta)\) for which the ceiling/capacity will be lowered, subject to feasibility constraints. More specifically, to construct \( \eta \), we start by setting \((U^K, Q^K) = (\bar{U}, \bar{Q})\). Then, we randomly choose some pair \((s, \theta)\) such that \(u^K_s < U_s, \theta < u^K_s, \theta + 1\) and \(q^K_s < Q_s < q^K_s + 1\). For the remaining \((s', \theta')\), \(u^{K-1}_{s', \theta'} = u^{K-1}_{s, \theta} + 1\) and \(q^{K-1}_{s'} = q^{K-1}_{s'} + 1\). For \((u^{K-2}, q^{K-2})\), we again randomly select another school-type pair different from \((s, \theta)\), and raise its ceiling and capacity by one (again, subject to the constraint that doing so does not violate the true ceilings and capacities). We continue in this manner until it is impossible to raise capacities any further without violating the true \((U, Q)\). This produces a sequence \(\eta = \{(U_1, Q_1), \ldots, (U^K, Q^K)\}\) which can then be used to run DQDA. We construct \( \eta \) “backwards”, starting from \((U^K, Q^K)\), only to simplify the coding. It can be done “forwards” as well.

To construct \( \rho \) for use in EDQDA, we start with the \( \eta \) just constructed (we begin with this \( \eta \) in order to make a fair comparison with DQDA). Starting from where we left off in the previous paragraph, we choose a pair \((s, \theta)\) randomly from those that have not yet reached their true type-specific ceiling, and raise this type-specific ceiling by 1 (note that just the ceilings are considered, not the capacities, since they are fixed in EDQDA). We continue to do so until we can no longer raise any \((s, \theta)\) ceiling further without violating the true \(U_s, \theta\). We then convert this sequence of ceiling-capacity vectors into the corresponding sequence of school-type pairs \( \rho \), as described in Section 6, which is then used to run EDQDA.
Additional simulation results

Below we present simulation results for additional parameter values. The first column corresponds to the low flexibility case, while the second corresponds to the high flexibility case. Each row corresponds to different values of the correlation parameter $\alpha$. Note that for image clarity, the vertical axes differ across figures.
Low flexibility

$\alpha = 0.195$

High flexibility

$\alpha = 0.195$

$\alpha = 0.26$

$\alpha = 0.325$
Low flexibility

Rank distribution for low flexibility and $\alpha = 0.39$

High flexibility

Rank distribution for high flexibility and $\alpha = 0.39$

Rank distribution for low flexibility and $\alpha = 0.455$

Rank distribution for high flexibility and $\alpha = 0.455$

Rank distribution for low flexibility and $\alpha = 0.52$

Rank distribution for high flexibility and $\alpha = 0.52$