Ex Post Implementation with Applications*

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ABSTRACT: This paper studies markets for heterogeneous goods using mechanism-design theory. For each combination of desirable properties, I derive an assignment process with these properties in the form of a corresponding direct-revelation game, or I show that it does not exist. Each participant’s utility is quasi-linear in money, and depend upon the allocation that he gets — thus, a participant’s privately known ‘type’ is multidimensional. The key properties are incentive compatibility, individual rationality, efficiency, and budget balance. The main results characterize mechanisms that are ex post incentive compatible in combination with other properties.

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1 Introduction

In this paper, we study a general heterogeneous goods market with transfers. This environment encompasses several classic models considered in market design literature: one-to-one matching markets, housing markets, roommate problems, seller-buyer markets with discrete heterogeneous goods, and partnership dissolution problems. Each agent’s preferences are quasi-linear in money and may depend on the privately held information of other agents. We focus on direct revelation mechanisms, and we are interested in finding mechanisms that satisfy the following properties at the ex post stage.

- Agents find it optimal to report their types truthfully (incentive compatibility).
- Agents benefit from participation (individual rationality).
- The mechanism does not run a deficit (budget feasibility).
- The mechanism does not run a deficit or create a surplus (budget balance).
- The sum of allocation utilities is maximized (efficiency).

Our main results establish necessary and sufficient conditions for the existence of ex post incentive compatible mechanisms in combination with the properties listed above. Ex post incentive compatibility states that each agent prefers truth-telling even when the agent knows the reports of other agents assuming these agents report truthfully. This property is desirable for at least three reasons. First, it allows the mechanism to perform well even when agents do not have common knowledge of the distribution of types (which is an unrealistic and crucial assumption for mechanisms that are only interim incentive compatible). Second, it allows designing the mechanism with minimal assumptions about the distribution of types. Third, it removes
any incentives to “game” the mechanism by either trying to delay reporting or spying on other agents (to learn what they plan to report).

In the case of private values, ex post incentive compatibility is equivalent to dominant strategy incentive compatibility for direct mechanisms. Dominant strategy incentive compatibility states that each agent prefers truth-telling regardless of what other agents report. This is an additional desirable property of mechanisms since it allows agents to optimize without forming any beliefs about the behavior of others.

Our first result (Theorem 1) provides a necessary and sufficient condition for the existence of an ex post incentive compatible, individually rational, and budget feasible mechanism.\(^1\) In other words, Theorem 1 characterizes when it is possible to implement any given allocation rule, not necessarily efficient, with a mechanism that satisfies these properties.

Previous results for particular models have shown that in markets with asymmetric information, efficient mechanisms may require either running a deficit (i.e., having the social planner subsidize the mechanism) or creating a surplus (to be collected by the social planner or a third party). For example, Myerson and Satterthwaite (1983) show that in a seller-buyer market with an overlap in valuations, it is necessary to subsidize the mechanism to achieve efficiency. On the other hand, it is well known that in a public goods model, an efficient mechanism may run a surplus (Green and Laffont (1979)). Theorem 1 may be viewed as a general characterization of environments in which a mechanism can achieve the desired properties without requiring a subsidy, and covers all allocation rules (not only the efficient ones).

Our second result (Theorem 2) provides a necessary and sufficient condition for the existence of a mechanism that is ex post incentive compatible and budget balanced.

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\(^1\)When we write individual rationality, budget feasibility, and budget balance without any qualifier, we mean “ex post.”
without individual rationality. To explain further, Theorem 2 characterizes when it is possible to implement any given allocation rule with a mechanism that satisfies these properties.

These two general results apply to both models with private or interdependent values, and they accommodate arbitrary correlation of types. However, the conditions we provide may be difficult to verify in a general interdependent values environment. But if agents have private values, then we can use existing results about Vickrey-Clarke-Groves mechanisms to rewrite the conditions for the efficient allocation rule (Corollaries 2 and 3). These conditions depend on the social surplus function and may be easier to check for some applications. In addition, we provide a sufficient condition for the existence of an incentive compatible, individually rational, efficient and budget feasible mechanism which states that an allocation which is efficient at some type profile should give a higher sum of utilities than the initial endowments for all type profiles (Proposition 1). This is useful to get positive existence results in our applications.

In the last part of the paper, we consider several applications of the general theory. The first application is a simple auction environment with interdependent values where we establish an already known existence result. The second application is a seller-buyer market for multiple heterogeneous goods. Buyers have general preferences over bundles of goods allowing substitutes or complements. Sellers’ marginal cost of production are commonly known and non-decreasing. In this market, there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism. Moreover, we provide an example which demonstrates that such a mechanism exists only if the marginal cost is non-decreasing. The third application is a housing market where each agent has a unique good to trade with unit demand with transfers (Shapley and Scarf (1974)). For the housing market, we
show that there does not exist a dominant strategy incentive compatible, efficient, and budget feasible mechanism in general. The last example is a roommates problem with transfers, where trading is restricted to pairs of agents (Gale and Shapley (1962)). Like before, there does not exist any mechanism with the required properties. However, for both the housing market and the roommates problem, we show that if agents prefer to trade with anyone rather than keeping their endowments, then there exists a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism.

Our paper is related to the ex post implementation literature which reduces the common knowledge assumptions in the spirit of the ‘Wilson Doctrine’ (Wilson (1987)). Ex post incentive compatibility is first introduced as uniform equilibrium in d’Aspremont and Gérard-Varet (1979) and as uniform incentive compatibility in Holmstrom and Myerson (1983) without characterizing implementable social choice functions. A characterization result is given in Chung and Ely (2002). In addition, Bergemann and Morris (2008) study the full ex post implementation problem, whereas Bergemann and Morris (2005) study when ex post implementation is equivalent to interim implementation for all type spaces. In the aforementioned papers implementation is equivalent to incentive compatibility without the other properties that we consider. For private values, ex post implementation is equivalent to dominant strategy implementation, so our paper is also related to dominant strategy implementation literature. Roberts (1979) and Bikhchandani et al. (2006) characterize social choice rules which are dominant strategy implementable. In contrast to this literature, we study when it is possible to implement an allocation rule with the additional properties.
2 General Model

In a heterogenous goods market with transfers, there is a finite set of agents $N$ and $k$ types of goods. Each agent $i \in N$ has an endowment $e_i \in \mathbb{Z}_+^k$ which specifies non-negative integer quantities for all types of goods. Let $e$ be the profile of endowments consisting of $e_i$ for all $i \in N$. Agent $i$’s type is $\theta_i \in \Theta_i$, where $\Theta_i$ is a connected subset of some Euclidean space. Similarly, $\theta \equiv \times_{i \in N} \theta_i$ is the type profile of agents and $\Theta \equiv \times_{i \in N} \Theta_i$ is the domain of types. This formulation allows types to be correlated. Each agent’s utility function is quasi-linear over allocation and monetary transfer. Hence, if agent $i$ receives allocation $x_i \in \mathbb{Z}_+^k$ and payment $t_i$, her utility is $u_i(x_i, \theta) + t_i$, where $u_i(x_i, \theta)$ is continuous in $\theta_i$. Therefore, we allow values to be interdependent, i.e., they may depend on signals of other agents. Let $\mathcal{A}^f$ denote a finite set of feasible allocations given exogenously.\(^2\) $\mathcal{A}^f$ is a useful tool to capture diverse economic environments. For example, in an exchange economy $\mathcal{A}^f$ may require the sum of allocations to be equal to the sum of endowments. Another example is a labor market where endowments may be positions that firms have or the labor that workers can supply and $\mathcal{A}^f$ requires that if a firm gets the endowment of a worker then the worker gets the endowment of the firm. See Section 4 for more examples.

A direct revelation mechanism for our formulation is a pair $(\mu, t)$ where $\mu : \Theta \rightarrow \mathcal{A}^f$ is an allocation function and $t : \Theta \rightarrow \mathbb{R}^{|N|}$ is a transfer function. For a given type profile $\theta$, $\mu_i(\theta) \in \mathbb{Z}_+^k$ is the allocation of agent $i$ and $t_i(\theta) \in \mathbb{R}$ is the transfer to agent $i$. Agent $i$’s utility from participation is,

$$v_i(\theta) \equiv [u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t_i(\theta).$$

\(^2\)The choice of modeling is inspired by Sonmez (1999). He studies the core in a general indivisible goods exchange problem without transfers.
By the revelation principle, it is sufficient to consider direct revelation mechanisms to check the existence of mechanisms with the desired properties listed below.

### 2.1 Properties

**Definition 1.** A direct revelation mechanism $(\mu, t)$ satisfies **ex post incentive compatibility** if, for all $i \in N$ and $\theta \in \Theta$,

$$v_i(\theta) \geq [u_i(\mu_i(\theta'_{-i}, \theta), \theta) - u_i(e_i, \theta)] + t_i(\theta'_{-i}, \theta - i)$$

for all $\theta'_{-i} \in \Theta_i$.

Ex post incentive compatibility is equivalent to an ex post no regret property, as no agent would like to change her report even if she were to know the reports of others. If $(\mu, t)$ is ex post incentive compatible, then it implements $\mu$ (with ex post incentives).

**Definition 2.** A direct revelation mechanism $(\mu, t)$ satisfies (ex post) **individual rationality** if, for all $i \in N$,

$$v_i(\theta) \geq 0$$

for all $\theta \in \Theta$.

That is, a mechanism is individually rational if agents have non-negative net utilities for all realized types.

**Definition 3.** A direct revelation mechanism $(\mu, t)$ satisfies (ex post) **efficiency** if

$$\mu(\theta) \in \arg \max_{\mu' \in A'} \sum_i u_i(\mu'_i, \theta),$$

for all $\theta \in \Theta$. 
In words, a mechanism is efficient if the allocation function maximizes the sum of allocation utilities over the set of feasible allocations.

**Definition 4.** A direct revelation mechanism \((\mu, t)\) satisfies (ex post) **budget feasibility** if

\[
\sum_{i} t_i(\theta) \leq 0
\]

for all \(\theta \in \Theta\).

Budget feasibility means that the mechanism does not run a deficit.

**Definition 5.** A direct revelation mechanism \((\mu, t)\) satisfies (ex post) **budget balance** if

\[
\sum_{i} t_i(\theta) = 0
\]

for all \(\theta \in \Theta\).

Budget balance requires the sum of transfers be exactly zero. Hence, the mechanism does not run a deficit or create surplus.

We only use the above definitions of individual rationality, efficiency, budget feasibility, and budget balance. Therefore, when we use them, we do not write “ex post” in front of the notion.

## 3 General Results

Let \((\mu, t)\) be an ex post incentive compatible direct revelation mechanism. In this section, we provide necessary and sufficient conditions for the existence of a transfer function \(t'\) such that \((\mu, t')\) is 1) ex post incentive compatible, individually rational, budget feasible and 2) ex post incentive compatible, and budget balanced.

The following lemma is useful in proving these results.
Lemma 1. Suppose \((\mu, t)\) is an ex post incentive compatible direct revelation mechanism. Then \((\mu, t')\) is an ex post incentive compatible direct revelation mechanism if and only if \(t_i(\theta) = t'_i(\theta) + g_i(\theta_{-i})\) for some function \(g_i\) for all \(i\).

All omitted proofs are in the Appendix.

The lemma tells us that if two mechanisms are ex post incentive compatible with the same allocation function, then the difference in transfers for any agent in two mechanisms cannot depend on the type of this agent. Moreover, it also states that changing the transfer function of an agent by a function which depends on the types of other agents preserves ex post incentive compatibility. This lemma is a slight generalization of Theorem 3 in Chung and Ely (2002).

If there exists an ex post incentive compatible mechanism \((\mu, t)\) which is not necessarily individually rational or budget feasible, the next theorem gives a necessary and sufficient condition using the utilities derived from \((\mu, t)\) for the existence of a transfer function \(t'\) such that \((\mu, t')\) is ex post incentive compatible, individually rational, and budget feasible.

Theorem 1. Suppose that \((\mu, t)\) is an ex post incentive compatible direct revelation mechanism. Then there exists a transfer function \(t'\) such that \((\mu, t')\) is ex post incentive compatible, individually rational, and budget feasible if and only if the following inequality holds:

\[
\sum_{\theta_i \in \Theta_i} \inf_{\theta_{-i}} v_i(\theta_i, \theta_{-i}) \geq \sum_i t_i(\theta), \text{ for all } \theta \in \Theta. \tag{1}
\]

Proof. “If” part: Let \(t'_i(\theta) = t_i(\theta) - \inf_{\theta_{-i}} v_i(\theta_i, \theta_{-i})\). By Lemma 1, \((\mu, t')\) is ex post incentive compatible. Add \(u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)\) to both sides of this equation to get \(v'_i(\theta) \equiv [u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t'_i(\theta) = v_i(\theta) - \inf_{\theta_{-i}} v_i(\theta_i, \theta_{-i})\) which must be
non-negative, so individual rationality is also satisfied. Finally, \( \sum_{i} t'_i(\theta) = \sum_{i} \left[ t_i(\theta) - \inf_{\theta_i \in \Theta_i} v_i(\theta_i, \theta_{-i}) \right] \) which is less than or equal to zero by assumption. Therefore, \((\mu, t')\) is budget feasible as well as incentive compatible and individually rational.

"Only if" part: Suppose that \((\mu, t')\) is ex post incentive compatible, individually rational, and budget feasible. By Lemma 1, \( t'_i(\theta) = t_i(\theta) + g_i(\theta_{-i}) \). Add \( u_i(\mu_i(\theta), \theta) - u_i(\epsilon_i, \theta) \) to both sides to get \( v'_i(\theta) = v_i(\theta) + g_i(\theta_{-i}) \). Since \((\mu, t')\) is ex post individually rational, \( g_i(\theta_{-i}) \geq - \inf_{\theta_i \in \Theta_i} v_i(\theta_i, \theta_{-i}) \). Finally, budget feasibility implies \( 0 \geq \sum_{i} t'_i(\theta) = \sum_{i} [t_i(\theta) + g_i(\theta_{-i})] \geq \sum_{i} [t_i(\theta) - \inf_{\theta_i \in \Theta_i} v_i(\theta_i, \theta_{-i})] \). Therefore, we get \( \sum_{i \theta_i \in \Theta_i} v_i(\theta_i, \theta_{-i}) \geq \sum_{i} t_i(\theta) \). 

The intuition for this result is as follows. To implement \( \mu \) with ex post incentives, we can only change the transfer function of agent \( i \) by a function that depends on \( \theta_{-i} \). Suppose that we charge each agent such an extra amount. The highest amount we can charge preserving ex post individual rationality is the minimum utility that agent \( i \) gets fixing \( \theta_{-i} \). If the sum of these charges can cover the sum of original transfers then there exists such a mechanism. Otherwise, there exists none. That is exactly what (1) checks.

Next we require budget balance instead of budget feasibility and drop the individual rationality requirement.

**Theorem 2.** Let \((\mu, t)\) be an ex post incentive compatible direct revelation mechanism and \( \tau(\theta) \) be the sum of transfers, i.e., \( \tau(\theta) = \sum_{i} t_i(\theta) \). Fix a type profile \( \theta \in \Theta \). Then there exists a transfer function \( t' \) such that \((\mu, t')\) is ex post incentive compatible and budget balanced if and only if the following holds:

\[
\sum_{j=0}^{|[N]|} (-1)^j \sum_{\{i_1, \ldots, i_j\} \subseteq N} \tau(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1}, \ldots, i_j) = 0, \quad \text{for all } \theta \in \Theta. \tag{2}
\]

\(^3\)With some abuse of notation, \( \theta_{-i_1, \ldots, i_j} \) is the vector of types for agents other than \( i_1, \ldots, i_j \). For
The above statement is valid for any particular choice of $\theta$.

**Proof.** "If" part: Suppose that (2) is satisfied. We construct $t'$ such that $(\mu, t')$ is ex post incentive compatible and budget balanced.

\[
t'_1(\theta) = t_1(\theta) + \sum_{j=1}^{\lvert N \rvert} (-1)^j \sum_{1 \in \{i_1, \ldots, i_j \}} \tau(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1, \ldots, i_j})
\]

\[
t'_2(\theta) = t_2(\theta) + \sum_{j=1}^{\lvert N \rvert} (-1)^j \sum_{1 \notin \{i_1, \ldots, i_j \}, 2 \in \{i_1, \ldots, i_j \}} \tau(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1, \ldots, i_j})
\]

\[
\vdots
\]

\[
t'_n(\theta) = t_n(\theta) + \sum_{j=1}^{\lvert N \rvert} (-1)^j \sum_{1, \ldots, n-1 \notin \{i_1, \ldots, i_j \}, n \in \{i_1, \ldots, i_j \}} \tau(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1, \ldots, i_j})
\]

By Lemma 1, $(\mu, t')$ is ex post incentive compatible. To show that it is also budget balanced, sum the above equalities to get:

\[
\sum_{i \in N} t'_i(\theta) = \sum_{i \in N} t_i(\theta) + \sum_{j=1}^{\lvert N \rvert} (-1)^j \sum_{\{i_1, \ldots, i_j \} \subseteq N} \tau(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1, \ldots, i_j}).
\]

By (2), \[\sum_{j=1}^{\lvert N \rvert} (-1)^j \sum_{\{i_1, \ldots, i_j \} \subseteq N} \tau(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1, \ldots, i_j}) = -\tau(\theta)\]. Moreover, \[\sum_{i \in N} t_i(\theta) = \tau(\theta)\] by definition. Therefore, we get that \[\sum_{i \in N} t'_i(\theta) = \tau(\theta) - \tau(\theta) = 0\].

"Only if" part: Suppose that $(\mu, t')$ is ex post incentive compatible and budget balanced. Then by Lemma 1, \[t_i(\theta) = t'_i(\theta) + g_i(\theta_{-i})\]. If we sum this over all agents we get \[\sum_i [t_i(\theta)] = \sum_i [t'_i(\theta) + g_i(\theta_{-i})]\] which is equivalent to \[\tau(\theta) = \sum_i g_i(\theta_{-i})\].

Define \(g_i^{(1)}(\theta_{-i})\) as follows: \(g_i^{(1)}(\theta_{-i}) = g_i(\theta_{-i}) - g_i(\theta_{1, \theta_{-1, i}})\) for \(i \neq 1\) and \(g_1^{(1)}(\theta_{-1}) = j = 0, \theta_{-i_1, \ldots, i_j} = \theta\).
\[ g_1(\theta_{-1}) + \sum_{i \neq 1} g_i(\theta_1, \theta_{-1,i}). \] Therefore, \( \sum_i g_i^{(1)}(\theta_{-i}) = \sum_i g_i(\theta_{-i}). \) Hence,

\[ \tau(\theta) = \sum_i g_i^{(1)}(\theta_{-i}). \tag{3} \]

Plug in \( \theta_1 = \theta_{-1} \) to get \( \tau(\theta_1, \theta_{-1}) = g_1^{(1)}(\theta_{-1}) + \sum_{i \neq 1} g_i^{(1)}(\theta_1, \theta_{-1,i}). \) By definition, \( g_i^{(1)}(\theta_1, \theta_{-1,i}) = 0, \ i \neq 1. \) Therefore, we get that \( \tau(\theta_1, \theta_{-1}) = g_1^{(1)}(\theta_{-1}). \) Using this equality, (3) can be rewritten as

\[ \tau(\theta) - \tau(\theta_1, \theta_{-1}) = \sum_{i \neq 1} g_i^{(1)}(\theta_{-i}). \]

Define \( g_i^{(j)}(\theta_{-i}) \) inductively on \( j \) by \( g_i^{(j)}(\theta_{-i}) = g_i^{(j-1)}(\theta_{-i}) - g_i^{(j-1)}(\theta_j, \theta_{-i,j}) \) for \( i \neq j \) and \( g_j^{(j)}(\theta_{-j}) = g_j^{(j-1)}(\theta_{-j}) + \sum_{i \neq j} g_i^{(j-1)}(\theta_j, \theta_{-i,j}). \) By induction on \( m \), it is easy to see that \( \tau(\theta) - \sum_{\{i_1\} \subseteq N'} \tau(\theta_{i_1}, \theta_{-i_1}) + \ldots + (-1)^m \sum_{\{i_1, \ldots, i_m\} \subseteq N'} \tau(\theta_{i_1}, \ldots, \theta_{i_m}, \theta_{-i_1}, \ldots, i_m) = \sum_{i \in N \setminus N'} g_i^{(m)}(\theta_{-i}) \) where \( N' = \{1, \ldots, m\}. \) Take \( N' = N \) to get:

\[ \sum_{j=0}^{\lfloor N \rfloor} (-1)^j \sum_{\{i_1, \ldots, i_j\} \subseteq N} \tau(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1}, \ldots, i_j) = 0. \]

To achieve budget balance, we need to charge each agent an extra amount such that the sum of the charges is equal to the sum of transfers, \( \tau(\theta) \). However, for agent \( i \), this extra charge can only be a function of \( \theta_{-i} \) to preserve ex post incentive compatibility. Therefore, we can achieve budget balance if and only if we can decompose \( \tau(\theta) \) additively into \( \lfloor N \rfloor \) functions where function \( i \) depends only on \( \theta_{-i} \). If such a decomposition is possible, then we can find a decomposition using only \( \tau \) by algebraic manipulations as in the proof, (2) gives the exact decomposition.
For a general environment, it may be hard to check (2). However, if one of the agents’ type has a degenerate distribution, i.e., when the type is a constant, then the condition is satisfied. Thus, this result can be applied to the important special case of auctions, where it is usually assumed that the auctioneer has a degenerate distribution.

**Corollary 1.** Let \((\mu, t)\) be an ex post incentive compatible direct revelation mechanism. Suppose that there exists agent \(i\) such that \(\theta_i\) has a degenerate distribution.

1. There exists a transfer function \(t'\) such that \((\mu, t')\) is ex post incentive compatible and budget balanced.

2. There exists a transfer function \(t'\) such that \((\mu, t')\) is ex post incentive compatible, individually rational, and budget balanced if and only if (1) holds.

If agent \(i\)’s type has a degenerate distribution, then this agent does not face any incentive issues. Therefore, agent \(i\) can be charged the residual balance without distorting incentives. This preserves ex post incentive compatibility and achieves budget balance. On the other hand, if we also require individual rationality on top of budget balance, then (1) becomes necessary and sufficient by the same reasoning.

### 3.1 Efficient Allocation with Private Values

In this subsection, we consider implementing the efficient allocation function. In order to apply Theorems 1 and 2, we need to find a transfer function to implement the efficient allocation function with ex post incentives. This may be hard for environments with interdependent values. However, if values are private, then we can use a Vickrey-Clarke-Groves (henceforth VCG) payment function.\(^4\) This enables us to

\(^4\)Thus, it follows the idea of efficient mechanism design by Vickrey (1961), Clarke (1971), and Groves (1973).
rewrite the conditions given in Theorems 1 and 2 for the efficient allocation function.

**Definition 6. (Private Values)** Agent $i \in N$ has private values if, for all $\theta_i \in \Theta_i$ and $x_i \in \mathbb{Z}_+^k$, $u_i(x_i, (\theta_i, \theta_{-i})) = u_i(x_i, (\theta'_i, \theta'_{-i}))$, for all $\theta_{-i}, \theta'_{-i} \in \Theta_{-i}$. When this holds the utility of agent $i$ is denoted by $u_i(x_i, \theta_i)$.

When values are private, ex post incentive compatibility is equivalent to the stronger notion of dominant strategy incentive compatibility for direct revelation mechanisms that we define below.

**Definition 7.** A direct revelation mechanism $(\mu, t)$ satisfies **dominant strategy incentive compatibility** if, for all $i \in N$, $\theta'_{-i} \in \Theta_{-i}$ and $\theta \in \Theta$,

$$[u_i(\mu_i(\theta, \theta'_{-i}), \theta) - u_i(e_i, \theta)] + t_i(\theta, \theta'_{-i}) \geq [u_i(\mu_i(\theta'), \theta) - u_i(e_i, \theta)] + t_i(\theta')$$

for all $\theta'_i \in \Theta_i$.

Dominant strategy incentive compatibility requires each agent to play truthfully even if the agent observes the reports of others without assuming that they report truthfully.

Suppose $\mu^e$ is an efficient allocation function and $SS(\theta)$ be the maximum value of sum of allocation utilities when the type profile is $\theta$, i.e.,

$$SS(\theta) = \sum_i [u_i(\mu^e_i(\theta), \theta_i) - u_i(e_i, \theta_i)].$$

Let $t_i(\theta) = \sum_{j \neq i} [u_j(\mu^e_j(\theta), \theta_j) - u_j(e_j, \theta_j)]$. Therefore,

$$v_i(\theta) = [u_i(\mu^e_i(\theta), \theta_i) - u_i(e_i, \theta_i)] + t_i(\theta) = \sum_{j \in \mathbb{N}} [u_j(\mu^e_j(\theta), \theta_j) - u_j(e_j, \theta_j)] = SS(\theta).$$

Therefore, $(\mu^e, t)$ is a VCG mechanism such that each agent’s net utility is equal to the social surplus. Moreover, $\sum_i t_i(\theta) = (|N| - 1)SS(\theta)$. Thus, Theorem 1 reduces to the following.
**Corollary 2.** Suppose that agents have private values. Let $SS(\theta)$ be the sum of allocation utilities for an efficient allocation function $\mu^e$. Then, there exists a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism if and only if the following holds:

$$\sum_\theta \inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) \geq (|N| - 1) SS(\theta), \text{ for all } \theta \in \Theta. \quad (4)$$

This corollary characterizes private value environments for which there exists a mechanism with the listed properties. Characterization of efficient mechanisms for interim implementation with budget balance are given in Krishna and Perry (1998), Makowski and Mezzetti (1994), and Williams (1999). Makowski and Mezzetti (1994) also study dominant strategy implementation, but they use ex ante budget balance. Kosenok and Severinov (2008) provide a similar result for a general allocation function in a model with discrete types.

The intuition for this result is the same as Theorem 1. Condition 4 requires the calculation of the social surplus and its infimum with respect to agents’ types. This may not be easy to do for some environments. In the next proposition we provide necessary or sufficient conditions which are easier to check.

**Proposition 1.** Suppose that agents have private values.

1. There exists a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism if for all $\theta$

$$\sum_i [u_i(\mu_i^e(\theta), \theta_i') - u_i(e_i, \theta_i')] \geq 0, \text{ for all } \theta' \in \Theta.$$  

2. There exists a dominant strategy incentive compatible, individually rational, ef-
cient, and budget feasible mechanism only if for all $\theta$

$$
\sum_i \inf_{\theta_i, s.t. \mu_i^\epsilon(\hat{\theta}_i, \theta_{-i}) = \mu_i^\epsilon(\theta)} [u_i(\mu_i^\epsilon(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] \geq 0.
$$

Part 1 gives a sufficient condition for the existence of a mechanism with the required properties. This condition states that an allocation that is efficient for some type profile gives a higher sum of allocation utilities than the initial endowments for all type profiles. On the other hand, Part 2 gives a sufficient condition. It states that the sum of utility differences where an agent gets the efficient allocation or the initial endowment should be nonnegative. In each summand, we only change the type of one agent such that the efficient allocation of that agent does not change.

For the efficient allocation function with private values, Theorem 2 reduces to Part 1 of the following.

**Corollary 3.** Suppose that agents have private values.

1. For each $i$ fix a type $\theta_i \in \Theta_i$. Then there exists a dominant strategy incentive compatible, efficient, and budget balanced mechanism if and only if the following holds:

$$
\sum_{j=0}^{\vert N \vert} (-1)^j \sum_{\{i_1, \ldots, i_j\} \subseteq N} SS(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1, \ldots, i_j}) = 0, \text{ for all } \theta \in \Theta. \tag{5}
$$

2. Let $\mu^\epsilon$ be an efficient allocation function. Suppose that for each agent $i$ there exists an agent $j$ such that for all $\theta \in \Theta$ and $\theta'_j \in \Theta_j$, $\mu_i^\epsilon(\theta) = \mu_i^\epsilon(\theta'_j, \theta_{-j})$.

Then there exists a dominant strategy incentive compatible, efficient, and budget balanced mechanism.

Part 1 characterizes environments with private values for which there exists a
dominant strategy incentive compatible, efficient, and budget balanced mechanism. Hurwicz and Walker (1990) also show that (5) is a necessary condition. Holmstrom (1977) provides another characterization result. His characterization states that a mechanism with the properties exists if and only if $SS(\theta)$ can be written as a sum of functions where each function depends on the types of all agents except one. Therefore, to check whether his characterization holds, one needs to verify the existence of these functions. On the other hand, (5) only involves the social surplus term, so it should be easier to check. Similarly, Laffont and Maskin (1980) study whether a mechanism with these properties exists in a stylized public goods model. They give a characterization in terms of a partial differential equation involving the value functions. For this special case, (5) is an equation involving the social surplus function, so no differentiability assumption is needed.

Part 2 gives a sufficient condition for the existence of such a mechanism. It states that for each agent $i$ there exists another agent $j$ such that agent $j$’s type does not change the efficient allocation of $i$. Therefore, when we expand (5) using the utility functions all terms cancel. This condition is weaker than the assumption that there exists an agent whose type has a degenerate distribution given in Corollary 1.

We ignore the individual rationality constraint in the corollary stated above. However, we can provide a sufficient condition for the existence of a mechanism with the properties listed above and individual rationality.

**Corollary 4.** Suppose the set of agents can be divided into two subsets $N_1$ and $N_2$ such that in each subset the efficient allocation is a reallocation of their endowments, and that this reallocation does not change with the type profile of agents in the other subset. Suppose also that (4) holds for $N_1$ and $N_2$, separately. If agents have private values, then there exists a mechanism which is dominant strategy incentive compatible.
individually rational, efficient, and budget balanced.

By assumption, there are two separate submarkets, and for each submarket (4) holds. By Corollary 2, there exists a dominant strategy incentive compatible, individually rational, and efficient mechanism for each submarket that creates a surplus. We can distribute the surplus from one submarket to the agents on the other submarket in any way. This does not thwart incentives, and individual rationality is preserved. Moreover, the final allocation is efficient for the whole economy by assumption, and budget balance is satisfied.

Now, let us consider some examples. For each example we check (4) and (5) to verify the existence of a mechanism with the required properties.

**Example 1.** There are three agents, sellers $S_1$, $S_2$, and buyer $B$. Sellers own the same car model to sell. Sellers costs, $\theta_{S_i} \in [0, 1]$ ($i = 1, 2$), and the buyer’s value, $\theta_B \in [2, 3]$, are their private information.

First note that,

$$SS(\theta) = \theta_B - \min\{\theta_{S_1}, \theta_{S_2}\}.$$  

To check condition (4), calculate the infimum of the social surplus with respect to each agent’s type. It is easy to see that the infimums are attained at $\theta_{S_1}^*(\theta_{-S_1}) = 1$, $\theta_{S_2}^*(\theta_{-S_2}) = 1$, and $\theta_B^*(\theta_{-B}) = 2$. Therefore,

$$\inf_{\theta_{S_1}} SS(\theta_{S_1}, \theta_{-S_1}) = SS(\theta_{S_1}^*(\theta_{-S_1}), \theta_{-S_1}) = \theta_B - \theta_{S_2}.$$  

And similarly,

$$\inf_{\theta_{S_2}} SS(\theta_{S_2}, \theta_{-S_2}) = SS(\theta_{S_2}^*(\theta_{-S_2}), \theta_{-S_2}) = \theta_B - \theta_{S_1},$$  

$$\inf_{\theta_B} SS(\theta_B, \theta_{-B}) = SS(\theta_B^*(\theta_{-B}), \theta_{-B}) = 2 - \min\{\theta_{S_1}, \theta_{S_2}\}.$$  

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Hence,

$$
(4) \iff \inf_{\theta_{S_1}} SS(\theta_{S_1}, \theta_{-S_1}) + \inf_{\theta_{S_2}} SS(\theta_{S_2}, \theta_{-S_2}) + \inf_{\theta_B} SS(\theta_B, \theta_{-B}) \geq 2SS(\theta) \\
\iff 2 + \min\{\theta_{S_1}, \theta_{S_2}\} \geq \theta_{S_1} + \theta_{S_2} \\
\iff 2 \geq \max\{\theta_{S_1}, \theta_{S_2}\}.
$$

The last inequality holds since $\theta_{S_1}, \theta_{S_2} \leq 1$. That means there exists a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism.

Instead of checking condition (5), we are going to take a more direct approach. Since $SS(\theta) = \theta_B - \min\{\theta_{S_1}, \theta_{S_2}\}$, $SS(\theta)$ can be decomposed additively into two functions where each function depends on at most two of the variables. Therefore, by Holmstrom (1977)’s characterization, there exists an efficient, dominant strategy incentive compatible, and budget balanced mechanism. Although his characterization is convenient here, we can only use it in simple setups. Indeed, in the next example the characterization is not useful.

In fact, by using a reverse auction we can satisfy all the properties simultaneously since the buyer’s type does not change the efficient allocation. Therefore, there also exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism.

**Example 2.** There are three agents, sellers $S_1$, $S_2$, and buyer $B$. Sellers own different car models to sell. Sellers costs, $\theta_{S_i} \in [0, 1]$ ($i = 1, 2$), and the buyer’s value, $\theta_B = (\theta_{B1}, \theta_{B2}) \in [2, 3]^2$, are their private information.

First note that,

$$SS(\theta) = \max\{\theta_{B1} - \theta_{S_1}, \theta_{B2} - \theta_{S_2}\}.$$
To check condition (4), calculate the infimum of the social surplus with respect to each agent’s type. It is easy to see that the infimums are attained at \( \theta^*_{S_1}(\theta_{-S_1}) = 1 \), \( \theta^*_{S_2}(\theta_{-S_2}) = 1 \), and \( \theta^*_B(\theta_{-B}) = (2, 2) \). Therefore,

\[
\inf_{\theta_{S_1}} SS(\theta_{S_1}, \theta_{-S_1}) = SS(\theta^*_{S_1}(\theta_{-S_1}), \theta_{-S_1}) = \max\{\theta_{B_1} - 1, \theta_{B_2} - \theta_{S_2}\}.
\]

And similarly,

\[
\inf_{\theta_{S_2}} SS(\theta_{S_2}, \theta_{-S_2}) = SS(\theta^*_{S_2}(\theta_{-S_2}), \theta_{-S_2}) = \max\{\theta_{B_1} - \theta_{S_1}, \theta_{B_2} - 1\},
\]

\[
\inf_{\theta_B} SS(\theta_B, \theta_{-B}) = SS(\theta^*_B(\theta_{-B}), \theta_{-B}) = 2 - \min\{\theta_{S_1}, \theta_{S_2}\}.
\]

Without loss of generality assume that \( \theta_{B_1} - \theta_{S_1} \geq \theta_{B_2} - \theta_{S_2} \). Hence,

\[
(4) \iff SS(\theta^*_{S_1}(\theta_{-S_1}), \theta_{-S_1}) + SS(\theta^*_{S_2}(\theta_{-S_2}), \theta_{-S_2}) + SS(\theta^*_B(\theta_{-B}), \theta_{-B}) \geq 2SS(\theta)
\]

\[
\iff 2 - \min\{\theta_{S_1}, \theta_{S_2}\} + \max\{\theta_{B_1} - 1, \theta_{B_2} - \theta_{S_2}\} \geq \theta_{B_1} - \theta_{S_1}
\]

\[
\iff 2 - \min\{\theta_{S_1}, \theta_{S_2}\} \geq (\theta_{B_1} - \theta_{S_1}) - \max\{\theta_{B_1} - 1, \theta_{B_2} - \theta_{S_2}\}.
\]

The last inequality holds since \( 2 - \min\{\theta_{S_1}, \theta_{S_2}\} \geq 1 - \theta_{S_1} \geq (\theta_{B_1} - \theta_{S_1}) - \max\{\theta_{B_1} - 1, \theta_{B_2} - \theta_{S_2}\} \). That means there exists a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism.

Since \( SS(\theta) = \max\{\theta_{B_1} - \theta_{S_1}, \theta_{B_2} - \theta_{S_2}\} \), it is not clear whether we can decompose \( SS(\theta) \) additively into functions where each function depends on the types of at most two agents. Hence, Holmstrom (1977)’s characterization cannot be applied directly.
Therefore, we check (5).

\[(5) \iff \max\{\theta_{B1} - \theta_{S1}, \theta_{B2} - \theta_{S2}\} - \max\{\theta_{B1}, \theta_{B2} - \theta_{S1}\} + \min\{\theta_{S1}, \theta_{S2}\} + \max\{\theta_{B1}, \theta_{B2}\} - \theta_{S1} - \theta_{S2} = 0.\]

If we let \(\theta_{S1} = \frac{1}{2}, \theta_{S2} = 0, \theta_{B1} = 2\frac{1}{2},\) and \(\theta_{B2} = 3,\) then the above equation becomes \(-\frac{1}{2} = 0.\) Hence, there does not exist a dominant strategy incentive compatible, efficient, and budget balanced mechanism.

In this example, despite that all agents’ types change the efficient allocation, there exists a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism. This mechanism creates an expected surplus. However, budget balance may not be satisfied in addition to other properties.

**Example 3.** There are six agents, four of which are sellers \(S_i, i = 1, 2, 3, 4,\) and two of which are buyers, \(B_i, i = 1, 2.\) \(S_1\) and \(S_2\) are selling different car models, whereas \(S_3\) and \(S_4\) are selling different SUV models. \(B_1\) is only interested in the cars, while \(B_2\) is only interested in the SUVs. We assume that each seller’s cost \(\theta_{S_i} \in [0, 1],\) \(i = 1, 2, 3, 4,\) and also that each buyer’s value \(\theta_{B_i} \in [2, 3]^2\) where each coordinate is the value for one of the vehicles that the buyer is interested in.

For this problem, there are two submarkets: the car market and the SUV market. Additionally, for each submarket (4) holds. Therefore, by Corollary 4, there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism.

In this example, although each agent’s type may change the efficient allocation rule, there exists a mechanism with all the required properties.
4 Applications

There are many potential applications of our general theory to specific markets. We do not include the following applications in our discussion: many-to-many matching markets with transfers, networks, coalition formation problems, and partnership dissolution problems. Nevertheless, we consider the following applications: auctions, seller-buyer markets with discrete heterogeneous goods, housing markets, and roommate problems with transfers. We hope these applications convince the reader that our methods can be applied to the others as well.

In the auctions application agents may have interdependent values. However, for the rest of the applications we focus on the case when agents have private values.

4.1 Auctions with Interdependent Valuations

In this subsection, we demonstrate how to apply Theorem 1 to a setup with interdependent values by considering an auction environment where buyers have one-dimensional types. To be more specific, we consider a simple one-item auction to get an implementation like the previous literature (see Crémer and McLean (1985) for pioneering work).

There are \( n \) buyers and a seller endowed with a good. The seller’s cost is zero. Buyers may have interdependent values. Buyer \( i \)'s value for the good is \( u_i(1, \theta) \) or simply \( u_i(\theta) \). \( \Theta_i \) is an interval in the real line. We also make the following assumptions.

1. \( u_i(\theta) \geq 0 \) for all \( i \) and \( \theta \).

2. \( u_j(\theta) \) is nondecreasing in \( \theta_i \) for all \( i, j \).
3. If \( u_i(\theta_i, \theta_{-i}) \geq u_j(\theta_i, \theta_{-i}) \) for some \( \theta_i, \theta_{-i} \) then \( u_i(\theta'_i, \theta_{-i}) > u_j(\theta'_i, \theta_{-i}) \) for all \( \theta'_i > \theta_i \).

Assumption 1 states that agents have nonnegative values for the object which should be nondecreasing with respect to any agent’s type which is Assumption 2. Finally, Assumption 3 states that if agent \( i \)'s value is greater than or equal to agent \( j \)'s for some type profile, then increasing agent \( i \)'s type makes this inequality strict.

For this setup we cannot use a VCG mechanism to implement the efficient allocation rule, because agents have interdependent values. However, we can still implement the efficient allocation rule via a transfer rule defined below.

**Lemma 2.** Suppose that Assumptions 1-3 hold. Let \( \mu^c \) be a deterministic efficient allocation rule. Then \( (\mu^c, t) \) is an ex post incentive compatible mechanism where \( t \) is given by:

\[
    t_i(\theta) = \begin{cases} 
        0 & \text{if } \mu^c_i(\theta) = 1, \\
        \sup_{\tilde{\theta}_i \text{ s.t. } \mu^c_i(\tilde{\theta}_i, \theta_{-i}) = 0} \max_{j \neq i} u_j(\tilde{\theta}_i, \theta_{-i}) & \text{otherwise.}
    \end{cases}
\]

The intuition of the equilibrium is as follows. If agent \( i \) loses the auction then she receives positive payments which is equal to the supremum of the winner’s value where the supremum is taken over her types for which she does not win the auction. Therefore, she does not benefit by lying and still losing the auction. Moreover, the payment is always higher than her true value, so she also does not benefit from lying and winning the auction. On the other hand, if agent \( i \) wins the auction she does not make or receive any payments. Hence, lying and still winning the auction is not beneficial. In addition, if agent \( i \) lies and loses the auction then her payment is not greater than her value since she can only lose the auction by understating her type which decreases the values of other agents as well.
This equilibrium is similar to the VCG mechanism in the private values case which gives each agent the whole social surplus. The main difference is the payment to a losing agent since this payment cannot depend on the type of that agent. Otherwise this agent can increase her payment without changing the allocation. In general, losers may have a higher utility from the social surplus.

Now that we have provided a transfer function that implements the efficient allocation with ex post incentives, we can apply Theorem 1.

**Corollary 5.** Suppose that Assumptions 1-3 hold. Then there exists an ex post incentive compatible, individually rational, efficient, and budget balanced mechanism.

It is already known in the literature that such a mechanism exists (see Dasgupta and Maskin (2000)). However, we establish this result via an application of Theorem 1 for demonstration.

### 4.2 Seller-Buyer Markets with General Preferences

In this application we consider a seller-buyer market where each seller can own multiple units of heterogenous goods and buyers can demand more than one unit. Sellers’ types are commonly known whereas buyers’ types are their private information. Thus, there is private information only on the buyers’ side. We only require that buyers prefer more to less, allowing goods to be substitutes or complements.

More formally, in a **seller-buyer market** there are \( m \) buyers and \( n \) sellers. Let \( B \) and \( S \) denote the set of buyers and sellers, respectively. Seller \( i \)'s endowment (or capacity), \( e_i \in \mathbb{Z}_+^k \), denotes the available vector of goods to sell for \( k \) types of goods, whereas buyers do not have any endowments. Moreover, each seller \( i \)'s type \( \theta_i \) has a degenerate distribution and the cost of providing bundle \( x_i \in \mathbb{Z}_+^k \) is \( u_i(x_i, \theta_i) \). Similarly, buyer \( i \) with type \( \theta_i \) has value \( u_i(x_i, \theta_i) \) for bundle \( x_i \in \mathbb{Z}_+^k \). The value for
owning the empty bundle is zero for all buyers, that is, for all \( i \in B \) and \( \theta_i \in \Theta_i \),
\[ u_i(0, \theta_i) = 0. \]
In addition, all buyers value more goods to less, i.e., for all \( i \in B \), \( \theta_i \in \Theta_i \) and \( x_i, y_i \in \mathbb{Z}^k_+ \) such that \( x_i \geq y_i \),
\[ u_i(x_i, \theta_i) \geq u_i(y_i, \theta_i). \]

A feasible allocation is a redistribution of goods such that each seller gets fewer
goods than their initial endowments. More formally,

\[ \mathcal{A}' = \{ x | \forall i \in B \ x_i \geq 0, \forall i \in S \ 0 \leq x_i \leq e_i, \text{ and } \sum_{i \in S \cup B} x_i = \sum_{i \in S} e_i \}. \]

**Proposition 2.** Suppose that each seller has an additively separable cost across dif-
ferent goods and that for each good the marginal cost is nondecreasing. Then there
exists a dominant strategy incentive compatible, individually rational, efficient, and
budget balanced mechanism for the seller-buyer markets.

We prove that (4) holds for seller-buyer markets in the Appendix which shows that
there exists a dominant strategy incentive compatible, individually rational, efficient,
and budget feasible mechanism. Now, distribute the surplus to the sellers in any way.
Since the sellers do not face incentive issues, the new mechanism will satisfy all the
properties in addition to budget balance.

In fact, under the assumptions stated, the VCG mechanism can be used for the
first part of the proof. It is well known in the literature that the VCG mechanism
is not budget feasible in general. Our contribution in this proposition is not the
reinvention of the VCG mechanism, but to come up with conditions where there exists
a mechanism with the required properties which happens to be the VCG mechanism.\(^5\)
The critical assumption is that sellers have non-decreasing marginal costs that are
commonly known. This implies that in the efficient allocation each extra unit which
\(^5\)In general the mechanism satisfying (4) has to be a Groves mechanism but not the VCG me-
chanism.
gets traded has a higher cost of production than the previous units. Without this assumption, the VCG mechanism may not be budget feasible which is demonstrated by the following example (in fact, no mechanism with the required properties exists).

**Example 4.** Suppose that there is one seller, $S$, with two identical goods and that there are three buyers, $B_1, B_2$, and $B_3$. The seller’s fixed cost is already sunk. The marginal cost of the first good is two, and the second good is $x$ where $9 > x > 1$. $B_1$ does not have any value for one good, but he values two goods at ten. $B_2$ and $B_3$ have a value of nine for any ($> 0$) number of goods. Therefore, $B_1$ has complementary values whereas $B_2$ and $B_3$ have substitutable preferences.

Let us calculate the VCG allocation and the payments made by the buyers. It is easy to check that in the efficient allocation $B_2$ and $B_3$ get one good each and the social surplus is $18 - (2 + x)$. To calculate the payments, suppose that $B_2$ was absent from the economy. Then, in the efficient allocation $B_3$ would get one good and the social surplus would be $9 - 2$. Therefore, $B_2$’s payment should be such that his net utility is the difference in the social surpluses calculated above. Hence, $9 - p_2 = (18 - (2 + x)) - (9 - 2)$. This gives $p_2 = x$. Similarly, $p_3 = x$. The VCG mechanism is budget feasible if:

$$p_2 + p_3 \geq 2 + x \iff x + x \geq 2 + x \iff x \geq 2.$$ 

Therefore, the VCG mechanism is budget feasible if the marginal cost is nondecreasing. In fact, a mechanism with the required properties exists if and only if (4). It is easy to check that (4) holds whenever $x \geq 2$.

The previous literature on auctions have shown that for certain ascending auctions we can attain these properties as long as buyers’ values satisfy certain restrictive conditions. For example Gul and Stacchetti (2000), and Ausubel and Milgrom (2002) assume that buyers’ values are substitutable, whereas Ausubel (2004) assumes that
buyers’ values are concave. On the contrary, we only assume that buyers like more

4.3 Housing Markets

In a housing market, there are \( n \) agents with unit demand for houses. Each agent

owns a separate house initially. \( \mathcal{A} \) is such that each agent is allocated one house.\(^{6}\)

Agent \( i \)'s type is \( \theta_i = (\theta_{i1}, \ldots, \theta_{in}) \) where \( \theta_{ij} \) is the value that agent \( i \) has for house \( j \).

For a housing market, we show that there does not exist a mechanism which

satisfies dominant strategy incentive compatibility, individual rationality, efficiency,

and budget feasibility if the type space is rich enough.

Corollary 6. If \( n \geq 2 \) and there exists \( c > 0 \) such that \( \Theta_i \supseteq [0, c]^n \) for all \( i \in N \), then

there does not exist a dominant strategy incentive compatible, individually rational,

efficient, and budget feasible mechanism for the housing market.

The proof in the Appendix provides an example where (4) fails.

Similarly, there does not exist a dominant strategy incentive compatible, efficient,

and budget balanced mechanism for the housing market.

Corollary 7. Suppose that \( n \geq 2 \) and that there exists \( c > 0 \) such that \( \Theta_i \supseteq [0, c]^n \)

for all \( i \in N \). Then there does not exist a dominant strategy incentive compatible,

efficient, and budget balanced mechanism for the housing market.

\(^{6}\)Shapley and Scarf (1974) introduce the housing market without the possibility of making transfers.
The intuition of the proof is as follows. There exists such a mechanism if and only if (5) holds. However, each term in (5) is a maximum function choosing one of the arguments unambiguously for a generic vector of valuations. Therefore, for a generic vector of values the right hand side of (5) is linear in utility terms which must be zero. Again, for a generic vector of values this equality holds only if the coefficient of each term on the left hand side is zero. However, when we increase the value of a utility term until it is selected by one of the maximum functions, the coefficient of this term can no longer be zero. Thus such a mechanism cannot exist.

On the other hand, if each agent’s value for any house is at least as much as the value for her own house we get a possibility result as an application of Proposition 1 (Part 1).

Corollary 8. Suppose that for all \( \theta \in \Theta, \theta_{ij} \geq \theta_{ii} \) for all \( i \) and \( j \). Then there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism for the housing market.

### 4.4 Roommates Problem with Transfers

In a roommates problem with transfers, there are \( n \) agents, each of which owns a unique good. \( \mathcal{A} \) is such that if agent \( i \) gets good \( j \), then agent \( j \) gets good \( i \).\(^7\) The interpretation is that each agent’s good is a permit to be roommates with that agent. Therefore, all agents are paired up as roommates or left alone. The model can also be viewed as a model of partnership formation in which only partnerships with two agents can form. Agent \( i \)'s type is \( \theta_i = (\theta_{i1}, \ldots, \theta_{in}) \) where \( \theta_{ij} \) is the value of forming a partnership with agent \( j \), or the value of good \( j \), for agent \( i \).

\(^7\)Gale and Shapley (1962) introduce the roommates problem without the possibility of making transfers.
If the type space is rich enough then there exists no dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism.

**Corollary 9.** If \( n \geq 2 \) and there exists \( c > 0 \) such that \( \Theta_i \supseteq [0, c]^n \) for all \( i \in N \), then there does not exist a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism for the roommates problem with transfers.

The proof in the Appendix constructs an example with two agents which can be embedded in larger economies.

However, if we assume that pairing up with anyone is as good as staying alone, then we get a possibility result.

**Corollary 10.** Suppose that for all \( \theta \in \Theta \), \( \theta_{ij} \geq \theta_{ii} \) for all \( i, j \in N \). Then there exists a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism for the roommates problem with transfers.

The deriving force for this corollary is the assumption that each agent prefers to be paired up rather than staying single, which might be a plausible assumption for some markets. The proof follows from Part 1 of Proposition 1. It turns out that we cannot require budget balance in this mechanism, even if individual rationality is not required when the type space is rich enough.

**Corollary 11.** Suppose that there are at least three agents and there exists \( c > 0 \) such that \( \Theta_i \supseteq [0, c]^{i-1} \times \{0\} \times [0, c]^{n-i} \). Then there exists no dominant strategy incentive compatible, efficient, and budget balanced mechanism for the roommates problem with transfers.

There exists such a mechanism if and only if (3) holds for the roommates problem with transfers. Since the type space is rich enough we can provide a counter example as in the proof of Corollary 7. We omit the proof.
5 Conclusion

We provide a characterization of ex post incentive compatible direct revelation mechanisms in combination with other properties that allow interdependent values and correlated types. To be specific, we give necessary and sufficient conditions for the existence of mechanisms with the following properties:

- ex post incentive compatibility, individual rationality, and budget feasibility,
- ex post incentive compatibility and budget balance.

Our model is general and includes several market design problems as applications (with the addition of monetary transfers) by restricting the set of feasible allocations, $A^f$. However, depending on the restriction, the answers that we get for the existence of efficient mechanisms with the above two lists of properties may change. In general, the first set of properties may be satisfied with an additional, and sometimes restrictive, assumption. However, the second set of properties generically fails.

It has often been noted that results for matching markets with transfers and without transfers have parallels. To quote a few, Roth et al. (1993) note that

... one of the oldest puzzles arising out of the game theoretic analysis of two-sided matching concerns the fact that virtually identical conclusions about the core arise from two apparently quite different models, namely the marriage model of Gale and Shapley (1962) and the linear assignment model of Shapley and Shubik (1972).

Similarly, Balinski and Gale (1990) state the following:

There is, by now, a substantial literature on these problems and one is struck by the fact that almost all results proved for the ordinal case have analogues in the cardinal case...
Given the above consensus, one might wonder *why* we study one-to-one matching markets with transfers under incomplete information. There are two main reasons for this.

The first reason is that, although the evidence thus far agrees with the above observation for models with complete information (see Roth and Sotomayor (1990)), results do not *always* carry over between these two models in the presence of incomplete information.\(^8\)

The second reason is that, allowing the possibility of making endogenous transfers allows us to ask questions specific to the markets with transfers. For example, whether it is possible to have budget feasible or budget balanced mechanisms, which is a meaningless question in the absence of transfers.

In the seller-buyer application, we have shown that there exists a dominant strategy incentive compatible, individually rational, efficient, and budget balanced mechanism. We can construct a direct revelation mechanism that has these properties. However, when it comes to applications this game might be hard to implement as each agent submits a bid for each possible combination of goods whose number increases exponentially with the number of goods. Thus, it might be desirable to find an indirect game, perhaps a simultaneous ascending auction, that implements this allocation. This remains an open question for future research.

\(^8\)For example, Roth and Rothblum (1999) show that in a *symmetric* environment the best response of firms for the worker-proposing deferred acceptance procedure is to use *truncation strategies*. Truncation strategies have been defined for markets with transfers by Day and Milgrom (2008). It can be shown with a simple example that in a one-to-one matching environment with transfers truncation strategies are not best responses.
6 Appendix: The Omitted Proofs

Proof of Lemma 1.

“Only if” part: This is Theorem 3 of Chung and Ely (2002). It holds because $\Theta_i$ is connected, $u_i$ is continuous in $\theta_i$, and $\mathcal{A}^f$ is finite.

“If” part: Suppose $(\mu, t)$ is ex post incentive compatible and $t'$ is such that $t_i(\theta) = t'_i(\theta) + g_i(\theta_{-i})$. Since $(\mu, t)$ is ex post incentive compatible, $v_i(\theta) \geq [u_i(\mu_i(\theta'_i, \theta_{-i}), \theta) - u_i(e_i, \theta)] + t_i(\theta'_i, \theta_{-i}) \forall i, \forall \theta_i, \forall \theta_{-i}$. Which is equivalent to $[u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t_i(\theta'_i, \theta_{-i})$. Substitute $t'_i(\theta) + g_i(\theta_{-i})$ for $t_i(\theta)$ and $t'_i(\theta, \theta_{-i}) + g_i(\theta_{-i})$ for $t_i(\theta'_i, \theta_{-i})$ and cancel $g_i(\theta_{-i})$ to get $v'_i(\theta) \equiv [u_i(\mu_i(\theta), \theta) - u_i(e_i, \theta)] + t'_i(\theta) \geq [u_i(\mu_i(\theta'_i, \theta_{-i}), \theta) - u_i(e_i, \theta)] + t'_i(\theta'_i, \theta_{-i})$. Hence, $(\mu, t')$ satisfies ex post incentive compatibility. ■

Proof of Corollary 1.

Suppose without loss of generality that agent 1’s type has a degenerate distribution.

Part 1: Define $t'$ as follows: $t'_i(\theta) = t_i(\theta) - \sum_{j \neq 1} t_j(\theta)$ and for all $i \neq 1$, $t'_i(\theta) = t_i(\theta)$. Since $t_j(\theta)$ is a function of $\theta_{-1}$ for $j \neq 1$, $(\mu, t')$ is ex post incentive compatible by Lemma 1. It is also budget balanced.

Part 2: Since budget balance implies budget feasibility, the existence of $(\mu, t')$ with the properties stated implies (1) by Theorem 1. For the other direction suppose that (1) holds. Then there exists an ex post incentive compatible, individually rational and budget feasible mechanism. Construct a new transfer function of agent 1 by giving the surplus created. Since $\theta_1$ has a degenerate distribution the new mechanism is also ex post incentive compatible. Moreover, it is individually rational and budget balanced. ■
Proof of Proposition 1.

**Part 1:** Let \( t_i(\theta) = SS(\theta) - \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta_{-i}) - [u_i(\mu^e_i(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] \). It is clear that \((\mu^e, t)\) is dominant strategy incentive compatible, individually rational, and efficient.

We are going to show that it is budget feasible, i.e., \( 0 \geq \sum_i t_i(\theta) = (|N| - 1) SS(\theta) - \sum_i \inf SS(\hat{\theta}_i, \theta_{-i}) \).

Let \( \mathcal{A}^f(i = \nu_i) \) be the set of feasible allocations such that agent \( i \) is allocated \( \nu_i \).

In general this may be an empty set. However, we will only consider \( \nu_i \) for which this set is non-empty.

Now, \( SS(\hat{\theta}_i, \theta_{-i}) \geq [u_i(\mu^e_i(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] + \max_{\mu \in \mathcal{A}^f(i = \mu^e_i(\theta))} \sum_j [u_j(\mu^e_j(\theta), \theta_j) - u_j(e_j, \theta_j)] \). On the right hand side, we fix the allocation of agent \( i \) to be \( \mu^e_i(\theta) \) and maximize the sum of allocation utilities over \( \mathcal{A}^f(i = \mu^e_i(\theta)) \). Since the left hand side is the unconstrained maximum, it must be greater than the right hand side. If we take the infimum of both sides with respect to \( \hat{\theta}_i \) we get \( \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta_{-i}) \geq \inf_{\hat{\theta}_i} [u_i(\mu^e_i(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] + \max_{\mu \in \mathcal{A}^f(i = \mu^e_i(\theta))} \sum_j [u_j(\mu^e_j(\theta), \theta_j) - u_j(e_j, \theta_j)] \). Note that the last term is greater than \( \sum_j [u_j(\mu^e_j(\theta), \theta_j) - u_j(e_j, \theta_j)] \) since \( \mu^e(\theta) \in \mathcal{A}^f(i = \mu^e_i(\theta)) \). Hence, we have shown that \( \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta_{-i}) \geq \inf_{\hat{\theta}_i} [u_i(\mu^e_i(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] + \sum_{j \neq i} [u_j(\mu^e_j(\theta), \theta_j) - u_j(e_j, \theta_j)] \). If we sum this over all agents we get:

\[
\sum_i \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta_{-i}) \geq \sum_i \inf_{\hat{\theta}_i} [u_i(\mu^e_i(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] + (|N| - 1) SS(\theta).
\]

Note that \( \sum_i \inf_{\hat{\theta}_i} [u_i(\mu^e_i(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] \geq 0 \) by assumption. Therefore, \( \sum_i \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta_{-i}) \geq (|N| - 1) SS(\theta) \) which completes the proof.

**Part 2:** Assume that \( \sum_i \inf_{\hat{\theta}_i, \mu^e_i(\theta_i, \theta_{-i})=\mu_i^e(\theta)} [u_i(\mu^e_i(\theta), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] < 0 \) for some \( \theta = \theta^* \). We are going to show that (4) fails for \( \theta^* \).

Now, \( \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta_{-i}) \leq \inf_{\hat{\theta}_i, \mu^e_i(\theta_i, \theta_{-i})=\mu_i^e(\theta^*)} [u_i(\mu^e_i(\theta^*), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] + \max_{\mu \in \mathcal{A}^f(i = \mu^e_i(\theta^*))} \sum_{j \neq i} [u_j(\mu_j, \theta^*) - u_j(e_j, \theta^*)] \), because the right hand side is a constrained infimum whereas the left hand
side is the unconstrained infimum. If we sum this over all agents, we get:

\[
\sum_i \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta^*_i) \leq \sum_i \inf_{\hat{\theta}_i, \mu^*_i(\hat{\theta}_i, \theta^*_i) = \mu^*_i(\theta^*)} [u_i(\mu^*_i(\theta^*), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] + (|N| - 1)SS(\theta^*)
\]

Since \( \sum_i \inf_{\hat{\theta}_i, \mu^*_i(\hat{\theta}_i, \theta^*_i) = \mu^*_i(\theta^*)} [u_i(\mu^*_i(\theta^*), \hat{\theta}_i) - u_i(e_i, \hat{\theta}_i)] < 0 \) by assumption, the last inequality implies \( \sum_i \inf_{\hat{\theta}_i} SS(\hat{\theta}_i, \theta^*_i) < (|N| - 1)SS(\theta^*) \). Thus, (4) fails which means that there exists no mechanism with the required properties. This completes the proof. ■

**Proof of Corollary 3.**

The first part is immediate from Theorem 2.

**Part 2:** For each agent \( i \), we show that \( u_i \) terms appearing in (5) cancel out. The terms with the endowment, \( u_i(e_i, \hat{\theta}_i) \), cancel out by counting. Now consider the terms with the efficient allocation, \( u_i(\mu^*_i(\theta), \hat{\theta}_i) \). Take any subset of agents \( j \notin \{i_1, \ldots, i_s\} \). Consider \( SS(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_{-i_1}, \ldots, \theta_{-i_s}) \) and \( SS(\theta_{i_1}, \ldots, \theta_{i_j}, \theta_j, \theta_{-j}, \theta_{-i_1}, \ldots, \theta_{-i_s}) \). Since agent \( i \)'s efficient allocation is the same when agent \( j \)'s type change, \( u_i \)'s in these two expressions have the same argument, and therefore, they are the same. In (5), they have different signs since one of them has one more fixed type. Since we can pair up any subset \( S \) which does not include \( j \) with \( S \cup \{j\} \) in the same manner, all the \( u_i \) terms cancel out and (5) holds. ■

**Proof of Lemma 2.**

Suppose that agents report truthfully. Let us show that agent \( i \) does not have incentives to change her report. There are two cases depending on whether agent \( i \) is the winner or not.

**Case 1 (agent \( i \) is the winner):** In this case agent \( i \)'s net utility is \( u_i(\theta_i, \theta_{-i}) \).
Any other report for which she wins the auction does not change her net utility. Suppose that \( \theta_i' \) is such that agent \( i \) loses the auction. Then, by Assumption 3, \( \theta_i' < \theta_i \) which implies that \( u_j(\theta) \geq u_j(\theta_i', \theta_{-i}) \) for all \( j \) by Assumption 2. Therefore, if we take the maximum of both sides then we get \( \max_{j \neq i} u_j(\theta) \geq \max_{j \neq i} u_j(\theta_i', \theta_{-i}) \). Finally if we take the supremum over both sides with respect to \( \theta_i' \) for which agent \( i \) loses the auction (which happens only when \( \theta_i' < \theta_i \)), we get \( \max_{j \neq i} u_j(\theta) \geq t_i(\tilde{\theta}_i, \theta_{-i}) \) where \( \mu_i^e(\tilde{\theta}_i, \theta_{-i}) = 0 \) for any such \( \tilde{\theta}_i \). The left hand side of this equation is less than or equal to \( u_i(\theta_i, \theta_{-i}) \) since agent \( i \) is the winner when the type profile is \( \theta \) and the right hand side is the maximum utility she could gain by lying and losing the auction. Therefore, agent \( i \) does not have a profitable deviation in this case.

**Case 2 (agent \( i \) is a loser):** In this case agent \( i \)'s net utility is \( t_i(\theta) = \sup_{\tilde{\theta}_i \text{ s.t. } \mu_i^e(\tilde{\theta}_i, \theta_{-i})=0} \max_{j \neq i} u_j(\tilde{\theta}_i, \theta_{-i}) \). Any report for which she is still a loser does not change her net utility. However, any report which makes her win the auction gets her a net utility of \( u_i(\theta_i, \theta_{-i}) \) which is less than or equal to her net utility under the original report. Therefore, agent \( i \) does not have a profitable deviation in this case.

**Proof of Corollary 5.**

Suppose that \((\mu^e, t)\) is the mechanism described in Proposition 2. By Theorem 1 such a mechanism exists if and only if \( \sum_i \inf_{\theta_i \in \Theta_i} v_i(\theta_i, \theta_{-i}) \geq \sum_i t_i(\theta), \forall \theta \in \Theta \) where \( v_i(\theta) \) is the net utility of agent \( i \) from \((\mu^e, t)\). Note that when \( i \) is a loser her payment does not depend on this agent’s type, so \( \inf_{\theta_i \in \Theta_i} v_i(\theta_i, \theta_{-i}) = t_i(\theta) \). Therefore, the inequality reduces to \( \inf_{\theta_i \in \Theta_i} v_i(\theta_i, \theta_{-i}) \geq 0 \) where \( \mu_i^e(\theta) = 1 \). The last inequality holds by Assumption 1.

**Proof of Proposition 2.** By Corollary 2, there exists a mechanism which is dominant strategy incentive compatible, individually rational, efficient, and budget
feasible if and only if

$$\sum_{i} \inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) \geq (|N| - 1)SS(\theta), \forall \theta \in \Theta.$$  

For $x \in \mathbb{Z}^k_+$ such that $x \leq \sum_{i \in S} e_i$, define

$$C(x|\theta_S) = \min_{y_1 + \ldots + y_n = x} \sum_{y_1 \leq e_1, \ldots, y_n \leq e_n} C_i(y_i, \theta_i).$$

$C(x|\theta_S)$ is the cost of providing vector $x$ when the type profile for the sellers is $\theta_S$. With this notation we can rewrite $SS(\theta)$ as $\sum_{i \in B} u_i(\mu_i^e(\theta), \theta_i) - C(\sum_{i \in B} \mu_i^e(\theta)|\theta_S)$.

**Claim:** $(m - 1)C(\sum_{i \in B} \mu_i^e(\theta)|\theta_S) \geq \sum_{i \in B} C(\sum_{j \neq i, j \in B} \mu_j^e(\theta)|\theta_S)$.

**Proof:** Fix a type of good. Let $r$ be the total number of times this good is counted in $\sum_{i \in B} \mu_i^e(\theta)$ and $r_i$ be the number of times it is counted in $\sum_{j \neq i, j \in B} \mu_j^e(\theta)$. Note that $r_i \leq r \forall i \in B$ and $\sum_{i \in B} r_i = (m - 1)r$. Without loss of generality assume that $r_i$ is increasing.

Let $x_i$ be the cost of supplying the $i$-th copy of the good, so $0 \leq x_1 \leq \ldots \leq x_r$. Thus, $(m - 1)C(\sum_{i \in B} \mu_i^e(\theta)|\theta_S) = (m - 1)(x_1 + \ldots + x_r)$ and $C(\sum_{j \neq i, j \in B} \mu_j^e(\theta)|\theta_S) = x_{1 + \ldots + x_{r_i}}$.

We want to show $(m - 1)(x_1 + \ldots + x_r) \geq \sum_{i = 1}^{m} (x_1 + \ldots + x_{r_i})$ to prove the claim. Let us count the sum of coefficients of $x_i$ where $j + 1 \leq i \leq n$. For the left hand side it is $(m - 1)(r - j)$. For the right hand side if $j + 1 \leq r_1$, then the sum of the coefficients is $(r_1 - j) + \ldots + (r_m - j) = (m - 1)r - mj$, less than the sum for the left hand side. Otherwise, if $r_i < j + 1 \leq r_{i+1}$ for some $i$, then the sum of the coefficients is $(r_{i+1} - j) + \ldots + (r_m - j) = (m - 1)r - (r_1 + \ldots + r_i) - (m - i)j$ which is less than or equal to the sum of the coefficients for the right hand side if and only if $r_1 + \ldots + r_i \leq ij$, which holds. Therefore, for any $0 \leq x_1 \leq \ldots \leq x_r$, $(m - 1)(x_1 + \ldots + x_r) \geq \sum_{i = 1}^{m} (x_1 + \ldots + x_{r_i})$, so the claim holds.
For $i \in S$, $\inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) = SS(\theta)$ since $\theta_i$ can take only one value (since it has a degenerate distribution). Thus,

$$\sum_{i \in S} \inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) = nSS(\theta).$$  \hspace{1cm} (6)

For $i \in B$, and any type profile we can allocate every other buyer according to $\mu^e(\theta)$. Hence, $\inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) \geq \sum_{j \neq i, j \in B} u_j(\mu^e_j(\theta), \theta_j) - C(\sum_{j \neq i, j \in B} \mu^e_j(\theta)|\theta_S)$ where the cost function only depends on the type profile of the buyers. If we sum this up over all $i \in B$ we get:

$$\sum_{i \in B} \inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) \geq (m - 1) \sum_{i \in B} u_j(\mu^e_j(\theta), \theta_j) - \sum_{i \in B} \sum_{j \neq i, j \in B} \mu^e_j(\theta)|\theta_S).$$  \hspace{1cm} (7)

If we add up (6) and (7) we get

$$\sum_{i \in N} \inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) \geq nSS(\theta) + (m - 1) \sum_{i \in B} u_j(\mu^e_j(\theta), \theta_j) - \sum_{i \in B} \sum_{j \neq i, j \in B} \mu^e_j(\theta)|\theta_S)$$

$$\geq nSS(\theta) + (m - 1) \sum_{i \in B} u_j(\mu^e_j(\theta), \theta_j) - (m - 1) C(\sum_{i \in B} \mu^e_i(\theta)|\theta_S)$$

$$= nSS(\theta) + (m - 1) SS(\theta) = (n + m - 1) SS(\theta).$$

The second inequality follows from the claim proven above. Hence, we have shown that a dominant strategy incentive compatible, individually rational, efficient, and budget feasible mechanism exists. To get the budget balance distribute, the surplus to sellers in any way. Since they have degenerate distributions, they do not face any incentive issues. Hence, budget balance is also satisfied in addition to other properties. ■
Proof of Corollary 6. There exists such a mechanism if and only if (4) holds. We show that this inequality is violated for an example with 2 agents. For any bigger economy this example can be embedded when the type profile of the other agents is fixed at 0.

Let $d$ be a constant such that $0 < d \leq c$. The first agent’s type is $(0, d)$- his value for the house he has is 0, his value for the other house is $d$. The second agent’s type is $(d, 0)$- her value for the house she owns is 0 and her value for the other one is $d$. In the efficient allocation agents swap their houses, so $SS(\theta) = 2d$. The social surplus is minimized when the first agent’s type is $(d, 0)$ in which case the social surplus is 0. Similarly, the social surplus is minimized when the second agent’s type is $(0, d)$ in which case the social surplus is 0. Therefore, $\sum \inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) = 0$. Hence, (4) fails. The conclusion follows.

Proof of Corollary 7. By Corollary 3 such a mechanism exists if and only if (5), that is,

$$\sum_{j=0}^{n} (-1)^j \sum_{\{i_1, \ldots, i_j\} \subseteq N} SS(0, \ldots, 0, \theta_{-i_1, \ldots, i_j}) = 0 \text{ for all } \theta.$$ 

Let $\Theta_0$ be the subset of $\Theta$ such that there does not exist a linear relationship among $\theta_{ij}$ with integer coefficients such as $2\theta_{11} = \theta_{12} + \theta_{22}$ excluding the trivial coefficients. Since each equation defines a hyperplane in the Euclidean space, the Lebesgue measure of the set of points for which one equation is satisfied is zero. Moreover, there is a countable number of such equations and $\Theta_0^c$ is the union of sets defined by these equations, so $\Theta_0^c$ has zero Lebesgue.

For any $\theta \in \Theta_0$ consider (5). Each term in the sum is determined uniquely by definition of $\Theta_0$. Therefore, we get an equation in terms of $\theta_{ij}$ with integer coefficients. Again, by definition of $\Theta_0$, no such equation holds unless the coefficients are all zero.
This means that for all $i, j$, $\theta_{ij}$ appears an even number of times in the sum matching the positive and negative appearances.

Now, take $\hat{\theta} \in \Theta_0 \cap [0, d]^n$ where $0 < d < c$. Since $\Theta_0^c$ has a zero Lebesgue measure, there exists such a point. Let $\hat{\theta}_{1i} < \hat{\theta}_{1j}$ for all $j \neq i$. Increase $\hat{\theta}_{1i}$ continuously until there is a tie in one of the terms in (5). Such a term exists since if it does not happen before there will be a tie in the term $SS(\theta_1, 0, \ldots, 0) = \max\{\theta_{11}, \ldots, \theta_{1n}\}$. Moreover, there can only be one tie at any point. Otherwise, there is a linear relationship between some coordinates of $\hat{\theta}$ with integer coefficients. Increase $\hat{\theta}_{1i}$ further by a small amount to $\theta'_{1i}$ so that the tie is broken for that term, other terms in the sum do not change and $(\theta'_{1i}, \hat{\theta}_{-1i}) \in \Theta_0$. This is possible since $\Theta_0^c$ is a union of countable number of hyperplanes none of which is $\theta_{1i} = \hat{\theta}_{1i}$. By construction, $\theta'_{1i}$ appears an odd number of times in (5) for $(\theta'_{1i}, \hat{\theta}_{-1i})$ contradicting $(\theta'_{1i}, \hat{\theta}_{-1i}) \in \Theta_0$.

**Proof of Corollary 9.** There exists such a mechanism if and only if (4) holds. We show that this inequality is violated for an example with 2 agents. For any bigger economy this example can be embedded when the type profile of the other agents is fixed at 0.

Let $d$ be a constant such that $0 < d \leq c$. The first agent’s type is $(0, d)$- his value for his good is 0 and his value for the other house is $d$. The second agent’s type is $(d, 0)$- her value for her good is 0 and her value for the other one is $d$. In the efficient allocation agents swap their goods, so $SS(\theta) = 2d$. The social surplus is minimized when the first agent’s type is $(d, 0)$ in which case the social surplus is 0. Similarly, the social surplus is minimized when the second agent’s type is $(0, d)$ in which case the social surplus is 0. Therefore, $\sum_i \inf_{\theta_i \in \Theta_i} SS(\theta_i, \theta_{-i}) = 0$. Hence, (4) fails. The conclusion follows. ■
References


