# The Optimality of Debt*

Pierre Chaigneau
Queen's University

Alex Edmans
LBS, CEPR, and ECGI

Daniel Gottlieb
LSE

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**Abstract**

Standard theories of debt consider a risk-neutral manager and a single contractible performance measure ("output"). It seems that debt may no longer be optimal if the manager is risk-averse and thus dislikes being the residual claimant, or if additional performance signals are available. This paper shows that debt remains the optimal contract under additional signals – they only affect the contractual debt repayment, but not the form of the contract. While debt may remain optimal under risk aversion, this may not be the case even if risk aversion is low, and even if there is no trade-off between incentives and risk sharing.

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*pierre.chaigneau@queensu.ca, aedmans@london.edu, d.gottlieb@lse.ac.uk.

The vast majority of firms issue debt. In some cases, such as most start-ups, debt is the only external source of financing. A large theoretical literature has therefore aimed to understand when debt contracts are optimal. Most justifications are based on moral hazard. In a costly state verification framework, Townsend (1979) and Gale and Hellwig (1985) show that debt contracts minimize audit costs while inducing truthful reporting of the firm's output. Hart and Moore (1998) find that debt allows for external funding even when output is not contractible and can be diverted by the manager. When output is contractible, Innes (1990) demonstrates that debt is optimal if the manager is protected by limited liability and a monotonicity constraint ensures that investors' payoff is non-decreasing in output. Intuitively, limited liability prevents investors from punishing the manager for low output, so they instead incentivize him by maximizing his rewards for high output. Due to the monotonicity constraint, the manager cannot gain more than one-for-one. He is thus the residual claimant, receiving equity; investors receive debt.

Two assumptions seem critical for generating this result. The first is that output is the only signal of effort and thus the only possible determinant of pay. When output $q$ is below the contractual repayment $q^*$, this is sufficiently bad news about effort that the manager is paid zero; otherwise, he receives the residual $q - q^*$. In reality, investors have access to multiple additional signals of performance, such as sales, profits, market share, credit ratings, or peer performance. If these signals are sufficiently good news, it may seem optimal to pay the manager a strictly positive amount even if $q < q^*$; a negative signal may mean that the manager should receive less than the residual even if $q > q^*$.

The second assumption is that the manager is risk-neutral, and so there is no trade-off between incentives and risk sharing. A debt contract provides the manager with the maximum possible incentives by making him the residual claimant. However, most managers are risk-averse in reality. The textbook of Tirole (2006) conjectures that debt is no longer optimal under risk aversion since the contract must then balance incentives with risk sharing.[1] Moreover, it seems that even an infinitessimal amount of risk aversion would change this result, just as the Borch Rule for risk-sharing requires the manager must be exactly risk-neutral to be the residual claimant and bear all the output volatility. As Tirole (2006) writes: "the theories ... are often criticized for their lack of robustness; it is also pointed out that they do not account for the diversity of capital structures that characterize modern corporations."

[1] He writes "We have assumed that the entrepreneur and the lenders are both risk neutral. Does the debt optimality result carry over to, say, entrepreneurial risk aversion? When the entrepreneur is risk averse, the optimal contract must, besides satisfying the lenders' breakeven constraint, aim at two targets: effort inducement and insurance. As is well-known (see, for example, Holmstrom 1979), these two goals are, in general, in conflict. Insuring the entrepreneur against variations in profit makes her unaccountable, and results in a low level of effort."

This paper studies whether debt remains optimal in a standard moral hazard setting even if there are other signals of performance, and even if the manager is risk-averse. Moreover, it analyzes under what conditions these features cause debt no longer to be optimal, thus accommodating the diversity of capital structures that Tirole (2006) highlights.

We first study the case in which the principal has access to a signal $s$ of effort in addition to output $q$, while retaining a risk-neutral manager. The signal could affect the optimal contract in two ways. First, debt might no longer be optimal. Debt is "bang-bang" in that the manager receives the lowest possible amount (zero) below $q^*$, and the highest possible amount (the residual) above it. It may seem that any informative signal will perturb the optimal contract so that the manager's payoff optimally lies between the extremes. In contrast, we show that debt remains the optimal contract even under strictly informative signals – and even if the signals are informative everywhere, i.e. provide information about effort regardless of the output level.

Second, the signal could change the face value of debt so that it depends on the signal, and so the contract becomes performance-sensitive debt. For example, a signal that indicates high effort (such as a high credit rating) could lower $q^*$ and increase the manager's payoff. Indeed, Holmström's (1979) informativeness principle showed that any informative signal affects the contract. However, we show that a signal may be informative almost everywhere, yet affect neither the form of the contract nor the debt repayment, so the contract remains standard rather than performance-sensitive debt. The difference from Holmström (1979) is that there are no contracting constraints in his model, and so the principal can always make use of a signal by changing the contract in response. When constraints bind, the contract cannot change. If $q < q^*$ and the signal suggests that the manager has shirked (i.e. that $q < q^*$ is due to low effort rather than bad luck), the principal cannot use the signal to reduce the payment since the manager is receiving zero anyway: the limited liability constraint binds. Likewise, for $q > q^*$, the principal cannot use the signal to increase the payment since the monotonicity constraint binds.

We derive a necessary and sufficient condition for a signal to affect the contract under contracting constraints. A signal can only change the contractual debt repayment, which in turn depends on the likelihood ratio of the event $q \geq q^*$ – unlike the standard likelihood ratio which concerns an individual output realization. Intuitively, with a binding monotonicity constraint, changing the debt repayment changes the wage for all $q \geq q^*$. Thus, a signal only affects the contract if it affects the likelihood ratio that $q \geq q^*$, i.e. is informative about whether output exceeding the debt repayment is the outcome of effort or luck. This is a stronger condition than in Holmström (1979): even if a signal is informative almost everywhere, it does not affect the contract if it is not informative about this specific event. For example, a signal that indicates that high output is due to high effort rather than good luck is informative, but not valuable since debt is fully repaid under high output anyway.

We next study the case in which the manager is risk-averse, but output is the only signal of performance. It is straightforward to demonstrate that the optimal contract pays the manager zero for outputs below the threshold, but for debt to be optimal we also need the slope above the threshold to be 1, the highest possible given the monotonicity constraint. On the one hand, the desire to provide incentives raises the slope; on the other hand, diminishing marginal utility reduces the effectivness of incentives and lowers the slope. We show that, if the standard likelihood ratio rises fast enough with output (more formally, if the semi-elasticity of the likelihood ratio to output exceeds the manager's risk aversion), the first force dominates and the optimal contract is debt. Since debt involves the maximum sensitivity of 1 for all $q > q^*$, the semi-elasticity of the likelihood ratio must exceed the manager's absolute risk aversion for all $q > q^*$. As a result, a low level of risk aversion is insufficient for debt to be optimal. For example, if output is normally distributed, the output sensitivity effect is negative: the semi-elasticity falls with output. Thus, absolute risk aversion needs to fall with wealth (and thus output) for the condition to be satisfied for all output levels. If the manager exhibits constant absolute risk aversion, this will not be the case even if he has a low risk aversion coefficient. This contrasts the intuition that, since Innes's (1990) result was derived under risk neutrality, it will continue to hold under risk aversion as long as it is sufficiently low.

Interestingly, we still require a condition on the likelihood ratio even if the participation constraint is non-binding – even if investors do not need to compensate the manager for the risk associated with a performance-sensitive contract, and so the trade-off between incentives and risk-sharing highlighted by Tirole (2006) does not apply. Instead, the manager's risk aversion might lead to debt no longer being optimal for a different reason. A debt contract gives the manager high-powered incentives at high output levels, i.e. for $q > q^*$. For high output levels, the manager is already receiving high pay; diminishing marginal utility means that he attaches less value to additional pay, which reduces the effectiveness of high-powered incentives. Only if the likelihood ratio rises sufficiently fast with output to offset this marginal utility effect do high-powered incentives remain effective so that debt remains optimal. Thus, contrary to conventional wisdom, debt may be suboptimal even if there is no trade-off between incentives and risk-sharing.

Finally, we consider the most general case of both a risk-averse manager and additional performance signals, and show that our results continue to apply. Debt remains the optimal contract as long as the likelihood ratio rises fast enough with output and risk aversion does not rise too fast with output, although now the likelihood depends on the signal, and the output level at which risk aversion is evaluated is a function of the debt repayment, which also depends on the signal. The robustness of debt is consistent with the financing decisions of mature firms, which Leary and Roberts (2010) show to be driven by moral hazard, rather than information asymmetry, and also young firms which frequently raise debt, as documented by Robb and Robinson (2014) and Hwang, Desai, and Baird (2019).

This paper is related to other research on the optimality of debt contracts in richer agency settings than Innes (1990). Hébert (2018) analyzes the case in which a risk neutral manager can affect the dispersion of output in addition to its mean, and shows that debt remains optimal because it is the least risky security. Georgiadis, Ravid, and Szentes (2022) relax assumptions on the cost of choosing output distributions, and find that debt is generally not optimal. In a continuous time principal-agent model with a risk neutral agent, DeMarzo and Sannikov (2006) show that the optimal contract involves debt. As is well-known, debt can also be optimal in models of financing that involve adverse selection (Myers and Majluf (1984), Nachman and Noe (1994)), including in models with an endogenous information environment (Yang (2020), Inostroza and Tsoy (2023)) or Knightian uncertainty (Malenko and Tsoy (2023)).

# 1 The Model

There are two parties, a risk neutral principal (board acting on behalf of investors), and an agent (manager). The manager exerts an unobservable effort $e \in [0, \bar{e}]$. As is standard, effort can be interpreted as any action that improves output but is costly to the manager, such as working rather than shirking, choosing projects that generate cash flows rather than private benefits, or not extracting rents. The manager's utility function $u(\cdot)$ is strictly increasing, weakly concave, and twice continuously differentiable. His cost of effort $C(\cdot)$ is strictly increasing, strictly convex, twice continuously differentiable in $[0, \bar{e})$, with $C(0) = C'(0) = 0$ and $\lim_{e \nearrow \bar{e}} C'(e) = +\infty$. His reservation utility is $\overline{U}$.

Effort affects the probability distribution of output $q$ and a signal $s$, which are both observable and contractible. Output is continuously distributed with full support on $(\underline{q}, \overline{q})$, where we can have $\underline{q} = -\infty$ and/or $\overline{q} = \infty$. To ensure that an optimal contract exists, we assume that the signal is discrete, $s \in \{s_1, ..., s_S\}$. The signal can have one or multiple dimensions (i.e., be a vector).

The signal is distributed according to the probability mass function $\phi_e^s := \Pr(\tilde{s} = s | \tilde{e} = e)$, which is strictly positive and twice continuously differentiable in $e$. Output is distributed according to the cumulative distribution function $F(q|e, s)$, which is twice continuously differentiable in $q$ and $e$ and has a strictly positive density $f(q|e, s)$. The joint distribution of output and the signal is $\mathbf{f}(q, s|e) = \phi_e^s f(q|e, s)$. We assume that the likelihood ratio of output, $\frac{\frac{\partial f}{\partial e}(q|e,s)}{f(q|e,s)}$, is strictly increasing in output $q$ ("MLRP"). For a given effort $e$, the likelihood ratio associated with the event $(\tilde{q} = q, \tilde{s} = s)$ is:

$$LR_s(q) := \frac{\frac{\partial \mathbf{f}}{\partial e}(q, s|e)}{\mathbf{f}(q, s|e)} = \frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|e, s)}{f(q|e, s)}. \tag{1}$$

As is the case for any standard unbounded distribution, we assume that $\lim_{q \nearrow \overline{q}} \frac{\partial \mathbf{f}}{\partial e}(q, s|e) = 0$, which implies that debt with an arbitrarily high repayment provides no incentives. Moreover, for all $s$ we assume that $\lim_{q \nearrow \overline{q}} LR_s(q) = \infty$, and $\lim_{q \searrow \underline{q}} LR_s(q) = -\infty$ when the support is unbounded below. These assumptions simplify expressions by ruling out corner solutions, but are not important for our results.

The principal has full bargaining power and offers the manager a schedule of payments $\{w_s(q)\}$ conditional on each realization of $(q, s)$. We follow Grossman and Hart (1983) and separate the principal's problem into two stages. The first stage determines the optimal contract and the associated cost of implementing each effort. Given this cost, the second stage determines which effort to implement. She solves the following program:

$$\max_{\{w_s(q)\}, \hat{e}} \sum_s \int_{\underline{q}}^{\overline{q}} (q - w_s(q))\, \mathbf{f}(q, s|\hat{e}) dq \tag{2}$$

subject to

$$\sum_s \int_{\underline{q}}^{\overline{q}} u(\bar{W} + w_s(q)) \mathbf{f}(q, s|\hat{e}) dq - C(\hat{e}) \geq \overline{U}, \tag{3}$$

$$\hat{e} \in \arg\max_e \sum_s \int_{\underline{q}}^{\overline{q}} u(\bar{W} + w_s(q)) \mathbf{f}(q, s|e) dq - C(e), \tag{4}$$

$$w_s(q) \geq 0,\ w_s(q) \leq \max\{0, q\}, \tag{5}$$

$$q - w_s(q) \text{ non-decreasing in } q. \tag{6}$$

The principal minimizes the expected payment (2) subject to the manager's individual rationality constraint ("IR") (3) and incentive compatibility constraint ("IC") (4). The limited liability constraints (5) stipulate that both investors and the manager are protected by limited liability, and so payments can neither be negative nor exceed the entire output.[2] The monotonicity constraint (6) states that a dollar increase in output cannot increase the wage by more than a dollar (else the manager would inject his own money into investors to increase output), or equivalently the payoff to the principal cannot decrease in output (else she would exercise her control rights to "burn" output).

As a preliminary result, Lemma 1 below presents a new condition for the validity of the first-order approach ("FOA") to the effort choice problem in the above program.[3] Let $H_e$

[2] When output can be negative, bilateral limited liability requires a third party – e.g., a creditor, supplier, or the government – to bear the loss.

[3] Innes (1990) assumes that the FOA is valid and mentions Rogerson's (1985) sufficient conditions for its

and $K_e$ be defined as:

$$\begin{aligned} H_e := \quad & \sum_s \int_{\underline{q}}^{\overline{q}} \frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e)dq, \\ K_e := \quad & \sum_s \int_0^{\overline{q}} q \max\left\{ \frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e), 0 \right\} dq. \end{aligned}$$

**Lemma 1** *Suppose that $u(\bar{W})H_e + u'(\bar{W})K_e < C''(e)$ $\forall e \in (0, \bar{e})$. Then the FOA is valid.*

The condition in Lemma 1 relies on the contracting constraints, the nonconvexity of the utility function, and the associated bounds on the manager's utility $u(\bar{W} + w_s(q))$ to derive an upper bound on the convexity of the expected utility with respect to effort. The FOA is valid if the cost of effort is more convex than this upper bound. We henceforth assume that the condition in Lemma 1 holds.

Note that not all efforts can be implemented. In particular, the constraint on the slope of the contract may prevent the principal from implementing high effort levels. Lemma 2 gives the conditions for an incentive-compatible contract to exist.

**Lemma 2** *The contract that maximizes effort incentives is a levered equity contract, $w_s(q) = \max\{q - q_s^e, 0\}$, in which $q_s^e > 0$ solves:*

$$\int_{q_s^e}^{\overline{q}} u'\left(\bar{W} + q - q_s^e\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e)dq = 0. \tag{7}$$

*Moreover, $LR(q_s^e, s|e) < 0$. With a risk neutral manager, we have $q_s^e = \underline{q}$ $\forall s$. When $\overline{U}$ is sufficiently low, an effort level $e > 0$ can be implemented if and only if:*

$$\sum_s \int_{\underline{q}}^{\overline{q}} u\left(\bar{W} + \max\left\{q - q_s^e, 0\right\}\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e)dq \geq C'(e).$$

Lemma 2 shows that the highest incentives that can be given to the manager involve him receiving levered equity, in which case investors receive debt. Throughout the paper, we will refer to this as a "debt contract" since our focus, and the focus of the literature to which we contribute, is on the financing contract. The debt repayment, $q_s^e$, is signal-dependent and given by equation (7). If the manager is risk-neutral, $q_s^e = \underline{q}$: the debt repayment is the lowest possible level, which corresponds to investors selling the firm to the manager. If the manager is risk-averse, $q_s^e > \underline{q}$. Lowering the debt repayment would lead to a higher wage both in states close to $q_s^e$ (which have a negative likelihood ratio, and are thus more

validity. However, these conditions rule out standard distributions, such as the normal distribution that we consider below.

likely to be realized if the manager shirks) and high-output states (with a positive likelihood ratio, and are thus more likely to be realized if the manager works). Due to diminishing marginal utility, the agent places less weight on the latter since his wage is already high, and so lowering $q_s^e$ towards $\underline{q}$ would reduce effort incentives. In the remainder of the paper, we assume that the conditions in Lemma 2 hold, so that effort $e$ can be implemented.

We now proceed as follows. Section 2 focuses on the case of a risk-neutral manager, allowing us to study the optimality of debt where the principal has access to additional signals of performance. Section 3 focuses on the case of no additional signals and returns to a general utility function, allowing us to study the optimality of debt under a risk-averse manager. Section 4 returns to the most general case of a risk-averse manager and additional performance signals, and shows that the results are a combination of Sections 2 and 3.

# 2 Additional Performance Signals

This section focuses on the case of a risk-neutral manager ($u(w) = w$) to allow us to study whether the presence of additional signals affects the optimality of debt. The principal's program is as in equations (2)-(6), for the case of $u(w) = w$.

## 2.1 The Optimality of Debt

Let

$$\overline{LR}_s(q) := \frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz}{\int_q^{\overline{q}} f(z|\hat{e}, s)dz} \tag{8}$$

denote the likelihood ratio associated with the event $(\tilde{q} \geq q, \tilde{s} = s)$. The likelihood ratio comprises two terms. The first, $\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s}$, captures how individually informative the signal is about effort. For example, if $s$ is profits, high profits indicate high effort. The second, $\frac{\int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz}{\int_q^{\overline{q}} f(z|\hat{e}, s)dz}$, captures the effect of effort on the output density not only at $q$, but over all outputs greater than $q$, conditional on the signal. For example, if the signal $s$ is peer firm performance, this likelihood ratio will be lower if peer performance is strong.

In Innes (1990), without an additional signal $s$, investors receive debt and the manager receives equity. The manager is paid zero if output is less than the contractual debt repayment $q^*$ (henceforth referred to as the "debt repayment"), and the residual $q - q^*$ otherwise, for the intuition given in the introduction. With an additional signal $s$, it is not clear that the optimal contract remains debt. It may seem that, for low outputs, if the signal is sufficiently indicative of effort (i.e. $\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s}$ is high), the manager should be paid a strictly positive amount, rather than zero. Conversely, it may seem that, for high outputs, if the signal is sufficiently indicative of shirking, the manager should be paid less than the residual. However, Proposition 1 below shows that the contract remains debt.[4]

**Proposition 1** *(Risk-neutral manager, additional performance signals): The optimal contract is* $w_s(q) = \max\{q - q_s^*, 0\}$*. For interior solutions, debt repayments* $\{q_s^*\}$ *are such that* $\overline{LR}_{s_i}\left(q_{s_i}^*\right) = \overline{LR}_{s_j}\left(q_{s_j}^*\right)$*, where* $\overline{LR}_s(q)$ *is strictly increasing in* $q$*.*

Proposition 1 shows that, instead of affecting the form of the optimal contract, which remains debt, the signal realization affects the debt repayment. The intuition is as follows. A negative signal means that it is optimal to pay the manager less, but this reduction can only occur for high output levels where the payment is strictly positive. Conceptually, this decrease could be achieved by lowering the slope of the manager's pay, but it turns out to be optimal instead to raise the debt repayment. Due to MLRP, it is more efficient to provide strong incentives for only high output levels than moderate incentives for a larger range of output levels. Conversely, if output is low, a positive signal only leads to a strictly positive payment if it raises the likelihood ratio in equation (8) above a threshold. Due to MLRP, it is efficient to provide the manager with the minimum possible payment (zero) over a wide range of output levels; thus, a positive signal should lead to a positive payment only at the top end of this range. Instead of affecting the optimal contract, the signal instead affects the optimal debt repayment and thus the "incentive zone" – the subset of outputs where the manager receives a strictly positive payment. Intuitively, the signal allows the principal to concentrate incentives in states of the world that are stronger positive signals of effort. The debt repayment $q_s^*$ is the level of output at which the likelihood ratio equals a cutoff, which is the inverse of the Lagrange multiplier associated with the IC (see equation (34) in Appendix 5).

Proposition 1 also shows that the optimal debt repayment depends on the likelihood ratio of the event $\tilde{q} \geq q$ conditional on signal $s$. Note that the relevant likelihood ratio $\overline{LR}_s(q)$ is over a range of outputs $\tilde{q} \geq q$, rather than at a single output level $\tilde{q} = q$. The firm cannot increase the payment at a specific output level in isolation without increasing it at all lower outputs, as this would violate the monotonicity constraint; similarly, it cannot decrease the payment at a specific output level in isolation without decreasing it at all higher outputs.

[4]One might think that we can apply the logic in Innes (1990) signal-by-signal to show that the optimal contract remains debt in the presence of an additional signal. However, the logic of Innes (1990) cannot be applied independently for each signal realization, because the probability of each signal realization depends on effort, which in turn depends on incentives provided across all signal realizations. For example, if the signal is a credit rating and shirking causes a low credit rating which in turn leads to a high debt repayment, the manager will increase effort to avoid the low rating.

## 2.2 Should Debt Be Performance-Sensitive?

Proposition 1 shows that the optimal contract remains debt in the presence of an additional signal $s$. However, the signal may affect the debt repayment $q_s^*$, and so debt becomes performance-sensitive. Proposition 2 studies the conditions under which this is the case.

**Proposition 2** *(Risk-neutral manager, effect of signal on debt repayment):*

*(i) The optimal contract is independent of the signal if and only if* $\overline{LR}_{s_i}(q^*) = \overline{LR}_{s_j}(q^*)$ *for all* $s_i$, $s_j$, *where* $q^*$ *solves the following equation:*

$$\sum_s \phi_e^s \int_{q^*}^{\overline{q}} (q - q^*) \frac{\partial f}{\partial e}(q|e,s)dq = C'(e).$$

*(ii) Given output* $q$, *the payment* $w_s(q)$ *is independent of the signal if* $q \leq \min_s \{q_s^*\}$.

*(iii) The debt repayment is zero under signal* $s$ *if* $\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s}$ *is sufficiently high.*

Part (i) of Proposition 2 asks whether a signal affects the contract ex ante – before observing output, would the principal like to make the contract contingent on the signal? It shows that limited liability requires us to refine the informativeness principle. From Proposition 1, a signal only affects the contract if it changes the principal's optimal choice of the debt repayment. She cannot change the contract for outputs below the debt repayment because the manager is already paid zero, nor for outputs above the debt repayment because the manager is already paid the residual. In turn, the principal will choose not to make the debt repayment depend on the signal, i.e. set $q_s^* = q^*\ \forall s$, if and only if the likelihood ratio that $q \geq q^*$ is the same across signals.

Note that the condition in part (i) is different from Holmström (1979). In his paper, a signal affects the contract if and only if it affects the likelihood $LR_s(q)$ of the event $(\tilde{q} = q, \tilde{s} = s)$, i.e. it is informative about the event that output equals any $q$. Here, it does so if and only if it affects the likelihood $\overline{LR}_s(q^*)$ of the event $(\tilde{q} \geq q^*, \tilde{s} = s)$, i.e. is informative about the event that output exceeds the contractual debt repayment $q^*$. As a result, a signal can be informative almost everywhere, yet have no value. A signal only changes the contract if it shifts probability mass from below $q^*$ to above $q^*$ (or vice-versa). A signal that redistributes mass within the left tail, or within the right tail, has zero value. A "smoking gun" indicates that a bad event (low output) is due to poor performance rather than bad luck, but the bad event will likely lead to the manager being fired and being paid zero anyway.[5] For instance, investors only noticed that Enron was adopting misleading accounting practices when it was already going bankrupt.

[5]The "smoking gun" could be generated by an audit that is only undertaken upon a bad event, in which case the signal realization is zero absent a bad event.

Part (i) of Proposition 2 has implications for when debt contracts should be performance-sensitive. In theory, the debt repayment could depend on many signals, but in practice it is often signal-independent. Proposition 2 potentially rationalizes this practice – even if signals are informative about effort, they should not enter the contract if they are only informative in the tails. In addition, Proposition 2 provides conditions under which the repayment *should* depend on additional signals, as in performance-sensitive debt – if and only if the signal is informative about effort conditional on output exceeding the promised repayment. In addition to studying the optimality of performance-sensitive debt, Proposition 2 also allows us to study the conditions under which the manager's equity claim should depend on performance milestones, as documented empirically by Kaplan and Strömberg (2003) for venture capital contracts.[6]

Part (ii) asks whether a signal is valuable ex post – after observing output, will the payment to the manager depend on the signal? In other words, while part (i) asks whether the optimal *contract* depends on the signal, part (ii) asks whether the optimal *payment* depends on the signal. If output is sufficiently low, the signal has no value since the manager will be paid zero even under the most favorable signal realization. Thus, even if the signal realization reduced the optimal debt repayment – i.e. changed the optimal contract – it would not change the actual payment as it remains zero. Part (ii) is relevant if signals are costly, and the principal can observe output before deciding whether to gather the signal.

Part (iii) shows that, if a signal is a sufficiently positive signal of effort, then $q_s^* = 0$. Intuitively, to provide strong incentives, the principal may be willing to completely forgive the debt in rare states that are very positive signals of effort. Indeed, $\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s}$ will be high when effort has a strong effect on the probability of observing signal $s$, and when the probability $\phi_{\hat{e}}^s$ of observing signal $s$ is low. Note that the debt repayment could not be zero in a model without an additional signal, as the principal would never obtain a return in any state. This also means that the debt repayment may be the same under two different signal realizations, if they are both sufficiently positive that the optimal debt repayment is zero.

We close with two examples that apply Proposition 2 to a real-world setting. First, we consider whether contracts should depend on $s$, a signal of economic conditions. Economic conditions are informative about effort – for any given level of output, a high $s$ suggests that the output was due to good economic conditions rather than effort, and so it increases the likelihood that the manager has shirked. However, Proposition 2 shows that economic conditions $s$ should only affect the contract if they affect the probability that $q \geq q^*$ under

[6] While the original informativeness principle in Holmström (1979) would suggest that contracts should depend on performance milestones, it does not generally deliver debt and equity as optimal contracts. Kaplan and Strömberg (2004) find that the debt and equity contracts used in venture capital are determined primarily by agency problems, not risk-sharing considerations.

high versus low effort. This will fail to hold if they affect the level of output but not the probability that output exceeds $q^*$.[7] For example, consider a start-up which is developing a major new software; the manager's effort affects the probability that the software is adopted by the industry. If the software is adopted, $q \geq q^*$ (regardless of economic conditions); if it is not adopted, $q < q^*$ (again, regardless of economic conditions). Economic conditions could affect the actual level of $q$ (both if the software is adopted and if it is not), but if they do not affect the probability that $q \geq q^*$, because they do not affect the likelihood that the software will be adopted, then they should not be included in the contract. In contrast, for an "everyday" software product, where the probability that $q \geq q^*$ does depend on economic conditions (as well as the manager's effort), then the debt repayment should depend on economic conditions.

As a second example, consider a firm whose production can break down due to a fault, whose probability can depend on managerial effort. If it does, then output is below $q^*$ (regardless of economic conditions); if it does not, then $q \geq q^*$ (regardless of economic conditions). As in the previous example, economic conditions could affect the actual level of $q$ (both if production breaks down and if it does not), but if they do not affect the probability that production breaks down, then they should not be included in the contract. In contrast, if demand depends on the state of the economy, rather than a breakdown, then the debt repayment should depend on economic conditions. In the first example, what matters is whether the signal is uninformative about the upside (developing new software); in this example what matters is whether the signal is uninformative about the downside (production breaking down).

Our rationale for performance-sensitive debt complements existing explanations. Manso, Strulovici, and Tchistyi (2010) model performance-sensitive debt as a mechanism to signal the firm's growth rate in an adverse selection model; there is no moral hazard. Bhanot and Mello (2006) and Koziol and Lawrenz (2010) show that performance-sensitive debt deters risk shifting. While none of these papers model an effort decision, Manso et al. (2010, Section 8) conjecture that performance-sensitive debt "could serve as an additional incentive for the firm's manager to exert effort" and Tchistyi (2009) shows that performance-sensitive debt can deter cash flow diversion.

Innes (1993) derives the optimal contract when profits (which correspond to $q$ in our setting) can be decomposed into output and the output price, i.e. the price is an additional signal that can be used in the contract. He shows that the optimal contract is a price-contingent commodity bond, which has similarities to performance-sensitive debt; however, the only signal that he analyzes is price (i.e. one component of output). We consider a broad

[7] It will also hold if they affect the probabilities (that $q \geq q^*$ under high and low effort) by the same proportion.

set of signals, including signals that are informative about the manager's effort, and signals that affect the output distribution in different ways to the price. Bensoussan, Chevalier-Roignant, and Rivera (2021) model performance-sensitive debt as a solution to debt overhang. Adam and Streitz (2016) test empirically whether performance-sensitive debt is used to reduce hold-up problems, which arise from the information the lender acquires over the course of the lending relationship, Adam et al. (2020) show empirically that overconfident managers issue performance-sensitive debt, and Asquith, Beatty, and Weber (2005) contrast the settings in which debt involves interest-decreasing versus interest-increasing provisions.

# 3 Risk-Averse Manager

This section considers the case of a risk-averse manager, and where output is the only signal that can be used in a contract. This allows us to study whether risk aversion affects the optimality of debt.

The model is as in Section 1 except that there is no longer a signal $s$; all the prior notation continues to apply with $s$ removed; for example, the probability density function of output is now $f(q|e)$ and the likelihood ratio associated with the event $\tilde{q} = q$ is now: $LR(q) = \frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)}$. Let $A_w \equiv -\frac{u''(w)}{u'(w)}$ denote the manager's coefficient of absolute risk aversion at wealth $w$.

Lemma 3 states that the optimal contract that induces effort $e^*$ involves a zero payment if output falls below a threshold.

**Lemma 3** *The optimal contract involves $w(q) = 0$ if and only if $q \leq q^*$.*

A debt contract not only involves a zero payment below a threshold, but the maximum possible slope of 1 above it. Proposition 3 gives a sufficient condition for this to be the case:

**Proposition 3** *(Risk-averse manager, no signal): If $\frac{\frac{d}{dq}LR(q)}{LR(q)} \geq A_{\bar{W}+q-q^*}$ for all $q > q^*$, and the debt contract with the highest incentive-compatible debt repayment $q^*$ satisfies the participation constraint:*

$$\int_{\underline{q}}^{q^*} u(\bar{W})f(q|e^*)dq + \int_{q^*}^{\overline{q}} u(\bar{W} + q - q^*)f(q|e^*)dq - C(e^*) \geq \overline{U}$$

$$\textit{where } q^* = \max\left\{\hat{q} \textit{ s.t. } \int_{\underline{q}}^{\hat{q}} u(\bar{W})\frac{\partial f}{\partial e}(q|e^*)dq + \int_{\hat{q}}^{\overline{q}} u(\bar{W} + q - \hat{q})\frac{\partial f}{\partial e}(q|e^*)dq = C'(e^*)\right\},$$

*then the optimal contract that induces effort $e^*$ is debt, i.e. $w(q) = \max\{q - q^*, 0\}$.*

The optimal slope of the contract for high outputs ($q > q^*$) balances two forces. The first is the "output sensitivity effect", captured by the rate of growth of the likelihood ratio – more precisely, the semi-elasticity of the likelihood ratio with respect to output, $\frac{d}{dq}[\ln LR(q|e)]$. If the likelihood ratio increases sufficiently fast with output, pay should increase rapidly with output, pushing the optimal slope of the contract towards its maximum of 1.

The second is the "marginal utility effect" – financial incentives should be greater if the manager values these incentives highly, i.e. his marginal utility remains high even though $q > q^*$ means that pay is already high. Just as the output sensitivity effect depends on the rate of growth (i.e. semi-elasticity) of the likelihood ratio, the marginal utility effect depends on the semi-elasticity of marginal utility, i.e. the change in marginal utility divided by marginal utility, which equals the coefficient of absolute risk aversion $A_{\bar{W}+q-q^*}$. If the manager exhibits decreasing absolute risk aversion, then marginal utility falls at a slower rate as his wealth rises, which also pushes the optimal slope of the contract towards its maximum value of 1. In contrast, if the manager exhibits increasing absolute risk aversion, then marginal utility falls at a faster rate as his wealth rises, which makes it costly to provide incentives that involve high payments. This tends to reduce the slope of the contract for very high outputs. Note that risk aversion matters because it affects the rate at which marginal utility changes as wealth increases, rather than due to the standard trade-off between incentives and risk-sharing.

Corollary 1 shows how the conditions in Proposition 3 simplify for the case of a normally distributed output.[8]

**Corollary 1** *Assume that $q$ follows a normal distribution with mean $e$ and variance $\sigma^2$, that the manager's risk tolerance satisfies $\frac{d}{dw}\left(-\frac{u'(w)}{u''(w)}\right) \geq 1$, and that $\overline{U} \leq u(0)$. The optimal contract is debt if:*

$$e - \frac{u'(\bar{W})}{u''(\bar{W})} \geq q^* > e. \tag{9}$$

We apply Corollary 1 to two utility functions. First, consider a constant relative risk aversion utility function with relative risk aversion coefficient $\gamma$. The condition on the manager's risk tolerance is simply $\gamma \leq 1$, and condition (9) becomes:

$$\frac{\bar{W}}{\gamma} + e \geq q^* > e.$$

Thus, for the optimal contract to be debt, the manager's risk aversion must be sufficiently low, and his initial wealth must be sufficiently high. These factors increase the manager's

[8]While Proposition 3 had a single inequality, recall that it had to hold for all $q > q^*$, which explains why expression (9) contains two inequalities.

risk tolerance, and $\gamma \leq 1$ also ensures that the sensitivity of the manager's risk tolerance to his wealth is sufficiently high – which is necessary for the condition in Proposition 3 to be satisfied for all $q \geq q^*$.

Second, consider hyperbolic risk aversion, where $A_w = \frac{1}{aw+b}$ for some parameters $a$ and $b$. Then, $-\frac{u'(w)}{u''(w)} = aw + b$, and so the condition on the sensitivity of the manager's risk tolerance holds if and only if $a \geq 1$. Condition (9) becomes:

$$e - \left(a\bar{W} + b\right) \geq q^* > e.$$

Thus, with CARA utility, which is characterized by $a = 0$ and $b > 0$, this condition is never satisfied, i.e. the optimal contract cannot be debt. This highlights how a low level of risk aversion is necessary but not sufficient for the optimal contract to be debt – what matters is how risk aversion changes with wealth. With the normal distribution, the output sensitivity effect is negative: $\frac{d}{dq}\left[\ln LR(q|e)\right] = \frac{1/\sigma^2}{\sigma^2/(q-e)} = \frac{1}{q-e}$, which is decreasing in $q$. Thus, risk aversion needs to be sufficiently decreasing with wealth for the condition in Proposition 3 to hold for all output levels. This contrasts the simple intuition that, since Innes's (1990) result was derived under risk neutrality, it will continue to hold under risk aversion as long as it is sufficiently low.

# 4 Additional Performance Signals, Risk-Averse Manager

This section considers the most general case of both a risk-averse manager and an additional performance signal. The principal's program is in equations (2)-(6).

Lemma 4 states that the optimal contract involves a zero payment if output falls below a threshold.

**Lemma 4** *For every $s$, there exists $q_s^* \leq q_s^e$ such that $w_s(q) = 0$ if and only if $q \leq q_s^*$.*

Proposition 4 gives a sufficient condition for the slope above $q_s^*$ to be 1, i.e. the optimal contract to be debt.

**Proposition 4** *(Risk-averse manager, additional performance signals): Let $\frac{\frac{d}{dq}LR_s(q)}{LR_s(q)} \geq A_{\bar{W}+q-q_s^*}$ for all $q > q_s^*$, and the debt contract with the highest incentive-compatible debt repayment $q_s^*$ state-by-state satisfies the participation constraint. Then, the optimal contract is debt.*

This condition is identical to Proposition 3, except for the dependence on the signal $s$. The intuition is the same: the optimal contract remains debt if the output sensitivity effect is sufficiently strong relative to the marginal utility effect.

Using Proposition 4, we can now determine when the signal realization affects the optimal contract, similar to part (i) in Proposition 2. The result is given in Proposition 5.

**Proposition 5** *(Risk-averse manager, effect of signal on debt repayment): Suppose the conditions for Proposition 4 hold. The optimal contract is independent of the signal if, for all* $s, \hat{s} \in \{s_1, \ldots, s_S\}$, $\overline{LR}_s^u(q^*) = \overline{LR}_{\hat{s}}^u(q^*)$ *where*

$$\overline{LR}_s^u(x) := \frac{\int_x^{\overline{q}} u'\left(\bar{W} + q - x\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e) dq}{\int_x^{\overline{q}} \mathbf{f}(q, s|e) dq}$$

*is the marginal utility-adjusted likelihood ratio of the event* $(\tilde{q} \geq x, \tilde{s} = s)$, *and* $q^*$ *and* $e^*$ *solve equations (3) and (4) without a signal.*

Proposition 5 generalizes the conditions for a signal to have value from the risk-neutral model in Proposition 2 (part (i)). The key difference is that now the condition also depends on the slope of the utility function, which affects the manager's incentive to exert effort. Intuitively, lowering the debt repayment $q_s^*$ increases the manager's payment for any $q > q_s^*$. Moreover, a manager values an additional dollar according his marginal utility in the relevant state, which is decreasing in wealth when the manager is risk-averse.

# 5 Conclusion

This paper shows that, in the presence of limited liability and monotonicity constraints, the optimal contract remains debt even if the principal has access to additional performance signals. While it may seem intuitive that a good signal should lead to the manager being paid even if output is low, and a bad signal should lead to him not being the residual claimant even if output is high, we show that the signal does not affect the form of the contract, but only the debt repayment. As a result, Holmström's (1979) informativeness principle needs to be refined in the presence of the above constraints – a signal is only valuable if it is informative about whether output exceeds the debt repayment. If this condition is satisfied, then performance-sensitive debt is optimal. If not, for example because the signal is only informative when output is high (or only when output is low), then the debt contract does not depend on the signal.

We also analyze the conditions under which the optimal contract remains debt when the manager is risk-averse. Simple intuition might suggest that this is the case if the manager's risk aversion is sufficient low, and also that risk aversion only matters if the participation constraint binds, so that investors need to compensate the manager for the risk imposed by a performance-sensitive contract. We show that neither conjecture is true. Even with a slack participation constraint, risk aversion matters because it affects the rate at which the manager's marginal utility changes with output and thus the effectiveness of financial incentives for high output levels. In addition, what matters is not the level of risk aversion in isolation, but relative to the rate at which the likelihood ratio rises with output. Since a debt contract involves the maximum possible slope of 1 for all output levels above the threshold, debt is only optimal if risk aversion is below this rate for all such output levels. For the normal distribution, this rate declines with output, and so debt is suboptimal under constant absolute risk aversion even if the risk aversion coefficient is low.

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## Proofs

**Proof of Lemma 1**

The FOA is valid if the following objective function is concave in $e$:

$$\sum_s \int_{\underline{q}}^{\bar{q}} u(\bar{W} + w_s(q))\mathbf{f}(q, s|e)dq - C(e).$$

Thus, the FOA is valid if:

$$\sum_s \int_{\underline{q}}^{\bar{q}} u(\bar{W} + w_s(q))\frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e)dq < C''(e) \qquad \forall e. \tag{10}$$

From equation (5), $w_s(q) \in [0, q]$ for all $q, s$. We have:

$$\begin{aligned} & \sum_s \int_{\underline{q}}^{\bar{q}} u(\bar{W} + w_s(q))\frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e)dq \\ \leq \; & \sum_s \int_{\underline{q}}^{\bar{q}} \max\left\{ \left(u(\bar{W}) + u'(\bar{W})\max\{q, 0\}\right)\frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e), u(\bar{W})\frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e)\right\} dq \end{aligned}$$

since $u$ is monotonically increasing and nonconvex. Therefore, a sufficient condition for equation (10) is:

$$u(\bar{W})\sum_s \int_{\underline{q}}^{\bar{q}} \frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e)dq + u'(\bar{W})\sum_s \int_0^{\bar{q}} q \max\left\{\frac{\partial^2 \mathbf{f}}{\partial e^2}(q, s|e), 0\right\} dq < C''(e) \quad \forall e. \tag{11}$$

Rewriting using the definition of $H_e$ and $K_e$ gives Lemma 1. ■

**Proof of Lemma 2:**

Consider the program that determines the highest incentive to perform effort:

$$\max_{w,x} \int_{\underline{q}}^{\bar{q}} \sum_s u\left(\bar{W} + w_s\left(q\right)\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e)dq,$$

subject to

$$\dot{w}_s(q) = x_s(q),$$

$$w_s(q) \geq 0, \; w_s(q) \leq q, \text{ and } x_s(q) \leq 1.$$

It is straightforward to show that the solution of this program has $w_s(0) \geq 0$ and $w_s(q)$ non-decreasing (see proof of Lemma 4). Therefore, we can replace the constraints by

$$\dot{w}_s(q) = x_s(q),$$

$$w_s(0) = 0,\ w_s(q) \leq q,\ \text{and } 0 \leq x_s(q) \leq 1.$$

The Hamiltonian associated with this program is

$$\mathcal{H}(w, x, \lambda, \mu, q) = \sum_s u\left(\bar{W} + w_s\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e) + \sum_s \lambda_s(q)x_s + \mu_x(q)\left(1 - x_s\right)$$

The optimality condition with respect to the control variables $x_s$ are:

- $\lambda_s(q) \geq 0$ if $x_s(q) = 1$
- $\lambda_s(q) = 0$ if $0 < x_s(q) < 1$
- $\lambda_s(q) \leq 0$ with if $x_s(q) = 0$

The optimality condition with respect to the state variables $w_s$ are:

$$\dot{\lambda}_s(q) = -u'\left(\bar{W} + w_s\left(q\right)\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e).$$

Note that, by MLRP, there exists $q_0$ such that $\frac{\partial \mathbf{f}}{\partial e}(q, s|e) > 0$ iff $q > q_s^0$. In any interval with $0 < \dot{w}_s(q) < 1$, we must have $\dot{\lambda}_s(q) = \lambda_s(q) = 0$. Since $u' > 0$, the sign of $\dot{\lambda}_s(q)$ must be the opposite of the sign of $\frac{\partial \mathbf{f}}{\partial e}(q, s|e)$:

$$\dot{\lambda}_s(q)\left\{\begin{array}{c} > \\ = \\ < \end{array}\right\}0 \iff q\left\{\begin{array}{c} < \\ = \\ > \end{array}\right\}q_s^0.$$

Thus, there is no interval with $\dot{w}_s(q) \in (0, 1)$.

The transversality condition is $\lim_{q\to\overline{q}} \lambda_s(q) = 0$. Since $\dot{\lambda}_s(q) < 0$ for $q > q_s^0$ and $\lim_{q\to\overline{q}} \lambda_s(q) = 0$, there must be $q_s^e < q_s^0$ such that $\lambda_s(q) > 0$ for all $q \in (q_s^e, \overline{q})$. That is, $x_s(q) = 1$ for all such $q$. Integration gives:

$$\lim_{q\to\overline{q}} \lambda_s(q) - \lambda_s(\underline{q}) = -\int_{\underline{q}}^{\overline{q}} u'\left(\bar{W} + w_s\left(q\right)\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e)dq$$

and, by transversality,

$$\lambda_s(\underline{q}) = \int_{\underline{q}}^{\overline{q}} u'\left(\bar{W} + w_s\left(q\right)\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e)dq$$

Let $V(q) \equiv u'\left(\bar{W} + w_s\left(q\right)\right)$, which is a decreasing function since $V'(q) = u''\left(\bar{W} + w_s\left(q\right)\right)w_s'\left(q\right) \leq$

0. It is straightforward to show that MLRP implies

$$\lambda_s(\underline{q}) = \int_{\underline{q}}^{\overline{q}} V\left(q\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e) dq \leq 0.$$

To see this, integrate by parts by letting $dv = \frac{\partial \mathbf{f}}{\partial e}(q, s|e)dq$ so that $v = \int_{\underline{q}}^{q} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q}$ and $V\left(q\right) = u$, $V'\left(q\right)dq = du$. Then

$$\lambda_s(\underline{q}) = \left[ V\left(q\right) \int_{\underline{q}}^{q} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q} - \int \int_{\underline{q}}^{q} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q}V'\left(q\right)dq \right]$$

$$= V\left(\overline{q}\right) \underbrace{\int_{\underline{q}}^{\overline{q}} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q}}_{=0} - \int_{\underline{q}}^{\overline{q}} \left[ \int_{\underline{q}}^{q} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q} \right] V'\left(q\right)dq$$

$$= - \int_{\underline{q}}^{\overline{q}} \left[ \int_{\underline{q}}^{q} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q} \right] V'\left(q\right)dq.$$

Since $\int_{\underline{q}}^{\overline{q}} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q} = \frac{d}{de} \underbrace{\left[ \int_{\underline{q}}^{\overline{q}} \mathbf{f}(\tilde{q}, s|e)d\tilde{q} \right]}_{=1} = 0$ and $\frac{\partial \mathbf{f}}{\partial e} > 0$ for all $q > q_s^0$, it follows that $\int_{\underline{q}}^{q} \frac{\partial \mathbf{f}}{\partial e}(\tilde{q}, s|e)d\tilde{q} < 0$. The result then follows from the fact that $V'\left(q\right) \leq 0$.

We have therefore shown that there exists $q_s^e < q_s^0$ such that

$$w_s(q) \equiv \max\{q - q_s^e, 0\}.$$

The thresholds $q_s^e$ are determined by $\lambda_s(q_s^e) = 0$. Integrating $\dot{\lambda}_s$, we find

$$\lambda_s(q_s^e) = \int_{q_s^e}^{\overline{q}} u'\left(\bar{W} + q - q_s^e\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e)dq = 0. \tag{12}$$

With a risk neutral agent, $u'$ is a constant, so that equation (12) yields $q_s^e = \underline{q}$ by definition of a density function. ■

**Proof of Proposition 1**

We first prove that the likelihood ratio $\overline{LR}_s(q)$ in equation (8) is increasing in $q$:

$$\frac{d}{dq} \left\{ \frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz}{\int_q^{\overline{q}} f(z|\hat{e}, s)dz} \right\} = \frac{-\frac{\partial f}{\partial e}(q|\hat{e}, s) \int_q^{\overline{q}} f(z|\hat{e}, s)dz + f(q|\hat{e}, s) \int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz}{\left( \int_q^{\overline{q}} f(z|\hat{e}, s)dz \right)^2}. \tag{13}$$

For $\frac{\partial f}{\partial e}(q|\hat{e}, s) \leq 0$, we have $-\frac{\partial f}{\partial e}(q|\hat{e}, s) \int_q^{\overline{q}} f(z|\hat{e}, s)dz \geq 0$. Moreover, $f(q|\hat{e}, s) \int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz >$

0 because of MLRP and $\int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz = 0$. In sum, the right-hand side ("RHS") of equation (13) is positive. For $\frac{\partial f}{\partial e}(q|\hat{e}, s) > 0$, the RHS of equation (13) is positive if and only if:

$$\begin{aligned}
& f(q|\hat{e}, s) \int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz \geq \frac{\partial f}{\partial e}(q|\hat{e}, s) \int_q^{\overline{q}} f(z|\hat{e}, s)dz \\
\Leftrightarrow \quad & \int_q^{\overline{q}} \frac{\frac{\partial f}{\partial e}(z|\hat{e}, s)}{\frac{\partial f}{\partial e}(q|\hat{e}, s)} dz \geq \int_q^{\overline{q}} \frac{f(z|\hat{e}, s)}{f(q|\hat{e}, s)} dz \\
\Leftrightarrow \quad & \int_q^{\overline{q}} \left[ \frac{\frac{\partial f}{\partial e}(z|\hat{e}, s)}{\frac{\partial f}{\partial e}(q|\hat{e}, s)} - \frac{f(z|\hat{e}, s)}{f(q|\hat{e}, s)} \right] dz \geq 0,
\end{aligned}$$

which holds because by MLRP we have $\frac{\frac{\partial f}{\partial e}(z|\hat{e},s)}{f(z|\hat{e},s)} \geq \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}$ for any $q \geq z$.

The rest of the proof is divided into two parts:

*Step 1. Conditional on each signal realization, the optimal contract is debt.*

*Step 1.a.* This part of the proof adapts the proof technique from Lemma 1 in Matthews (2001) to a setting with continuous output and an additional signal. Let $(W_s^*)_{s \in \{1,\ldots,S\}}$ (henceforth denoted by $(W_s^*)$ for brevity) be a feasible payment schedule that induces effort $\hat{e}$. For a given signal realization $s'$, consider an alternative payment schedule which is the same as $(W_s^*)$ for any signal other than $s'$, and $W_{s'}^{q_{s'}} = \max\{0, q - q_{s'}\}$ for a given $s'$. The contractual debt repayment $q_{s'}$ is chosen so that the payment schedules contingent on signal $s'$, $W_{s'}^*$ and $W_{s'}^{q_{s'}}$, have the same expected payment under effort $\hat{e}$:

$$\int_{\underline{q}}^{\overline{q}} W_{s'}^*(q)\mathbf{f}(q, s'|\hat{e})dq = \int_{\underline{q}}^{\overline{q}} W_{s'}^{q_s}(q)\mathbf{f}(q, s'|\hat{e})dq. \tag{14}$$

It is straightforward to show that $W_{s'}^{q_{s'}}$ exists and is unique. We will first show that, for a given $s'$, replacing $W_{s'}^*$ by $W_{s'}^{q_{s'}}$ increases effort.

For a given $s'$, define:

$$W_{s,s'}^{**}(q) := \begin{cases} W_s^*(q) & \text{for } s \neq s' \\ W_s^{q_s}(q) & \text{for } s = s' \end{cases}. \tag{15}$$

In what follows we will compare the original payment schedule $(W_s^*)$ to the payment schedule $\left(W_{s,s'}^{**}\right)$ as defined in equation (15). Let $e_{s'}^D$ be an optimal effort for the agent when the payment schedule is $\left(W_{s,s'}^{**}\right)$ instead of $(W_s^*)$:

$$e_{s'}^D \in \arg\max_{e \in [0,\bar{e}]} \sum_s \int_{\underline{q}}^{\overline{q}} W_{s,s'}^{**}\mathbf{f}(q, s|e)dq - C(e).$$

Since the agent chooses $\hat{e}$ when the payment schedule is $(W_s^*)$ and $e_{s'}^D$ when it is $(W_{s,s'}^{**})$, we must have:

$$\sum_s \int_{\underline{q}}^{\overline{q}} W_s^{**}(q)\mathbf{f}(q,s|e_{s'}^D)dq - C(e_{s'}^D) \geq \sum_s \int_{\underline{q}}^{\overline{q}} W_{s,s'}^{**}(q)\mathbf{f}(q,s|\hat{e})dq - C(\hat{e}),$$

and

$$\sum_s \int_{\underline{q}}^{\overline{q}} W_s^{*}(q)\mathbf{f}(q,s|\hat{e})dq - C(\hat{e}) \geq \sum_s \int_{\underline{q}}^{\overline{q}} W_s^{*}(q)\mathbf{f}(q,s|e_{s'}^D)dq - C(e_{s'}^D).$$

Combining these two inequalities, we obtain

$$\sum_s \int_{\underline{q}}^{\overline{q}} \left[W_{s,s'}^{**}(q) - W_s^*(q)\right] \left[\mathbf{f}(q,s|e_{s'}^D) - \mathbf{f}(q,s|\hat{e})\right] dq \geq 0.$$

Using equation (15), this rewrites simply as:

$$\int_{\underline{q}}^{\overline{q}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \left[\mathbf{f}(q,s') - \mathbf{f}(q,s'|\hat{e})\right] dq \geq 0. \tag{16}$$

Since both contracts have the same expected value under effort $\hat{e}$ by construction, and $W_{s'}^{q_{s'}}$ pays the lowest feasible amount for $q < q_{s'}$ and has the highest possible slope for $q > q_{s'}$, there exists $\bar{q}_{s'} \geq q_{s'}$ such that

$$W_{s'}^{q_{s'}}(q) \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} W_{s'}^*(q) \text{ for all } q \begin{Bmatrix} \leq \\ \geq \end{Bmatrix} \bar{q}_{s'}. \tag{17}$$

We will now show by contradiction that $\hat{e} \leq e_{s'}^D$. Suppose that $\hat{e} > e_{s'}^D$. Then:

$$\begin{aligned}
0 &\leq \textstyle\int_{\underline{q}}^{\overline{q}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \left[\frac{\mathbf{f}(q,s'|e_{s'}^D)}{\mathbf{f}(q,s'|\hat{e})} - 1\right] \mathbf{f}(q,s'|\hat{e})dq \\
&= \textstyle\int_{\underline{q}}^{\overline{q}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \frac{\mathbf{f}(q,s'|e_{s'}^D)}{\mathbf{f}(q,s'|\hat{e})}\mathbf{f}(q,s'|\hat{e})dq - \underbrace{\int_{\underline{q}}^{\overline{q}} \left[W_{s'}^{D}(q) - W_{s'}^*(q)\right] \mathbf{f}(q,s'|\hat{e})dq}_{=0} \\
&= \textstyle\int_{\underline{q}}^{\bar{q}_{s'}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \frac{\mathbf{f}(q,s'|e^D)}{\mathbf{f}(q,s'|\hat{e})}\mathbf{f}(q,s'|\hat{e})dq + \int_{\bar{q}_{s'}}^{\overline{q}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \frac{\mathbf{f}(q,s'|e_{s'}^D)}{\mathbf{f}(q,s'|\hat{e})}\mathbf{f}(q,s'|\hat{e})dq \\
&< \textstyle\int_{\underline{q}}^{\bar{q}_{s'}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \frac{\mathbf{f}(\bar{q}_{s'},s'|e_{s'}^D)}{\mathbf{f}(\bar{q}_{s'},s'|\hat{e})}\mathbf{f}(q,s'|\hat{e})dq + \int_{\bar{q}_{s'}}^{\overline{q}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \frac{\mathbf{f}(\bar{q}_{s'},s'|e_{s'}^D)}{\mathbf{f}(\bar{q}_{s'},s'|\hat{e})}\mathbf{f}(q,s'|\hat{e})dq \\
&= \textstyle\frac{\mathbf{f}(\bar{q}_{s'},s'|e_{s'}^D)}{\mathbf{f}(\bar{q}_{s'},s'|\hat{e})} \int_{\underline{q}}^{\overline{q}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^*(q)\right] \mathbf{f}(q,s'|\hat{e})dq = 0,
\end{aligned}$$

where, for every $s$, the first line divides and multiplies the expression inside the integral in equation (16) by $\mathbf{f}(q,s'|\hat{e})$; the second line adds a term that equals zero (due to equation (14)); the third line splits the integral between outputs lower and higher than $\bar{q}_{s'}$; the fourth line uses MLRP supposing that $\hat{e} > e^D_{s'}$ and equation (17); the fifth line uses equation (14). These inequalities give us a contradiction ($0 < 0$), showing that $\hat{e} \leq e^D_{s'}$.

For a given initial contract $(W^*_s)$, repeat the same procedure for every $s \in \{s_1, \ldots, s_S\}$ which is such that the payment schedule under this signal realization does not take the form of debt. The resulting contract, which we denote by $(W^D_s)$, is a debt contract, i.e. the payment schedule takes the form of debt for every $s$. Since the procedure weakly increased the implemented effort for every $s$, the effort implemented by this debt contract, denoted by $e^D$, is weakly larger than the effort $\hat{e}$ to be induced (this directly follows from the fact that the left-hand side ("LHS") of the IC is additive across signals).

*Step 1.b., nonbinding participation constraint.*

We now show how to modify this contract to implement the same effort as the initial contract, $\hat{e}$, at a lower cost. Since the resulting contract will still be a debt contract, it satisfies the contracting constraints in equations (5) and (6).

By assumption, the contract $(W^*_s)$ is incentive compatible and the FOA holds, so that:

$$\sum_s \int_{\underline{q}}^{\overline{q}} W^*_s(q) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) dq \quad = \quad C'(\hat{e}). \tag{18}$$

Let $\varepsilon$ be an arbitrarily large constant which satisfies the following two conditions: (i) $\varepsilon > \max\{q_1, \ldots, q_S\}$, and (ii):

$$\sum_s \int_{\varepsilon}^{\overline{q}} (q - \varepsilon) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) dq \quad < \quad C'(\hat{e}). \tag{19}$$

There exists $\varepsilon$ that satisfies condition (19) because of the assumption that $\lim_{q \nearrow +\overline{q}} \frac{\partial \mathbf{f}}{\partial e}(q, s|e) = 0$. Consider the subset of $\{s_1, \ldots, s_S\}$ such that:

$$\int_{\varepsilon}^{\overline{q}} (q - \varepsilon) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) dq < \int_{\underline{q}}^{\overline{q}} W^*_s(q) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) dq, \tag{20}$$

and denote this subset by $\mathcal{S}$. $\mathcal{S}$ is nonempty (if it were, summing over signals in equation (20) and comparing with equation (19) would yield the contradiction that equation (18) does not hold).

For any $s \in \mathcal{S}$, we claim and establish below that there exists $\hat{q}_s \geq q_s$ which solves:

$$\int_{\hat{q}_s}^{\overline{q}} (q - \hat{q}_s) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) dq = \int_{\underline{q}}^{\overline{q}} W^*_s(q) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) dq. \tag{21}$$

For a given $s \in \mathcal{S}$, using the IC with the FOA and the results on effort under the two payment schedules $W_s^*$ and $W_s^{q_s}$ established in Step 1.a. gives the following equation:

$$\int_{q_s}^{\overline{q}} (q - q_s) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e})dq \ \ \geq \ \ \int_{\underline{q}}^{\overline{q}} W_s^*(q) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e})dq \tag{22}$$

For each signal $s \in \mathcal{S}$, there are two cases. If, for a given $s$, equation (22) holds as an equality, then set $\hat{q}_s = q_s$, so that equation (21) holds. If, for a given $s$, equation (22) holds as a strict inequality, then for this $s$, there is $\hat{q}_s \in (q_s, \varepsilon)$ such that equation (21) holds because of the intermediate value theorem, which for a given $s$ we apply on the interval $[q_s, \varepsilon]$. The theorem applies because of equation (20), equation (22) as a strict inequality, and $\int_z^{\overline{q}} (q - z) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e})dq$ is a continuous function of $z$.

First, if $\mathcal{S} = \{s_1, \ldots, s_S\}$ or if

$$\sum_{\tilde{s} \notin \mathcal{S}} \int_{\varepsilon}^{\overline{q}} (q - \varepsilon) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq + \sum_{\tilde{s} \in \mathcal{S}} \int_{\hat{q}_{\tilde{s}}}^{\overline{q}} (q - \hat{q}_{\tilde{s}}) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq = C'(\hat{e}), \tag{23}$$

where for each $s \in \mathcal{S}$, the contractual debt repayment $\hat{q}_s$ is implicitly defined in equation (21), then for any $s \in \mathcal{S}$ use the payment schedule:

$$W_s^{\hat{q}_s}(q) = \max\{0, q - \hat{q}_s\}, \tag{24}$$

and for any $s \notin \mathcal{S}$ the contractual debt repayment is set at $\varepsilon$.

Second, if $\mathcal{S} \subset \{s_1, \ldots, s_S\}$ and the condition in equation (23) does not hold, then let the signals in $\mathcal{S}$ be ordered such that $\mathcal{S} = \{s_1^{\mathcal{S}}, \ldots, s_N^{\mathcal{S}}\}$, with $N \geq 1$ (since $\mathcal{S}$ is nonempty). Denote by $\mathcal{S}^c$ the complement of $\mathcal{S}$. For any $s \in \mathcal{S}^c$, set the contractual debt repayment at $\varepsilon$. If

$$\sum_{\tilde{s} \in \mathcal{S}^c \cup \{s_1^{\mathcal{S}}\}} \int_{\varepsilon}^{\overline{q}} (q - \varepsilon) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq + \sum_{\tilde{s} \in \mathcal{S} \setminus \{s_1^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\overline{q}} (q - \hat{q}_{\tilde{s}}) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq < C'(\hat{e}), \tag{25}$$

then let $\check{q}_{s_1^{\mathcal{S}}}$ be implicitly defined by:

$$\sum_{\tilde{s} \in \mathcal{S}^c} \int_{\varepsilon}^{\overline{q}} (q - \varepsilon) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq + \sum_{\tilde{s} \in \mathcal{S} \setminus \{s_1^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\overline{q}} (q - \hat{q}_{\tilde{s}}) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq + \int_{\check{q}_{s_1^{\mathcal{S}}}}^{\overline{q}} \left(q - \check{q}_{s_1^{\mathcal{S}}}\right) \frac{\partial \mathbf{f}}{\partial e}(q, s_1^{\mathcal{S}}|\hat{e})dq = C'(\hat{e}).$$

$\check{q}_{s_1^{\mathcal{S}}}$ exists and is larger than $\hat{q}_{s_1^{\mathcal{S}}}$ by application of the intermediate value theorem to the interval $[\hat{q}_{s_1^{\mathcal{S}}}, \varepsilon]$, with equations (25) and (26):

$$\sum_{\tilde{s} \in \mathcal{S}^c} \int_{\varepsilon}^{\overline{q}} (q - \varepsilon) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq + \sum_{\tilde{s} \in \mathcal{S}} \int_{\hat{q}_{\tilde{s}}}^{\overline{q}} (q - \hat{q}_{\tilde{s}}) \frac{\partial \mathbf{f}}{\partial e}(q, \tilde{s}|\hat{e})dq > C'(\hat{e}). \tag{26}$$

We get equation (26) because of equation (18), and because for signals in $\mathcal{S}$, the contractual debt repayment $\hat{q}_s$ satisfies equation (21), for signals in $\mathcal{S}^c$ the condition in equation (20) does not hold, and equation (23) does not hold here (see above). If condition (25) holds, then set the contractual debt repayment of signal $s_1^{\mathcal{S}}$ at $\check{q}_{s_1^{\mathcal{S}}}$, and set the contractual debt repayment at $\hat{q}_s$ for other signals in $\mathcal{S}$. If condition (25) does not hold, then set $\hat{q}_{s_1^{\mathcal{S}}} = \varepsilon$, repeat the same steps with signal $s_2^{\mathcal{S}}$ (we omit explicit formulation of these steps for brevity), and continue repeating these steps to additional signals in $\mathcal{S}$ until, for a signal $s_i^{\mathcal{S}}$, with $i \leq N$, condition

$$\sum_{\tilde{s}\in\mathcal{S}^c\cup\{s_1^{\mathcal{S}},\ldots,s_i^{\mathcal{S}}\}} \int_{\varepsilon}^{\overline{q}} (q-\varepsilon)\,\frac{\partial \mathbf{f}}{\partial e}(q,\tilde{s}|\hat{e})dq + \sum_{\tilde{s}\in\mathcal{S}\setminus\{s_1^{\mathcal{S}},\ldots,s_i^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\overline{q}} (q-\hat{q}_{\tilde{s}})\,\frac{\partial \mathbf{f}}{\partial e}(q,\tilde{s}|\hat{e})dq < C'(\hat{e}) \quad (27)$$

is satisfied, in which case set the contractual debt repayment of signal $s_i^{\mathcal{S}}$ at $\check{q}_{s_i^{\mathcal{S}}}$, which is implicitly defined by:

$$\begin{aligned}\sum_{\tilde{s}\in\mathcal{S}^c\cup\{s_1^{\mathcal{S}},\ldots,s_{i-1}^{\mathcal{S}}\}} \int_{\varepsilon}^{\overline{q}} (q-\varepsilon)\,\frac{\partial \mathbf{f}}{\partial e}(q,\tilde{s}|\hat{e})dq + &\sum_{\tilde{s}\in\mathcal{S}\setminus\{s_1^{\mathcal{S}},\ldots,s_i^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\overline{q}} (q-\hat{q}_{\tilde{s}})\,\frac{\partial \mathbf{f}}{\partial e}(q,\tilde{s}|\hat{e})dq \\ &+ \int_{\check{q}_{s_i^{\mathcal{S}}}}^{\overline{q}} \left(q-\check{q}_{s_i^{\mathcal{S}}}\right)\frac{\partial \mathbf{f}}{\partial e}(q,s_i^{\mathcal{S}}|\hat{e})dq = C'(\hat{e}).\end{aligned}$$

$\check{q}_{s_i^{\mathcal{S}}}$ exists and is larger than $\hat{q}_{s_i^{\mathcal{S}}}$ because of the same arguments used above. Because of equation (19), condition (27) will be satisfied for a signal $s_i^{\mathcal{S}}$, with $i \leq N$. If $i < N$, for signals $s \in \{s_{i+1}^{\mathcal{S}}, \ldots, s_N^{\mathcal{S}}\}$ in $\mathcal{S}$, set the contractual debt repayment to $\hat{q}_s$ as in equation (21).

In sum, for each given $s$, the new contract is a debt contract with a repayment of either $\hat{q}_s$ or $\check{q}_s$ or $\varepsilon$, such that $\hat{q}_s \geq q_s$ if $\hat{q}_s$ exists, $\check{q}_s > \hat{q}_s \geq q_s$ if $\check{q}_s$ and $\hat{q}_s$ exist, and $\varepsilon > q_s$. Since by construction the debt contract $(W_s^D)$ with contractual debt repayments $q_s$ has the same cost as the initial contract $(W_s^*)$, and the cost of a debt contract for the principal at a given $s$ is decreasing in the contractual debt repayment at this signal $s$, the new debt contract achieves the same effort $\hat{e}$ as the initial contract $(W_s^*)$ at a lower cost.

*Step 1.b., binding participation constraint.*

We now show that inducing a higher effort at the same cost increases the principal's objective function. We know that the agent will accept the new contract since both contracts are associated with the same expected payment under effort $\hat{e}$, as established above, and the agent prefers to exert a higher effort, i.e. the agent's expected utility is higher under this higher level of effort.

We have considered an initial contract $W_s(q)$ that elicits effort $\hat{e}$. We have shown in step 1.a. that there exists contract $W_s^D$ that satisfies contracting constraints and elicits higher effort $e^D$ which is such that $\mathbb{E}[W_s^D(q)|\hat{e}] = \mathbb{E}[W_s(q)|\hat{e}]$.

The change in the principal's objective function from switching to the initial contract to the debt contract is:

$$\sum_s \int_{\underline{q}}^{\overline{q}} q\mathbf{f}(q,s|e^D)dq - \sum_s \int_{\underline{q}}^{\overline{q}} q\mathbf{f}(q,s|\hat{e})dq - \left(\sum_s \int_{\underline{q}}^{\overline{q}} W_s^D(q)\mathbf{f}(q,s|e^D)dq - \sum_s \int_{\underline{q}}^{\overline{q}} W_s(q)\mathbf{f}(q,s|\hat{e})dq\right) \quad (28)$$

Since by construction of the debt contract we have:

$$\sum_s \int_{\underline{q}}^{\overline{q}} W_s^D(q)\mathbf{f}(q,s|\hat{e})dq = \sum_s \int_{\underline{q}}^{\overline{q}} W_s(q)\mathbf{f}(q,s|\hat{e})dq,$$

we can rewrite the change in the principal's objective function in equation (28) as:

$$\sum_s \int_{\underline{q}}^{\overline{q}} q\mathbf{f}(q,s|e^D)dq - \sum_s \int_{\underline{q}}^{\overline{q}} q\mathbf{f}(q,s|\hat{e})dq - \left(\sum_s \int_{\underline{q}}^{\overline{q}} W_s^D(q)\mathbf{f}(q,s|e^D)dq - \sum_s \int_{\underline{q}}^{\overline{q}} W_s^D(q)\mathbf{f}(q,s|\hat{e})dq\right) \quad (29)$$

where $W_s^D(q)$ can be written as $W_s^D(q) = \max\{q - q_s^*, 0\}$. The expression in equation (29) is positive if:

$$\frac{\partial}{\partial e}\mathbb{E}[q - W_s^D(q)|e] = \sum_s \left(\int_{\underline{q}}^{\overline{q}} q\frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq - \int_{q^*}^{\overline{q}} (q - q_s^*)\frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq\right) > 0 \quad (30)$$

for all $e \in (\hat{e}, e^D)$. The expression in equation (30) is equal to zero if $q_s^* = 0$ for all $s$. If $q_s^* > 0$ for some $s$, we have:

$$\sum_s \frac{d}{dq_s^*}\left\{\int_{\underline{q}}^{\overline{q}} q\frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq - \int_{q_s^*}^{\overline{q}} (q - q_s^*)\frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq\right\} = \sum_s \int_{q_s^*}^{\overline{q}} \frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq > 0$$

by MLRP. In sum, the expression in equation (30) is positive.

*Step 2. Determining the optimal debt repayment.*

Since any debt contract satisfies bilateral LL and monotonicity, and since we assumed that the condition for the FOA in Lemma 1 holds, the firm's program becomes:

$$\min_{\{q_s\}_{s=1,\ldots,S}} \sum_s \int_{q_s}^{\overline{q}} (q - q_s)\,\mathbf{f}(q,s|\hat{e})dq. \quad (31)$$

subject to

$$\sum_s \int_{q_s}^{\overline{q}} (q - q_s)\, \mathbf{f}(q, s|\hat{e})dq - C(\hat{e}) \geq \overline{U} - \bar{W} \tag{32}$$

$$\sum_s \int_{q_s}^{\overline{q}} (q - q_s)\, \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e})dq = C'(\hat{e}), \tag{33}$$

where $\frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) = \frac{\partial \phi^s_{\hat{e}}}{\partial e} f(q|\hat{e}, s) + \phi^s_{\hat{e}} \frac{\partial f}{\partial e}(q|\hat{e}, s)$. The likelihood ratio can be rewritten as follows:

$$\begin{aligned}
\overline{LR}_s(q) &= \frac{\int_q^{\overline{q}} \left[\frac{\partial \phi^s_{\hat{e}}}{\partial e} f(z|\hat{e}, s) + \phi^s_{\hat{e}} \frac{\partial f}{\partial e}(z|\hat{e}, s)\right] dz}{\int_q^{\overline{q}} \phi^s_{\hat{e}} f(z|\hat{e}, s)dz} \\
&= \frac{\int_q^{\overline{q}} \frac{\partial \phi^s_{\hat{e}}}{\partial e} f(z|\hat{e}, s)dz}{\int_q^{\overline{q}} \phi^s_{\hat{e}} f(z|\hat{e}, s)dz} + \frac{\int_q^{\overline{q}} \phi^s_{\hat{e}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz}{\int_q^{\overline{q}} \phi^s_{\hat{e}} f(z|\hat{e}, s)dz} \\
&= \frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\int_q^{\overline{q}} \frac{\partial f}{\partial e}(z|\hat{e}, s)dz}{\int_q^{\overline{q}} f(z|\hat{e}, s)dz}
\end{aligned}$$

For each fixed $\kappa$ and signal realization $s$, construct the threshold $q^*_s(\kappa)$ as follows:

$$q^*_s(\kappa) := \begin{cases} 0 & \text{if } \overline{LR}_s(0) > \kappa \\ \overline{LR}_s^{-1}(\kappa) & \text{if } \overline{LR}_s(0) \leq \kappa \end{cases}. \tag{34}$$

When the participation constraint is nonbinding, the cutoff $\kappa$ is implicitly determined by the binding incentive constraint for a given $\hat{e}$:

$$\sum_s \int_{q^*_s(\kappa)}^{\overline{q}} (q - q^*_s(\kappa)) \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e})dq = C'(\hat{e}). \tag{35}$$

When the participation constraint is binding, the cutoff $\kappa$ and the equilibrium effort are implicitly determined by the binding incentive constraint and the binding participation constraint. The necessary first-order conditions associated with the program in equations (31), (32), and (33) are equation (34), the binding IC:

$$\sum_s \int_{q^*_s(\kappa)}^{\overline{q}} (q - q^*_s(\kappa))\, \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e})dq = C'(\hat{e}), \tag{36}$$

and possibly the binding participation constraint, where $\kappa := \frac{1-\lambda}{\mu}$, where $\lambda$ and $\mu$ are the Lagrange multipliers associated with the participation constraint and the incentive constraint, respectively.

Each $\kappa$ determines $q_s^*(\kappa)$ according to equation (34). From the Intermediate Value Theorem, there exists $\kappa$ that solves equation (36): as $\kappa \searrow -\infty$, the LHS of (36) exceeds $C'(\hat{e})$ since then $q_s^*(\kappa) = 0 \; \forall s$ and

$$\sum_s \int_{\underline{q}}^{\overline{q}} q \frac{\partial \mathbf{f}}{\partial e}(q, s|\hat{e}) dq \geq C'(\hat{e})$$

by Lemma 2 with a risk neutral agent, and it converges to $0 < C'(\hat{e})$ as $\kappa \nearrow +\infty$. Moreover, $\kappa$ must be unique since our conditions for the validity of the FOA ensure that the agent's program has a unique solution. ■

**Proof of Proposition 2**

Start with part (i) of the Proposition. From Proposition 1, there are two possible cases in which the optimal contract does not depend on the signal ($q_{s_1}^* = ... = q_{s_S}^* = q^*$): an interior solution $q^* \in (0, \overline{q})$ and a boundary solution $q^* = 0$ or $q^* = \overline{q}$. Using the conditions from equation (34) for an interior solution establishes:

$$\overline{LR}_{s_i}(q^*) = \overline{LR}_{s_j}(q^*) = \kappa \;\; \forall s_i, s_j. \tag{37}$$

where $\kappa$ is determined by (35).

We now verify that the solution cannot be at the boundary. For a boundary solution we need either $\overline{LR}_s(0) > \kappa$ for all $s$ or $\lim_{q \to \overline{q}} \overline{LR}_s(q) < \kappa$ for all $s$. In the first case, the firm always receives zero, which contradicts the optimality of implementing high effort (since the firm can always obtain strictly positive profits by paying zero in all states and implementing low effort). In the second case, the manager always receives zero, violating equation (35) as the IC is not satisfied.

For part (ii) of the Proposition, if $q \leq \min_s \{q_s^*\}$ then $w_s(q) = 0 \; \forall s$, i.e., $w_s(q)$ is independent of $s$.

For part (iii), given signal realization $s$, according to the optimal contract in Proposition 1 and to equation (34), the debt repayment is zero if $\overline{LR}_s(q)$ is above $\kappa$ for any $q$, where $\kappa$ is implicitly defined in equation (35). Given that the second term in the likelihood ratio $\overline{LR}_s(q)$ in equation (8) is increasing in $q$ (as established in the proof of Proposition 1) and is bounded from below by 0, a sufficient condition for the payment to be the zero under signal $s$ is that the first term in the likelihood ratio $\overline{LR}_s(q)$ in equation (8) be above $\kappa$. ■

**Proof of Proposition 3:**

With the condition for the FOA in Lemma 1 and without additional signals, the optimal contract solves the program:

$$\min_{\{w(q)\},e} \int_{\underline{q}}^{\overline{q}} w\left(q\right) f(q|e)dq$$

$$\text{subject to} \qquad \int_{\underline{q}}^{\overline{q}} u\left(\bar{W}+w\left(q\right)\right) f(q|e)dq - C(e) \geq \overline{U},$$

$$\int_{\underline{q}}^{\overline{q}} u\left(\bar{W}+w\left(q\right)\right) \frac{\partial f}{\partial e}(q|e)dq = C'(e),$$

$$w(q) \geq 0 \text{ and } \dot{w}(q) \leq 1.$$

To write this as an optimal control problem, we introduce the auxiliary variable $x(q) \equiv \dot{w}(q)$ and adjoin the objective with the IC (as in an isoperimetric problem). Doing so, we obtain the following program:

$$\max_{\{w(q),x(q)\}} \int_{\underline{q}}^{\overline{q}} \left[-w\left(q\right) f(q|e) + \theta u\left(\bar{W}+w\left(q\right)\right) f(q|e) + \xi u\left(\bar{W}+w\left(q\right)\right) \frac{\partial f}{\partial e}(q|e)\right] dq$$

subject to

$$\dot{w}(q) = x(q),$$

$$w(q) \geq 0,\ w(q) \leq q, \text{ and } x(q) \leq 1.$$

Note that $w(q) \geq 0$ and $w(q) \leq q$ are (first-order) pure state constraints. We use Theorem 3.60 from Grass et al. (2008). The Hamiltonian $\mathcal{H}$ and Lagrangian $\mathcal{L}$ are:

$$\begin{aligned} \mathcal{H}(w,x,\lambda,q) &= -wf(q|e) + \theta u\left(\bar{W}+w\left(q\right)\right) f(q|e) + \xi u\left(\bar{W}+w\right) \frac{\partial f}{\partial e}(q|e) + \lambda(q)x \\ \mathcal{L}(w,x,\lambda,\mu,\eta,q) &= \mathcal{H}(w,x,\lambda,q) + \mu(q)\left(1-x\right) + \nu(q)w + \varpi(q)(q-w) \end{aligned}$$

The optimality condition with respect to the control variable is:

$$\frac{\partial \mathcal{L}}{\partial x} = 0 \quad \therefore \quad \lambda(q) = \mu(q),$$

and, by CS, $\mu(q) \geq 0$ with $=$ if $x(q) < 1$. Thus, $\lambda(q) \geq 0$ with $=$ if $x(q) < 1$. The equation of motion for the costate variable is:

$$\dot{\lambda}(q) = -\frac{\partial \mathcal{L}}{\partial w} \quad \therefore \quad \dot{\lambda}(q) = f(q|e) - \theta u'\left(\bar{W}+w\left(q\right)\right) f(q|e) - \xi u'\left(\bar{W}+w\right) \frac{\partial f}{\partial e}(q|e) - \nu(q) + \varpi(q),$$

and, by CS, $\nu(q) \geq 0$ with $=$ if $w(q) > 0$. Thus,

$$\dot{\lambda}(q) \leq f(q|e) - \theta u'\left(\bar{W}+w\left(q\right)\right) f(q|e) - \xi u'\left(\bar{W}+w\right) \frac{\partial f}{\partial e}(q|e) + \varpi(q)$$

with = if $w(q) > 0$.

By MLRP, it can be shown that there exists $q^*$ such that $w(q) = 0$ if and only if $q \leq q^*$. At junction points, the costate variable may jump downwards:

$$\lambda(q^+) \leq \lambda(q^-).$$

By monotonicity, there is at most one junction point $q^* \geq 0$, which is the point in which LL stops binding: $w(q) = 0$ for $q \leq q^*$ and $w(q) > 0$ for $q > q^*$. Moreover, since $x(q) = 0 < 1$ for all $q < q^*$, we must have $\lambda(q) = 0$ for all $q < q^*$, implying that $\lim_{q \searrow q^*} \lambda(q) \leq 0$. But recall that $\lambda(q) \geq 0$ for all $q$. Therefore, $\lambda$ must be continuous at $q^*$ ($\lim_{q \searrow q^*} \lambda(q) = 0$).

There is also the condition that the Hamiltonian

$$H(q) \equiv \mathcal{H}(w(q), x(q), \lambda(q), q) = -w(q)f(q|e) + \theta u\left(\bar{W} + w\left(q\right)\right) f(q|e) + \xi u\left(\bar{W} + w(q)\right) \frac{\partial f}{\partial e}(q|e) + \lambda(q)x(q)$$

is continuous. However, this condition is trivially satisfied here. Since $w$ is continuous, it becomes:

$$\lambda(q^-)x(q^-) = \lambda(q^+)x(q^+).$$

Moreover, as seen before, $\lambda(q^+) = \lambda(q^-) = 0$ at the junction point $q^*$, so this automatically holds. The transversality condition is $\lim_{q \to \bar{q}} \lambda_s(q) = 0$.

Combining all conditions, we have:

- $\lambda(q) \geq 0$ with = if $\dot{w}(q) < 1$.
- $\dot{\lambda}(q) \leq f(q|e) - \theta u'\left(\bar{W} + w\left(q\right)\right) f(q|e) - \xi u'\left(\bar{W} + w(q)\right) \frac{\partial f}{\partial e}(q|e) + \varpi(q)$ with = if $w(q) > 0$.
- $\lambda$ is continuous.
- $\lim_{q \to \bar{q}} \lambda(q) = 0$.

Note that since $\dot{\lambda}(q) = 0$ and $w(q) = 0$ for $q < q^*$, we must have

$$\dot{\lambda}(q) \leq f(q|e) - \theta u'\left(\bar{W}\right) f(q|e) - \xi u'\left(\bar{W}\right) \frac{\partial f}{\partial e}(q|e),$$

which can be rearranged as:

$$\frac{1}{u'\left(\bar{W}\right)} \geq \theta + \xi \frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)} \text{ for all } q \leq q^*.$$

**Lemma 5** *In any interval with $w(q) \in (0, q)$ and $\dot{w}(q) < 1$, we have*

$$\dot{w}\left(q\right) = \frac{\frac{d}{dq}\left[\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)}\right] \Big/ \left(\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)} + \frac{\theta}{\xi}\right)}{A_{\bar{W}+w(q)}}.$$

**Proof.** Suppose there exists an interval with $w(q) \in (0, q)$ and $\dot{w}(q) < 1$. Since $w(q) \in (0, q)$, we must have

$$\dot{\lambda}(q) = f(q|e) - \theta u'\left(\bar{W} + w\left(q\right)\right) f(q|e) - \xi u'\left(\bar{W} + w\left(q\right)\right) \frac{\partial f}{\partial e}(q|e)$$

and, because $\dot{w}(q) < 1$, we must have $\lambda(q) = 0$ so that $\dot{\lambda}(q) = 0$. Combining both conditions and rearranging, we obtain:

$$\frac{1}{\xi} = u'\left(\bar{W} + w\left(q\right)\right)\left(\underbrace{\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)}}_{LR(q|e)} + \frac{\theta}{\xi}\right).$$

Because the LHS is not a function of $q$, the derivative of the RHS must be zero:

$$u''\left(\bar{W} + w\left(q\right)\right)\left(\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)} + \frac{\theta}{\xi}\right)\dot{w}\left(q\right) + u'\left(\bar{W} + w\left(q\right)\right)\frac{d}{dq}\left[\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)}\right] = 0.$$

Rearranging, gives:

$$\dot{w}\left(q\right) = -\frac{u'\left(\bar{W} + w\left(q\right)\right)}{u''\left(\bar{W} + w\left(q\right)\right)}\frac{\frac{d}{dq}\left[\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)}\right]}{\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)} + \frac{\theta}{\xi}}. \tag{38}$$

■

It suffices to show that there is no interval of outputs with $w(q) > 0$ and $\dot{w}(q) < 1$. By the previous lemma, this is true if

$$\dot{w}\left(q\right) = -\frac{u'\left(\bar{W} + w\left(q\right)\right)}{u''\left(\bar{W} + w\left(q\right)\right)}\frac{\frac{d}{dq}\left[\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)}\right]}{\frac{\frac{\partial f}{\partial e}(q|e)}{f(q|e)} + \frac{\theta}{\xi}} \geq 1$$

at all $q > q^*$. At $q = q^*$, this condition rewrites as:

$$-\frac{u'\left(\bar{W}\right)}{u''\left(\bar{W}\right)}\frac{\frac{d}{dq}\left[\frac{\frac{\partial f}{\partial e}(q^*|e)}{f(q^*|e)}\right]}{\frac{\frac{\partial f}{\partial e}(q^*|e)}{f(q^*|e)} + \frac{\theta}{\xi}} \geq 1. \tag{39}$$

If the condition in equation (39) is satisfied, suppose that the condition in Proposition 3 is satisfied for $q \in [q^*, \hat{q})$ for a given $\hat{q} > q^*$. Then $w(\hat{q}) = \hat{q} - q^*$, and the condition at $q = \hat{q}$ is:

$$\dot{w}\left(\hat{q}\right) = -\frac{u'\left(\bar{W} + \hat{q} - q^*\right)}{u''\left(\bar{W} + \hat{q} - q^*\right)} \frac{\frac{d}{dq}\left[\frac{\frac{\partial f}{\partial e}(\hat{q}|e)}{f(\hat{q}|e)}\right]}{\frac{\frac{\partial f}{\partial e}(\hat{q}|e)}{f(\hat{q}|e)} + \frac{\theta}{\xi}} \geq 1. \tag{40}$$

The optimal contract is debt if the condition in equation (40) is satisfied for any $\hat{q} \geq q^*$. This condition involves Lagrange multipliers, which we address in the last part of the proof below.

Consider the first step in Grossman and Hart (1983). To start, solve the optimization problem in equations (2)-(6) without the participation constraint (equation (3)). In this relaxed problem, the optimal contract is debt if the condition in equation (40) is satisfied with $\theta = 0$. There are two cases. First, if the condition in equation (40) is not satisfied with $\theta = 0$, then the optimal contract that solves the optimization problem in equations (2)-(6) is not debt. Indeed, a necessary condition for the optimal contract to be debt is that it be debt with $\theta = 0$. Second, if the condition in equation (40) with $\theta = 0$ is satisfied, then verify if the participation constraint in equation (3) is satisfied with the debt contract thus derived. If it is satisfied, then the participation constraint is nonbinding, i.e. $\theta = 0$, and the optimal contract that solves the optimization problem in equations (2)-(6) is debt. If it is not satisfied, then $\theta > 0$ and the optimal contract is not debt.

We now prove this latest claim. For $q^* > q^e$ (where $q^e$ is defined in Lemma 2), by MLRP:

$$\begin{aligned} &\frac{\partial}{\partial q^*}\left\{\int_{\underline{q}}^{q^*} u(\bar{W})\frac{\partial f}{\partial e}(q|e^*)dq + \int_{q^*}^{\overline{q}} u(\bar{W} + q - q^*)\frac{\partial f}{\partial e}(q|e^*)dq\right\} \\ &= -\int_{q^*}^{\overline{q}} u'(\bar{W} + q - q^*)\frac{\partial f}{\partial e}(q|e^*)dq < 0, \end{aligned}$$

whereas the inequality is reversed for $q^* < q^e$, again by MLRP. Thus, there are at most two debt contracts that induce a given $e^*$. Denote the debt repayment under these contracts as $q_1^*$ and $q_2^*$, where both $q_1^*$ and $q_2^*$ are functions of $e^*$, with $q_1^* < q_2^*$. We now show that the debt contract with debt repayment $q_1^*$ is dominated.

**Lemma 6** *The debt contract with debt repayment $q_1^*$ is not optimal.*

**Proof.** The proof shows that this contract is dominated by another contract. Consider a stochastic contract with payment $\max\{q - (q_1^* + \epsilon), 0\}$ with probability $p$, and fixed payment $w$ with probability $1 - p$, where both $\epsilon$ and $w$ are positive and arbitrarily small.

The change in incentives is independent of $w$ and only depends on $\epsilon$ and $p$ (this follows from $\int_{\underline{q}}^{\overline{q}} u(\bar{W} + w)\frac{\partial f}{\partial e}(q|e^*)dq = u(\bar{W} + w)\int_{\underline{q}}^{\overline{q}} \frac{\partial f}{\partial e}(q|e^*)dq = 0$ by definition of a density function). Note that while setting $p > 0$ decreases incentives, setting $\epsilon > 0$ increases incentives (given that $q_1^* < q^e$).

The change in the agent's expected utility is:

$$(1-p)u(\bar{W}+w) + p\left(\int_{\underline{q}}^{q_1^*+\epsilon} u(\bar{W})f(q|e^*)dq + \int_{q_1^*+\epsilon}^{\bar{q}} u(\bar{W}+q-(q_1^*+\epsilon))f(q|e^*)dq\right)$$
$$-\left(\int_{\underline{q}}^{q_1^*} u(\bar{W})f(q|e^*)dq + \int_{q_1^*}^{\bar{q}} u(\bar{W}+q-q_1^*)f(q|e^*)dq\right)$$

For arbitrarily small $\epsilon$ and $w$, this can be approximated as:

$$(1-p)\left(u(\bar{W})+u'(\bar{W})w\right) - p\epsilon\int_{q_1^*}^{\bar{q}} u'(\bar{W}+q-q_1^*)f(q|e^*)dq$$
$$-(1-p)\left(\int_{\underline{q}}^{q_1^*} u(\bar{W})f(q|e^*)dq + \int_{q_1^*}^{\bar{q}} u(\bar{W}+q-q_1^*)f(q|e^*)dq\right) \tag{41}$$

The change in the expected cost of the contract for the principal is:

$$(1-p)w + p\int_{q_1^*+\epsilon}^{\bar{q}} (q-(q_1^*+\epsilon))f(q|e^*)dq - \int_{q_1^*}^{\bar{q}} (q-q_1^*)f(q|e^*)dq$$

For an arbitrarily small $\epsilon$, this can be approximated as:

$$(1-p)w - p\epsilon\int_{q_1^*}^{\bar{q}} f(q|e^*)dq - (1-p)\int_{q_1^*}^{\bar{q}} (q-q_1^*)f(q|e^*)dq$$

Setting the change in expected cost to zero and rearranging gives:

$$w = \epsilon\frac{p}{1-p}\int_{q_1^*}^{\bar{q}} f(q|e^*)dq + \int_{q_1^*}^{\bar{q}} (q-q_1^*)f(q|e^*)dq \tag{42}$$

Equation (41) is positive if and only if:

$$u(\bar{W})+u'(\bar{W})w > \epsilon\frac{p}{1-p}\int_{q_1^*}^{\bar{q}} u'(\bar{W}+q-q_1^*)f(q|e^*)dq$$
$$+\left(\int_{\underline{q}}^{q_1^*} u(\bar{W})f(q|e^*)dq + \int_{q_1^*}^{\bar{q}} u(\bar{W}+q-q_1^*)f(q|e^*)dq\right)$$

With $u'' < 0$, we have $u(\bar{W}+q-q_1^*) < u(\bar{W}) + u'(\bar{W})(q-q_1^*)$ and $u'(\bar{W}+q-q_1^*) < u'(\bar{W})$

for $q > q_1^*$. Thus, a sufficient condition for equation (41) to be strictly positive is:

$$
\begin{aligned}
& u(\bar{W}) + u'(\bar{W})w \geq \epsilon \frac{p}{1-p} \int_{q_1^*}^{\bar{q}} u'(\bar{W}) f(q|e^*) dq \\
& + \left( \int_{\underline{q}}^{q_1^*} u(\bar{W}) f(q|e^*) dq + \int_{q_1^*}^{\bar{q}} \left( u(\bar{W}) + u'(\bar{W})(q - q_1^*) \right) f(q|e^*) dq \right) \\
\Leftrightarrow \quad & w \geq \epsilon \frac{p}{1-p} \int_{q_1^*}^{\bar{q}} f(q|e^*) dq + \int_{q_1^*}^{\bar{q}} (q - q_1^*) f(q|e^*) dq
\end{aligned}
$$

Compare with equation (42). By a continuity argument, given $\epsilon$ and $p$, there exists $w$ that leaves the expected utility of the agent unchanged while lowering the cost of the contract. Finally, incentives for effort $e^*$, which are independent from $w$ as already established, can be adjusted by changing $\epsilon$ and $p$. ■

In sum, if the debt contract with debt repayment $q_2^*$ that elicits $e^*$ does not satisfy the participation constraint, then there is no debt contract that induces $e^*$, satisfies the participation constraint, and is not dominated. Therefore, in this case, the optimal contract is not a debt contract. ■

**Proof of Corollary 1**

Consider a normal distribution, which has PDF:

$$
f(q|e) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[ -\frac{1}{2} \left( \frac{q-e}{\sigma} \right)^2 \right].
$$

The likelihood ratio is

$$
LR(q|e) = \frac{f_e(q|e)}{f(q|e)} = \frac{q-e}{\sigma^2},
$$

so that

$$
\ln LR(q|e) = \ln(q-e) - 2\ln\sigma.
$$

Differentiation, gives

$$
\frac{d}{dq}\left[ \ln LR(q|e) \right] = \frac{1}{q-e}.
$$

The condition for the optimal contract to be debt then becomes

$$
\frac{1}{A_{\bar{W}+w(q)}} \frac{1}{q-e} \geq 1 \quad \Leftrightarrow \quad 1 \geq A_{\bar{W}+w(q)})\,(q-e)
$$

for $q \in (q_L, q_H)$. Suppose that this condition holds for $q \in [q^*, q_L]$, so that the optimal contract is debt. To obtain a contradiction and conclude that the optimal contract is not debt, it suffices to find $q \in (q_L, q_H)$ with

$$1 < A_{\bar{W}+w(q)} (q - e).$$

Note that $w(q_L) = q_L - q^*$, where $q_L \geq q^* > e$. Evaluating the condition at $q_L$ gives

$$e \geq q_L - \frac{1}{A_{\bar{W}+q_L-q^*}}.$$

Thus, to obtain a contradiction, it suffices to show that this inequality holds for all $q_L \geq q^*$. At $q_L = q^*$, the condition becomes:

$$e \geq q^* - \frac{1}{A_{\bar{W}}}.$$

Note that if, for all $q_L \geq q^*$,

$$\frac{d}{dq_L}\left(q_L - \frac{1}{A_{\bar{W}+q_L-q^*}}\right) \leq 0,$$

then the inequality holding at $q^*$ implies that it holds for all $q_L \geq q^*$. This condition holds if for all $w \geq \bar{W}$,

$$1 \leq \frac{d}{dw}\left(-\frac{u'(w)}{u''(w)}\right).$$

■

**Proofs of Lemma 4 and Proposition 4.**

We introduce the auxiliary variables $x_s(q) \equiv \dot{w}_s(q)$ and adjoin the objective with the IC (as in an isoperimetric problem) to write the program as an optimal control problem:

$$\max_{\{w(q),x(q)\}} \int_{\underline{q}}^{\bar{q}} \sum_s \left[-w_s(q)\,\mathbf{f}(q,s|e) + \theta u\left(\bar{W} + w_s(q)\right)\mathbf{f}(q,s|e) + \xi u\left(\bar{W} + w_s(q)\right)\frac{\partial \mathbf{f}}{\partial e}(q,s|e)\right] dq$$

subject to

$$\dot{w}_s(q) = x_s(q),$$

$$w_s(q) \geq 0,\ w_s(q) \leq q,\ \text{and } x_s(q) \leq 1.$$

Note that $w_s(q) \geq 0$ and $w_s(q) \leq q$ are first-order pure state constraints. From Theorem 3.60 from Grass et al. (2008), we construct the Hamiltonian $\mathcal{H}$ and Lagrangian $\mathcal{L}$ functions:

$$\mathcal{H}(w,x,\lambda,q) = \sum_s \left[-w_s \mathbf{f}(q,s|e) + \theta u\left(\bar{W}+w_s(q)\right)\mathbf{f}(q,s|e) + \xi u\left(\bar{W}+w_s(q)\right)\frac{\partial \mathbf{f}}{\partial e}(q,s|e) + \lambda_s(q)x_s\right]$$
$$\mathcal{L}(w,x,\lambda,\mu,\eta,q) = \mathcal{H}(w,x,\lambda,q) + \sum_s \mu_s(q)\left(1-x_s\right) + \nu_s(q)w_s + \varpi_s(q)(q-w_s)$$

The optimality conditions with respect to the control variables $x_s$ are:

$$\frac{\partial \mathcal{L}}{\partial x_s} = 0 \quad \therefore \quad \lambda_s(q) = \mu_s(q),$$

and, by complementary slackness, $\mu_s(q) \geq 0$ with $=$ if $x_s(q) < 1$. Thus,

$$\lambda_s(q) \geq 0 \tag{43}$$

with $=$ if $x_s(q) < 1$. The optimality conditions with respect to the state variables $w_s$ are:

$$\frac{\partial \mathcal{L}}{\partial w_s} = -\dot{\lambda}_s(q) \quad \therefore \quad \dot{\lambda}_s(q) = \mathbf{f}(q,s|e) - \theta u'\left(\bar{W}+w_s(q)\right)\mathbf{f}(q,s|e) - \xi u'\left(\bar{W}+w_s(q)\right)\frac{\partial \mathbf{f}}{\partial e}(q,s|e) - \nu_s(q) + \varpi_s(q).$$

By complementary slackness, $\nu_s(q) \geq 0$ with $=$ if $w_s(q) > 0$, so that

$$\dot{\lambda}_s(q) \leq \mathbf{f}(q,s|e) - \theta u'\left(\bar{W}+w_s(q)\right)\mathbf{f}(q,s|e) - \xi u'\left(\bar{W}+w_s(q)\right)\frac{\partial \mathbf{f}}{\partial e}(q,s|e) + \varpi_s(q), \tag{44}$$

with $=$ if $w_s(q) > 0$.

A junction point $q$ has $w_s(q) = 0$ and either $w_s(q+\epsilon) > 0$ or $w_s(q-\epsilon) > 0$ for all $\epsilon > 0$ sufficiently small. The costate variables $\lambda_s(\cdot)$ are continuous except possibly at junction points, where it may jump downwards:

$$\lambda_s(q^+) \leq \lambda_s(q^-).$$

The transversality condition is $\lim_{q \to \bar{q}} \lambda_s(q) = 0$.

**Claim 1** *The optimal contract $w_s(\cdot)$ is non-decreasing in output.*

**Proof.** Suppose $w_s(q_L) > w_s(q_H) \geq 0$ for $q_L < q_H$. Then, there must exist $\hat{q}_L$ and $\hat{q}_H$ with $q_L < \hat{q}_L < \hat{q}_H < q_H$ with $w_s(\hat{q}_L) > w_s(\hat{q}_H) \geq 0$, $\dot{w}_s(\hat{q}_L) < 1$, and $\dot{w}_s(\hat{q}_H) < 1$. Then, from conditions (43) and (44), we have:

$$\mathbf{f}(\hat{q}_L,s|e) - \theta u'\left(\bar{W}+w_s(\hat{q}_L)\right)\mathbf{f}(\hat{q}_L,s|e) - \xi u'\left(\bar{W}+w_s(\hat{q}_L)\right)\frac{\partial \mathbf{f}}{\partial e}(\hat{q}_L,s|e) = 0$$

and

$$\mathbf{f}(\hat{q}_H, s|e) - \theta u'\left(\bar{W} + w_s\left(\hat{q}_H\right)\right)\mathbf{f}(\hat{q}_H, s|e) - \xi u'\left(\bar{W} + w_s(\hat{q}_H)\right)\frac{\partial \mathbf{f}}{\partial e}(\hat{q}_H, s|e) \geq 0.$$

Rearrange these conditions as

$$\frac{1}{\xi} = u'\left(\bar{W} + w_s(\hat{q}_L)\right)\left(\frac{\theta}{\xi} + \frac{\frac{\partial \mathbf{f}}{\partial e}(\hat{q}_L, s|e)}{\mathbf{f}(\hat{q}_L, s|e)}\right) \geq u'\left(\bar{W} + w_s(\hat{q}_H)\right)\left(\frac{\theta}{\xi} + \frac{\frac{\partial \mathbf{f}}{\partial e}(\hat{q}_H, s|e)}{\mathbf{f}(\hat{q}_H, s|e)}\right).$$

Since $u$ is concave and $w_s(\hat{q}_L) > w_s(\hat{q}_H)$, we must have that $u'\left(\bar{W} + w_s(\hat{q}_L)\right) \leq u'\left(\bar{W} + w_s(\hat{q}_H)\right)$. Therefore, the previous inequality implies

$$\frac{\frac{\partial \mathbf{f}}{\partial e}(\hat{q}_L, s|e)}{\mathbf{f}(\hat{q}_L, s|e)} \leq \frac{\frac{\partial \mathbf{f}}{\partial e}(\hat{q}_H, s|e)}{\mathbf{f}(\hat{q}_H, s|e)},$$

violating (strict) MLRP. ■

By monotonicity, there is at most one junction point $q_s^* \geq 0$ for each signal realization. In that junction point, the LL stops binding: $w_s(q) = 0$ for $q \leq q_s^*$ and $w_s(q) > 0$ for $q > q_s^*$. Moreover, since $x_s(q) = 0 < 1$ for all $q < q_s^*$, we must have $\lambda_s(q) = 0$ for all $q < q_s^*$, implying that $\lim_{q \searrow q_s^*} \lambda_s(q) \leq 0$. But recall that $\lambda_s(q) \geq 0$ for all $q$. Therefore, $\lambda_s$ must be continuous at $q_s^*$ ($\lim_{q \searrow q^*} \lambda_s(q) = 0$).

Combining all optimality conditions, we have:

- $\lambda_s(q) \geq 0$ with $=$ if $\dot{w}_s(q) < 1$.
- $\dot{\lambda}_s(q) \leq \mathbf{f}(q, s|e) - \theta u'\left(\bar{W} + w_s(q)\right)\mathbf{f}(q, s|e) - \xi u'\left(\bar{W} + w_s(q)\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e) + \varpi_s(q)$ with $=$ if $w_s(q) > 0$.
- $\lambda_s$ is continuous.
- $\lim_{q \to \bar{q}} \lambda_s(q) = 0$.

Note that since $\dot{\lambda}_s(q) = 0$ and $w_s(q) = 0$ for $q < q_s^*$, we must have

$$\dot{\lambda}_s(q) \leq \mathbf{f}(q, s|e) - \theta u'\left(\bar{W}\right)\mathbf{f}(q, s|e) - \xi u'\left(\bar{W}\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e),$$

which can be rearranged as:

$$\frac{1}{u'\left(\bar{W}\right)} \geq \theta + \xi\frac{\frac{\partial \mathbf{f}}{\partial e}(q, s|e)}{\mathbf{f}(q, s|e)} \quad \text{for all } q \leq q_s^*.$$

**Lemma 7** *In any interval of outputs with $w_s(q) \in (0, q)$ and $\dot{w}_s(q) < 1$, we have*

$$\dot{w}_s(q) = \frac{\frac{d}{dq}\left[\frac{\frac{\partial \mathbf{f}}{\partial e}(q,s|e)}{\mathbf{f}(q,s|e)}\right] \Big/ \left(\frac{\frac{\partial \mathbf{f}}{\partial e}(q,s|e)}{\mathbf{f}(q,s|e)} + \frac{\theta}{\xi}\right)}{A_{\bar{W}+w_s(q)}}.$$

**Proof.** Consider an interval with $w_s(q) \in (0, q)$ and $\dot{w}_s(q) < 1$. Since $w_s(q) \in (0, q)$, we must have:

$$\dot{\lambda}_s(q) = \mathbf{f}(q, s|e) - \theta u'\left(\bar{W} + w_s(q)\right)\mathbf{f}(q, s|e) - \xi u'\left(\bar{W} + w_s(q)\right)\frac{\partial \mathbf{f}}{\partial e}(q, s|e)$$

and, because $\dot{w}_s(q) < 1$, we must have $\lambda_s(q) = 0$ so that $\dot{\lambda}_s(q) = 0$. Combining both conditions and rearranging, we obtain:

$$\frac{1}{\xi} = u'\left(\bar{W} + w_s(q)\right)\left(\frac{\frac{\partial \mathbf{f}}{\partial e}(q, s|e)}{\mathbf{f}(q, s|e)} + \frac{\theta}{\xi}\right).$$

Because the LHS is not a function of $q$, the derivative of the RHS must be zero:

$$u''\left(\bar{W} + w_s(q)\right)\left(\frac{\frac{\partial \mathbf{f}}{\partial e}(q, s|e)}{\mathbf{f}(q, s|e)} + \frac{\theta}{\xi}\right)\dot{w}_s(q) + u'\left(\bar{W} + w_s(q)\right)\frac{d}{dq}\left[\frac{\frac{\partial \mathbf{f}}{\partial e}(q, s|e)}{\mathbf{f}(q, s|e)}\right] = 0.$$

Rearranging, gives:

$$\dot{w}_s(q) = -\frac{u'\left(\bar{W} + w_s(q)\right)}{u''\left(\bar{W} + w_s(q)\right)}\frac{\frac{d}{dq}\left[\frac{\frac{\partial \mathbf{f}}{\partial e}(q,s|e)}{\mathbf{f}(q,s|e)}\right]}{\frac{\frac{\partial \mathbf{f}}{\partial e}(q,s|e)}{\mathbf{f}(q,s|e)} + \frac{\theta}{\xi}}.$$

■

To conclude the proof of the proposition, it suffices to show that there is no interval of outputs with $w_s(q) > 0$ and $\dot{w}_s(q) < 1$. By the previous lemma and the same steps at the end of the proof of Proposition 3, this is true if

$$\dot{w}_s(q) = -\frac{u'\left(\bar{W} + q - q_s^*\right)}{u''\left(\bar{W} + q - q_s^*\right)}\frac{\frac{d}{dq}\left[\frac{\frac{\partial \mathbf{f}}{\partial e}(q,s|e)}{\mathbf{f}(q,s|e)}\right]}{\frac{\frac{\partial \mathbf{f}}{\partial e}(q,s|e)}{\mathbf{f}(q,s|e)} + \frac{\theta}{\xi}} \geq 1. \tag{45}$$

This condition involves Lagrange multipliers, which we address in the last part of the proof below.

Consider the first step in Grossman and Hart (1983). To start, solve the optimization problem without the participation constraint. In this relaxed problem, the optimal contract is debt if the condition in equation (45) is satisfied with $\theta = 0$. There are two cases. First, if the condition in equation (45) is not satisfied with $\theta = 0$, then the optimal contract is not debt. Indeed, a necessary condition for the optimal contract to be debt is that it be debt with $\theta = 0$. Second, if the condition in equation (45) with $\theta = 0$ is satisfied, then verify if the participation constraint is satisfied with the debt contract thus derived. If it is satisfied, then the participation constraint is nonbinding, i.e. $\theta = 0$, and the optimal contract is debt. If it is not satisfied, then $\theta > 0$ and the optimal contract is not debt. Indeed, for a given $s$, with $q_s^* > q_s^e$ (where $q_s^e$ is defined in Lemma 2), by MLRP:

$$\frac{\partial}{\partial q_s^*}\left\{\int_{\underline{q}}^{q_s^*} u(\bar{W})\frac{\partial \mathbf{f}}{\partial e}(q,s|e^*)dq + \int_{q_s^*}^{\overline{q}} u(\bar{W}+q-q_s^*)\frac{\partial \mathbf{f}}{\partial e}(q,s|e^*)dq\right\}$$
$$= -\int_{q_s^*}^{\overline{q}} u'(\bar{W}+q-q_s^*)\frac{\partial \mathbf{f}}{\partial e}(q,s|e^*)dq < 0,$$

whereas the inequality is reversed for $q_s^* < q_s^e$, again by MLRP. Thus, there are at most two debt repayments that provide the same level of effort incentives in a given state $s$. Denote these debt repayment as $q_{s1}^*$ and $q_{s1}^*$. The proof that a debt contract with debt repayment $q_{s1}^*$ in state $s$ is dominated is as in the proof of Proposition 3. ■

**Proof of Proposition 5**

The conditions in Proposition 4 hold so that the optimal contract is debt. The principal's program is:

$$\min_{\{q_s^*,e\}} \sum_s \int_{q_s^*}^{\overline{q}} (q-q_s^*)\,\mathbf{f}(q,s|e)dq \tag{46}$$

subject to

$$\sum_s \int_{q_s^*}^{\overline{q}} u\left(\bar{W}+q-q_s^*\right)\mathbf{f}(q,s|e)dq - C(e) \geq \overline{U} \tag{47}$$

$$\sum_s \int_{q_s^*}^{\overline{q}} u\left(\bar{W}+q-q_s^*\right)\frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq = C'(e). \tag{48}$$

The Lagrangian is given by:

$$\begin{aligned}\mathcal{L} \;=\; & -\sum_s \int_{q_s^*}^{\overline{q}} (q-q_s^*)\,\mathbf{f}(q,s|e)dq + \theta\sum_s \int_{q_s^*}^{\overline{q}} u\left(\bar{W}+q-q_s^*\right)\mathbf{f}(q,s|e)dq \\ & +\lambda\sum_s \int_{q_s^*}^{\overline{q}} u\left(\bar{W}+q-q_s^*\right)\frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq\end{aligned}$$

The first-order condition with respect to $q_s^*$ is:

$$\int_{q_s^*}^{\overline{q}} \mathbf{f}(q,s|e)dq \leq \theta\int_{q_s^*}^{\overline{q}} u'\left(\bar{W}+q-q_s^*\right)\mathbf{f}(q,s|e)dq + \lambda\int_{q_s^*}^{\overline{q}} u'\left(\bar{W}+q-q_s^*\right)\frac{\partial \mathbf{f}}{\partial e}(q,s|e)dq$$

with $=$ if $q_s^* > \underline{q}$. With some algebraic manipulations, we can rewrite this condition as:

$$\frac{\theta}{\lambda} \frac{\int_{q_s^*}^{\overline{q}} u'\left(\bar{W} + q - q_s^*\right) \mathbf{f}(q, s|e) dq}{\int_{q_s^*}^{\overline{q}} \mathbf{f}(q, s|e) dq} + \frac{\int_{q_s^*}^{\overline{q}} u'\left(\bar{W} + q - q_s^*\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e) dq}{\int_{q_s^*}^{\overline{q}} \mathbf{f}(q, s|e) dq} \geq \frac{1}{\lambda}$$

where $\theta = 0$ if the participation constraint does not bind. In particular, for all signals with interior thresholds, we have

$$\frac{\theta}{\lambda} \frac{\int_{q_s^*}^{\overline{q}} u'\left(\bar{W} + q - q_s^*\right) \mathbf{f}(q, s|e) dq}{\int_{q_s^*}^{\overline{q}} \mathbf{f}(q, s|e) dq} + \frac{\int_{q_s^*}^{\overline{q}} u'\left(\bar{W} + q - q_s^*\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e) dq}{\int_{q_s^*}^{\overline{q}} \mathbf{f}(q, s|e) dq} = \frac{1}{\lambda}$$

Let

$$\psi(x) \equiv \frac{\theta}{\lambda} \mathbb{E}\left[u'\left(\bar{W} + q - x\right) \mid q > x, s\right] + \overline{LR}_s^u(x)$$

where $\theta = 0$ if the participation constraint does not bind, and

$$\overline{LR}_s^u(x) := \frac{\int_x^{\overline{q}} u'\left(\bar{W} + q - x\right) \frac{\partial \mathbf{f}}{\partial e}(q, s|e) dq}{\int_x^{\overline{q}} \mathbf{f}(q, s|e) dq}$$

is the marginal utility-adjusted likelihood ratio of the event $(\tilde{q} \geq x, \tilde{s} = s)$. Assuming that the solution is interior for all signals, the optimal contract is not a function of the signal if

$$\psi(q_s^*) = \psi(q^*)$$

for all signal realizations $s$, where $q^*$ and $e^*$ solve (46)-(48) without a signal (i.e. when the signal $\tilde{s}$ follows a degenerate distribution). ■