Mathematical Underpinnings of the Office Selection Protocol

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4. Some Mathematical Underpinnings

Key to the office selection protocol is the existence of an adequate match algorithm designed to optimize some reasonable measure of rank-weighted happiness. We can represent each faculty member and each office as a node in a graph, where the edges in the graph correspond to possible assignments of faculty to offices. Since faculty rank only offices (and not other faculty), the graph is clearly *bipartite*, that is, no faculty node is connected to anything but office nodes (and vice versa). An assignment of faculty to offices corresponds to solving the matching problem for bipartite graphs, that is, selecting a subset of these edges such that no faculty node is connected to more than one office node (and vice versa). Fortunately, there exist several efficient algorithms for this problem [Bondy76, Kozen92].

Note that there are some special properties of this particular bipartite graph. First, the number of office nodes is always at least as large as the number of faculty nodes. This is because, presumably, the College will see to it that faculty making required moves have at least *some* office space, while faculty making elective moves also "contribute" their current offices to the graph.

The second important property is that, for any subset of faculty nodes, the number of corresponding office nodes (*i.e.*, the size of the set of feasible matchings for this subset of faculty) is at least as large as the number of faculty. This property follows from the fact that faculty making required moves must rank order all vacant offices, and faculty making elective moves will rank order at least their own office (and, more likely, at least two offices, since otherwise they'd have no reason to enter the selection protocol). This property is important, because it is both a necessary and sufficient condition to ensure that at an assignment that "saturates" all faculty (*i.e.*, uniquely assigns each faculty member to an unoccupied office) exists [Hall35].

Now if the edges of the graph are assigned "weights" corresponding to the contribution that edge makes to the rank-weighted happiness metric in any eventual match, we can restate the problem as the *optimal assignment problem*, where the weights of the selected edges are maximized. While one could simply enumerate all $n!$ possible matchings (where n is the number of faculty and $m \geq n$ is the number of vacant offices), more efficient algorithms for this problem are well known [Kuhn55, Munkres57].

The only remaining issue, then, is how to best compute the weights assigned to the edges of the bipartite graph, where an edge weight $w_{i,j}$ corresponds to an edge linking faculty member i (where

$0 < i \le n$ and. as before, n is the number of faculty in the protocol) with office j (where $0 < j \le m \le n$ and, as before, m is the number of vacant offices). Recall the ith faculty member is assigned a rank priority f_i between 1 (highest priority) and n (lowest priority) in Step 4 of the protocol, and that each faculty member assigns an office rank priority $o_{i,j}$ for at least some of the offices on the vacancy list. Edge weights are then set according to the following formula:

$$w_{i,j} = \left(1 - \frac{f_i - 1}{n}\right) \times \left(1 - \frac{o_{i,j} - 1}{m}\right)$$

Thus, for example, the edge corresponding to the top ranking faculty member's first choice office has unit weight.

5. An Example

An example should help make this clear. Consider six faculty named Alpha, Beta, Gamma, Delta, Epsilon, and Zeta, where Alpha is a full professor, Beta, Gamma and Delta are Associate Professors, and Epsilon and Zeta are Assistant Professors. Assume that Beta is making a required move, since his current office (Rm 110) has an expiring lien (Eta is finishing a term as DEO and is returning to her original office), and that Epsilon is a new hire. Assume further that there are a total of eight vacant offices (Rm 101 through Rm 108) in the pool, including the four offices currently occupied by Alpha (Rm 101), Gamma (Rm 103), Delta (Rm 104), and Zeta (Rm 105). Beta's current office is not included in the vacancy pool, since it is about to be reoccupied by Eta. Assume further that Rm 102 has an existing lien held by Theta, the current Associate Dean.

At this point (Step 3), each faculty member ranks the vacant offices in terms of personal preference (see table below, where current occupancy is indicated by bold face).

Office Preference Rankings

Alpha	Beta	Gamma	Delta	Epsilon	Zeta
108	101	108	106	108	108
106	103	**103**	**104**	106	**105**
101	107			107	
	108			101	
	106			104	
	105			105	
	104			102	
	102			103	

Note that Alpha, Gamma, Delta, and Zeta are ranking fewer than the eight vacant offices (since they are making elective moves), and that in each case the lowest ranked office is their current office. This ensures that they will never have to leave the office they currently occupy, unless they are moving to a more desirable office. Beta and Epsilon, however, must rank all eight vacant offices since they are in required move situations. Note also that Rm 102 remains relatively unpopular, undoubtedly because of the existing lien. While it may well be a desirable office, knowledge that any assignment to Rm 102 will only be temporary causes Rm 102 to appear well down in the rankings by faculty making required moves, and to be entirely absent from the rankings of faculty making elective moves.

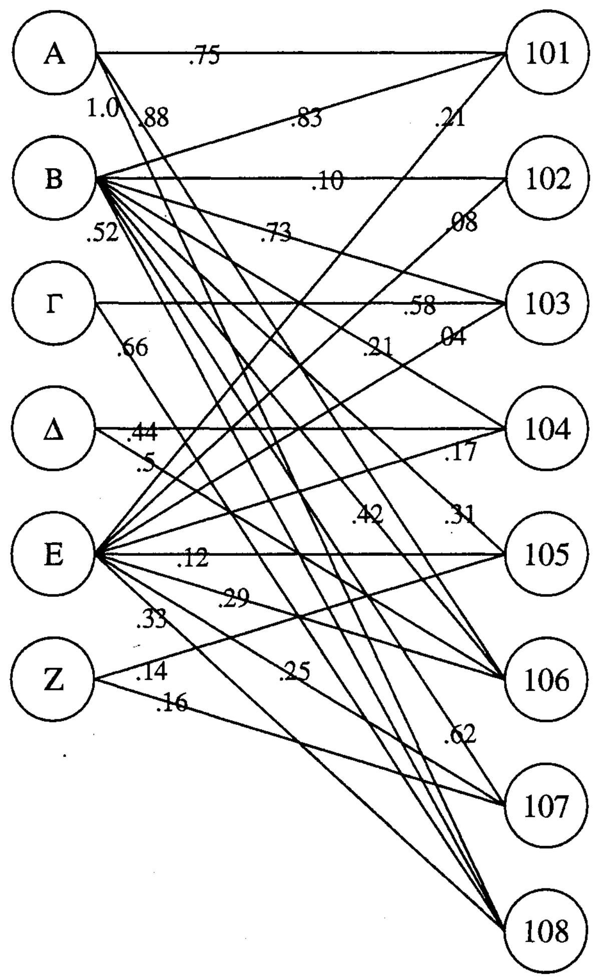
Following the office ranking step, the Dean's office generates a priority ranking for the faculty participants. Since Alpha is the only full professor, they will be first in line. Beta, Gamma, and Delta are equal-ranking Associate Professors, but since Beta is involved in a required move, he ranks before Gamma and Delta, who are randomly ordered. In a similar fashion, Epsilon takes priority over Zeta given that Epsilon, as a new hire, is making a required move. Thus without loss of generality, we can assume the Dean's office ranking, in this example, is isomorphic to alphabetical order on Greek characters.

Next, weights are assigned to each individual edge in the bipartite graph. Let's look as Delta as a representative example. Since Delta has ranked only two offices (106 and 104, Delta's current office), we can easily compute the appropriate weights (recall Delta is fourth in the Dean's office ranking, so $f_\Delta = 4$).

$$w_{\Delta,106} = \left(1 - \frac{f_\Delta - 1}{6}\right) \times \left(1 - \frac{o_{\Delta,106} - 1}{8}\right) = (\frac{1}{2}) \times (1) = 0.5$$

$$w_{\Delta,104} = \left(1 - \frac{f_\Delta - 1}{6}\right) \times \left(1 - \frac{o_{\Delta,104} - 1}{8}\right) = (\frac{1}{2}) \times (\frac{7}{8}) = 0.4375$$

Computing the remaining weights as just shown, the resulting bipartite graph looks like:



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At this point, we're ready to find the optimal match on the bipartite graph. Applying the matching algorithm, the solution generated is:

Solution		
Alpha	108	Alpha's first choice
Beta	101	Beta's first choice, vacated by Alpha
Gamma	103	No change (108 goes to Alpha)
Delta	106	Delta's first choice, was vacant
Epsilon	107	Epsilon's third choice (108 goes to Alpha & 106 goes to Delta)
Zeta	105	No change (108 goes to Alpha)

Since the only office in the current selection with an existing lien (Rm 102) remains unassigned, all assignments in the solution are considered permanent assignments. The Dean's office staff would now arrange a convenient moving schedule to accommodate this assignment.

References

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