The Value of Health and Longevity with Stochastic Health Risk and Partial Annuitization

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Abstract

We determine the value of health and longevity improvements for retirees in a calibrated life-cycle model with stochastic health risk and access to a limited menu of retirement products. In such a setting, consumers optimally annuitize only a fraction of their wealth and adjust their consumption choices to their health trajectory. These two aspects substantially affect their valuation of improvements to health and longevity. First, the ability to adjust consumption mitigates the adverse consequences of a health shock to a certain extent. Moreover, lower annuity holdings decrease the value of mortality reductions at very old ages, as consumers tend to spend down much of their non-annuitized wealth before then—which is especially the case for consumers with worse health. In combination, these aspects substantially reduce the value consumers place on aggregate health and longevity improvements, with our estimates undercutting estimates from the literature by 40% or more. We also document the complementarity of annuity income, e.g. from social security, and investments in health and longevity.

1 Introduction

This paper develops a life-cycle framework to study optimal consumption choices among retirees in the presence of stochastic health risk and a limited menu of retirement products. We calibrate the model based on an established economic-demographic microsimulation framework for elderly Americans, the Future Elderly Model.[1] We then use the model to estimate the aggregate value of health and longevity improvements. Our estimates for the value of improvements associated with a reduction in cancer for the retired population are at least 40% lower than estimates based on the prevailing method from the literature, which assumes deterministic health and complete annuitization. We show that this reduction is a result of three related effects: (i) Due to the incompleteness of annuity markets, consumers only hold a fraction of their wealth in annuities, leading to decreasing optimal consumption patterns—which in turn reduce the value of health and longevity improvements in old ages. (ii) In a setting with stochastic health, individuals can adjust their consumption decisions following negative health shocks, tempering the impact of health shocks on lifetime utility and therefore the willingness to pay to avoid them. And, (iii), individuals with poorer health conditions that benefit most from health improvements own fewer assets upon retirement, so that their willingness-to-pay is lower. We also investigate the relationship of annuity income, e.g. from social security, and investments in health and longevity, documenting that they are strong complements.

The American health-care sector is massive, with health care expenditures accounting for roughly 20% of U.S. gross domestic product and more than 25% of government spending relating to health care (Nunn et al., 2020). Public and private organizations invest hundreds of billions US dollars in health research and medical technologies (Research-America, 2022; Webster, 2023). Given these magnitudes and also for an appropriate allocation of public resources, it is important to estimate the value of potential advancements in health and longevity. In their seminal contribution, Murphy and Topel (2006) take a "first step" toward evaluating the social returns of medical Research and Development (R&D) by estimating the value of health and life improvements in an economic framework, which suggests that returns to historical and potential future expenditures may be quite large. For instance, they estimate that a 1% reduction in cancer mortality would be worth about $500 billion, substantiating large public investments in a possible "war on cancer."

We revisit the Murphy and Topel (2006) analysis by extending their framework in two

[1] The FEM uses the Health and Retirement Study (HRS) as a host dataset and draws on a variety of other data sources to complement the HRS data. It has been used in more than 55 peer-reviewed manuscripts that explore a range of policy questions, see https://healthpolicy.usc.edu/future-elderly-model.

important dimensions. First, we introduce stochastic health to our model framework, i.e., people tend to fall sick before dying of a chronic condition such as cancer or heart disease, and adjust their consumption choices accordingly. As a consequence, there is substantial variation in consumers' health and wealth trajectories, in contrast to the single deterministic consumption and wealth path in Murphy and Topel (2006). Second, we constrain the menu of retirement products (general life annuities). This is more relevant in our stochastic setting, because a complete market as in Murphy and Topel (2006) would require an unrealistically broad set of retirement products, given the rich set of possible trajectories and difficulties in defining and verifying health states. As a consequence, individuals will only annuitize a fraction of their wealth, especially if annuities contain a loading and if there are correlated medical costs (Davidoff et al., 2005; Reichling and Smetters, 2015).

These innovations complicate the solution of consumer's optimization problem and the derivation of the value of a marginal improvement in health or longevity. We integrate ideas from Leung (1994) and Parpas and Webster (2013) to derive a closed-form expression for a consumer's optimal consumption path, under certain regularity assumptions that imply regularity of the possible consumption paths. The solution entails "kink points" in the age-consumption profiles, after which the consumer solely relies on annuity income to finance future consumption. We present a numerical algorithm for solving for these kink points, and, thus, for optimal consumption paths. Unlike conventional approaches to solving general stochastic optimal consumption problems, our algorithm allows for an iterative solution health-state by health-state. For the derivation of the consumer's value of a marginal health or mortality improvement, we extend the ideas from Bauer et al. (2023) to allow for life-contingent annuity income.

For the calibration of the model, we rely on the FEM, a widely published microsimulation model that perfectly fits our setting. The FEM generates transition probabilities for the stochastic evolution of health states, which are defined via combinations of chronic conditions and impairments to activities of daily living (ADLs), and it provides corresponding mortality, medical spending, and quality of life profiles. The FEM has been used by many researchers to study policy questions that relate to elderly health and medical spending (e.g., Goldman et al., 2010, 2013; Reif et al., 2021, among many others). We rely on a version of the model with five health states, where state 1 corresponds to no chronic diseases or impairments, and state 5 corresponds to multiple chronic conditions and multiple impairments. Individuals in different health states face very different longevity prospects. For instance, the life expectancy for a 65-year-old (75-year-old) in health state 1 is almost 20 years (14 years), while a 65-year-old (75-year-old) in health state 5 is only expected to live for a little over 9 years (6 years). We rely on relevant literature to calibrate our preference and financial market parameters.

Optimal consumption paths based on the model are multi-faceted, due to the variation in underlying health and wealth trajectories. Under out-of-pocket medical spending from the FEM and realistic loadings on private-market annuities, consumers only purchase a limited amount of (flat) immediate annuities at retirement, breaking the full annuitization result from Yaari (1965) and in line with Reichling and Smetters (2015).[2] We also show that the willingness-to-pay for marginal improvements to health and longevity, and particularly the value of statistical life (VSL), significantly increases in annuity income for relatively low annuitization levels, illustrating the complimentarity of public health and pension programs.

For analyzing the value of improvements to health and longevity, in addition to our baseline cohort, via the FEM we obtain population transition and mortality rates after a hypothetical intervention against cancer. In this hypothetical population, life expectancy at age 65 (75) increases by about 1.6% (3%) for individuals in the healthy state 1 and by 5.1% (11.8%) for impaired individuals in the least healthy state 5. We derive the value of this health intervention by integrating the marginal values over the 2010 U.S. population. Here, we determine the value based on two approaches. We determine the aggregate willingness-to-pay, which is affected by the marginal value placed on the intervention as well as the marginal utility of consumption in a given age/health-state/wealth combination. And we determine the aggregate marginal value, which we normalize by a common marginal utility of a representative consumer.

We first derive the value for this intervention using the Murphy and Topel (2006) analysis that assumes deterministic health and full annuitization, which amounts to roughly USD 1.2 trillion. This magnitude is in line with the results in Murphy and Topel (2006, Table 8), although the figures are not perfectly comparable, because of the difference in intervention and the fact that they consider the entire population, whereas we limit our analyses to the elderly. We then determine the value of the intervention using our model setting, which accounts for: (i) a limited menu in retirement products (fixed annuity) resulting in optimal partial annuitization; (ii) stochastic health; and (iii) differences in consumer wealth levels by health-state. We obtain an aggregate willingness-to-pay of about 0.5 trillion—which considers differences in marginal utilities of consumption—and an aggregate marginal value of about 0.7 trillion—which uses the same marginal utility of

[2]There are other aspects that limit annuity take-up that are not in our model, such as bequest motives (Lockwood, 2012). Altogether, these factors can explain existing low annuitization levels, as other authors have pointed out (Inkmann et al., 2011, e.g.).

consumption as in the deterministic setting. Hence, the estimates from the stochastic model are substantially lower—55% and 40%, respectively—though they are still substantive. We show that all factors (i)-(iii) contribute to this decrease, and at least for the aggregate willingness-to-pay the effect stemming from all these factors is roughly on-par. We conclude the paper by discussing potential consequences of our findings.

The paper is structured as follows. Section 2 formalizes the model, derives the solution for consumers' optimal consumption paths, and determines the expressions for the value of health and longevity improvements. Section 3 presents an algorithm that implements the theoretical solution for optimal consumption paths. Section 4 introduces the data and calibration used in our quantitative analysis. Section 5 presents the results from our quantitative analysis. We first illustrate optimal consumption and annuity choice. We then discuss the relationship of annuity income and consumer valuations of improvements to health and longevity. And, finally, we show that incomplete markets and stochastic health risks substantially reduce the aggregate value of health and longevity improvements. Section 6 concludes.

2 Model

We introduce a setting where individuals face incomplete insurance markets and stochastic health risks such as illness and death. We aim to quantify how people value medical advances that improve their well-being and extend their lifetimes. Section 2.1 outlines our life-cycle model and derives a closed-form solution for the optimal consumption path. Section 2.2 derives a consumer's marginal value of improvements in health and longevity and, particularly, VSL as a special case. Section 2.3 explores how these marginal values can be used to assess the aggregate value of interventions that improve health and longevity.

2.1 Stochastic Health and Incomplete Retirement Markets

We follow Bauer et al. (2023) and model well-being and stochastic health risk via a continuous-time Markov chain with a finite state space. Within this framework, each distinct state corresponds to specific health conditions. Individuals are at risk of experiencing health shocks, which can induce changes in their health conditions modeled as transitions to different health states. Let Y_t denote the health state of an individual at time t with values in the state space $S = \{1, 2, \cdots, n, n+1\}$. We assume that the severity of health conditions progresses as the states increase. Specifically, state 1 represents perfect health, state n denotes the most adverse health conditions, and state $n+1$ corresponds to the absorbing state of death. For analytical convenience, we further assume that individuals can transition only to higher-order states.[3]

Let $\lambda_{ij}(t)$ be the instantaneous transition rate, or transition intensity, from state i to state j at time t, and let $\mu_i(t)$ be the instantaneous mortality rate in state i at time t, i.e., $\mu_i(t) = \lambda_{i,n+1}(t)$. Then the probability of an individual in health state i surviving for t years without transitioning to any other states is:

$$\tilde{S}(t,i) = \exp\left\{-\int_0^t \mu_i(s)ds\right\}\exp\left\{-\int_0^t \sum_{j>i,j\neq n+1} \lambda_{ij}(s)ds\right\}.$$

In this setting, a complete retirement market would require a very broad set of securities, since payments at a given date may depend on the entire health history. Offering such securities would require a firm and uniform definition of health states, as well as a device for state verification, which is unrealistic. Therefore, in line with real-world retirement markets, we substantially limit the available retirement securities. More specifically, borrowing an approach from Reichling and Smetters (2015), in addition to a riskless savings vehicle, we assume the consumer has an option at time zero to purchase a flat lifetime annuity that pays out $\bar{a}_{Y_0} \geq 0$ in all health states and has a price markup of $\xi \geq 0$. The consumer cannot finance the purchase of the annuity using future earnings or sell their annuity after the purchase at time zero.

The net present value of the lifetime annuity generally will change following a transition to a new health state, because a fixed payout is worth more to a person with higher life expectancy. We define the value of a one-dollar annuity at time t in state i as:

$$a(t,i) = \mathbb{E}\left[\int_t^T e^{-r(s-t)}\exp\left\{-\int_t^s \mu(u)du\right\}ds \middle| Y_t = i\right],$$

where r is the (market) riskless interest rate.

Let $\tau = \inf\{t \geq 0 : Y_t = n+1\}$ be a random time that represents time of death for an individual. Let $W(t)$ and $C(t)$ be the wealth and consumption at time t for an individual, and let $q_i(t)$, $m_i(t)$, and $r_i(t)$ be the quality of life, income, and (individual) rate of interest at time t,[4] which we assume are exogenous and solely depend on state $Y_t = i$, respectively. Denote the utility from consumption expenditure $C(t)$ with quality of life

[3]This, generally, does not mean that people cannot recover from illness. In principle, our model can accommodate transitions from sick to healthy states by introducing states that correspond to recovery.

[4]We differentiate between the market and the individual interest rate to potentially accommodate health-state specific costs (see Bauer et al., 2023).

$q_i(t)$ by $u(C(t), q_i(t))$. Let ρ be the rate of time preference. Moreover, suppose that utility is time separable and exponentially discounted. We assume throughout the paper that individuals are risk averse and the utility function $u(C(t), q_i(t))$ is twice-differentiable. That is, $u(c,q) \in C^{2\times 2}$, $u_c > 0$, and $u_{cc} < 0$. We further assume that $u_c(\cdot, q)$ diverges to positive infinity as consumption approaches zero for all q, so that optimal consumption is always positive.

Following Yaari (1965) and Leung (1994), we abstract from bequest motives and normalize the utility of death to 0 (see also Rosen (1988) for the importance of this assumption). The consumer's optimization problem is:

$$V(0, W_0, Y_0) = \max_{C(t), \bar{a}_{Y_0}} \mathbb{E}_\tau \Big[\mathbb{E} \Big[\int_0^\tau e^{-\rho t} u(C(t), q_{Y_t}(t))\, dt \Big| Y_0, W_0 \Big] \Big] \tag{1}$$

subject to:

$$\begin{aligned} W(0) &= W_0 - (1+\xi)\bar{a}_{Y_0} a(0, Y_0), \\ W(t) &\geq 0 \;\forall\; t, \\ \frac{\partial W(t)}{\partial t} &= r_{Y_t}(t)\, W(t) - C(t) + m_{Y_t}(t) + \bar{a}_{Y_0}, \end{aligned}$$

where W_0 and Y_0 are initial wealth and state, respectively.

Let T be the maximum possible lifetime. Denote $S_t(s)$ as the probability of an individual surviving s years at time t. By integrating over the possible time of death, it follows that the optimal value function at time t given health state Y_t and initial wealth $W(t)$ is:

$$V(t, W(t), Y_t, \bar{a}) = \max_{C(t+s), s\geq 0} \mathbb{E} \Big[\int_0^{T-t} S_t(s) e^{-\rho s} u(C(t+s), q_{Y_s}(t+s))\, ds \Big| Y_t, W(t) \Big]. \tag{2}$$

Parpas and Webster (2013) show that a stochastic optimization problem of the form (2) in a Markov chain setting with a finite number of states can be reformulated as a deterministic problem by taking the value function in higher states, $V(t, w, j, \bar{a})$, $j > i$, as exogenous. The intuition is that one can solve the problem state by state, starting with the highest state. In particular, applying their result, the optimization problem (1) can be alternatively written as:

$$V(0, W_i(0), i, \bar{a}) = \max_{C_i(t)} \int_0^T \tilde{S}_0(t, i) e^{-\rho t} \Big(u(C_i(t), q_i(t)) + \sum_{j>i} \lambda_{ij}(t) V(t, W_i(t), \bar{a}, j) \Big) dt \tag{3}$$

subject to:

$$
\begin{aligned}
W_i(0) &= W_0 - (1+\xi)\overline{a}\,a(0,i),\\
W_i(t) &\geq 0,\\
\frac{\partial W_i(t)}{\partial t} &= r_i(t)W_i(t) - C_i(t) + m_i(t) + \overline{a},
\end{aligned}
$$

where $W_i(t)$ and $C_i(t)$ denote wealth and consumption in the (deterministic) path where the consumer stays in state i throughout her lifetime. The annuity level $\overline{a}$ is a parameter that may (or may not) be optimized once the optimal consumption problem is solved.

Given that V and its partial derivatives are continuous, we can show that $V(t, W_i(t), \overline{a}, i)$ corresponding to Equation (3) satisfies the following Hamiltonian-Jacobi-Bellman (HJB) system of equations (see Bauer et al., 2023, Lemma 1):

$$
\begin{aligned}
\rho V(t, W_i(t), i) =& \max_{C_i(t)} \Bigg\{ u(C_i(t), q_i(t)) + \frac{\partial V(t, W_i(t), \overline{a}, i)}{\partial W_i(t)}[rW_i(t) - C_i(t) + m_i(t) + \overline{a}] \\
&+ \frac{\partial V(t, W_i(t), \overline{a}, i)}{\partial t} + \sum_{j>i} \lambda_{ij}(t)[V(t, W_i(t), \overline{a}, j) - V(t, W_i(t), \overline{a}, i)] \Bigg\},\ i = 1, \cdots, n,
\end{aligned}
\tag{4}
$$

where $V(t, W_i(t), \overline{a}, i)$ denotes the value function of the deterministic optimization problem (Bertsekas, 2005, Proposition 3.2.1).[5]

The key advantage of the reformulation as a deterministic problem is that we can apply the standard Pontryagin's maximum principle for the solution of the problem. In particular, the present value Hamiltonian corresponding to the optimization problem (3) takes the following form:

$$
\begin{aligned}
H(W_i(t), C_i(t), p_i(t), \eta_i(t)) =& e^{-\rho t}\,\tilde{S}(t,i)\left(u(C_i(t), q_i(t)) + \sum_{j>i} \lambda_{ij}(t)\, V(t, W_i(t), \overline{a}, j) \right) \\
&+ p_i(t)[r_i(t)\, W_i(t) - C_i(t) + m_i(t) + \bar{a}] + \eta_i(t) W_i(t),\ i = 1, \cdots, n
\end{aligned}
\tag{5}
$$

where $\eta_i(t)$ is a multiplier for the non-borrowing constraint $W_i(t) \geq 0$ and $p_i(t)$ is the costate variable.

Lemma 1. *The first order conditions corresponding to Equation (5) are (Hartl, 1984):*

[5]The intuition behind the result in Parpas and Webster (2013) is that this equation is also results as the HJB of the stochastic optimal control problem (2).

(i) $\frac{\partial H}{\partial C_i(t))} = e^{-\rho t}\tilde{S}(t,i)\frac{\partial u(C_i(t),q_i(t))}{\partial C_i(t)} - p_i(t) = 0,$

(ii) $\frac{\partial H}{\partial W_i(t)} = -p_i'(t) = r_i(t)p_i(t) + \eta_i(t) + e^{-\rho t}\tilde{S}(t,i)\sum_{j>i}\lambda_{ij}(t)\frac{\partial}{\partial W_i(t)}V(t, W_i(t), \overline{a}, j),$

(iii) $\eta_i(t) \geq 0, \quad \eta_i(t)W_i(t) = 0,$

(iv) $p_i(T) = p + q,$

(v) $p \geq 0, \quad pW_i(T) = 0, \quad q$ *unrestricted in sign.*

Proof. See Appendix A. ■

Incomplete annuity markets and life-cycle income complicate the analysis by introducing the possibility of multiple sets of non-interior solutions within and across states. (See the right panel in Figure 6 for an example.) For convenience of exposition, we focus on the case where future income is nondecreasing over time and the growth rate of consumption is weakly declining, as illustrated by the left panel in Figure 6. Prior empirical work suggests this case is a reasonable description for the typical consumer nearing retirement.[6] We do not take a stance on the reason underlying the (weakly) negative growth rate in consumption, but we note that it arises in our model under a wide variety of typical parameterizations.

Since we assume quality of life is exogenous, we write the utility function $u(C_i(t), q_i(t))$ as $u(C_i(t))$. Our assumptions on the utility function imply that the marginal utility of consumption is continuous and strictly monotonic. It follows that the inverse of marginal utility of consumption exists, which we denote as $u_c^{-1}(\cdot)$. We obtain:

Proposition 2. *Suppose that annuity markets are incomplete as described above, consumption growth is weakly declining ($\frac{\dot{c}_i}{c_i} \leq 0 \ \forall i$), and that income, $m_i(t)$, is nondecreasing in t. For any risk-averse individual, the optimal consumption path in state i then is:*

$$C_i^*(t) = u_c^{-1}\Bigg(e^{\rho t}\frac{1}{\tilde{S}(t,i)}\Bigg[exp\left\{-\rho T_i^* + \int_t^{T_i^*} r_i(s)ds\right\}\tilde{S}(T_i^*, i)u_c(m_i(T_i^*) + \bar{a}) \\ + e^{-\int_0^t r_i(s)ds}\int_t^{T_i^*} exp\left\{-\rho s + \int_0^s r_i(u)du\right\}\tilde{S}(s,i)\sum_{j>i}\lambda_{ij}(s)\frac{\partial}{\partial W_i(s)}V\left(s, W_i(s), \overline{a}, j\right)ds\Bigg]\Bigg),\ t \in (0, T_i^*) \tag{6}$$

[6] A typical consumption profile is constrained by low income at early ages, increasing during middle ages when income is high, and then declines during retirement until consumption equals the consumer's pension. This inverted U-shape for the age profile of consumption has been widely documented across different countries and goods (Carroll and Summers, 1991; Banks et al., 1998; Fernandez-Villaverde and Krueger, 2007).

where $T_i^* \in (0, T)$ is the greatest lower bound that satisfies $W_i(t) = 0 \; \forall \; t \in [T^*, T]$. For $t \geq T_i^*$, the optimal consumption expenditure equals income, i.e., $C_i^*(t) = m_i(t) + \bar{a} \; \forall \; t \geq T_i^*$.

Proof. See Appendix A. ∎

2.2 Value of Health and Longevity

To estimate the value of health and longevity, we follow Rosen (1988). Let ε be a marginal reduction, e.g. due to a medical advance, that affects both transition and mortality rates. Let $\delta_{ij}(t)$ be a perturbation on the transition rates $\lambda_{ij}(t)$, $0 \leq t \leq T$, which characterizes the medical advance. We consider a consumer whose longevity prospects are subject to a ε-perturbation in the direction of $\delta_{ij}(t)$. The sojourn probability in state i is then (Bauer et al., 2023):

$$\tilde{S}^{\varepsilon}(i,t) = \exp\left[-\int_0^t \sum_{j>i} \Big(\lambda_{ij}(s) - \varepsilon\delta_{ij}(s)\Big) ds\right], \text{ where } \varepsilon > 0. \tag{7}$$

We extend the analysis in Bauer et al. (2023) to derive the marginal value of the ε-perturbation:

Proposition 3. *The marginal utility of life extension in state i is given by:*

$$\begin{aligned}\frac{\partial V}{\partial \varepsilon}\bigg|_{\varepsilon=0} = &\int_{t_0}^{T} e^{-\rho t}\tilde{S}(i,t)\left[\left(\int_{t_0}^{t}\sum_{j>i}\delta_{ij}(s)ds\right)\left(u(c_i(t),q_i(t)) + \sum_{j>i}\lambda_{ij}(t)V(t,W_i(t),\bar{a},j)\right) - \sum_{j>i}\delta_{ij}(t)V(t,W_i(t),\bar{a},j)\right]dt \\ &- \frac{\partial V}{\partial W_i(0)}(1+\xi)\bar{a}\int_{t_0}^{T} e^{-rt}\tilde{S}(i,t)\left[\left(\int_{t_0}^{t}\sum_{j>i}\delta_{ij}(s)ds\right)\left(1 + \sum_{j>i}\lambda_{ij}(t)a(t,j)\right) - \sum_{j>i}\delta_{ij}(t)a(t,j)\right]dt,\end{aligned} \tag{8}$$

Proof. See Appendix A. ∎

Consider a special case where only the mortality rate in state i, $\lambda_{i,n+1}(t)$ is perturbed and it is perturbed at time $t = 0$. The perturbations $\delta_{ij}(t) = 0 \; \forall j < n+1$ and $\delta_{i,n+1}(t)$ is equal to the Dirac delta function (Rosen, 1988). Then, dividing equation (8) by the marginal utility of consumption yields the so-called *value of statistical life* (VSL) (Murphy and Topel, 2006):

Corollary 4. *VSL in state i at time 0 is equal to:*

$$VSL(i) = \frac{V(0, W_i(0), \bar{a}, i)}{u_c(c_i(0), q_i(0))} - (1+\xi)\bar{a}\, a(0,i) \tag{9}$$

The second term in equation (9)—sometimes referred to as "net savings"—represents the marginal cost to the annuity pool from saving a life and arises because the price of an annuity is linked to survival (Murphy and Topel, 2006).[7] VSL under incomplete markets captures elements of both the uninsured and fully insured cases. When annuities are absent ($\overline{a} = 0$), equation (9) simplifies to the uninsured case where VSL equals the first term in equation (9), which was studied in Bauer et al. (2023). Similarly, full annuitization is optimal when $\xi = 0$, $r = \rho$, and quality of life and future income are constant, in which case equation (9) simplifies to the complete markets case given by:[8]

$$VSL = \frac{V(0, \overline{W}_i(0), i)}{u_c(C_i(0), q_i(0))} - W_0,$$

which is the expression in Murphy and Topel (2006).

2.3 Aggregate Value of a Health Intervention

To study the aggregate value from a health intervention resulting from an investment in medical technologies and health research, we look at two measures: (i) aggregate social surplus, or the society's total willingness to pay for health and longevity improvements; and, (ii), aggregate utility with equal weights for all individuals. Aggregate surplus (i) is frequently used by economists when the goal is to decide how to allocate resources across different people. For example, how much federal fund should be invested in health research that focuses on less complex conditions among the young where people are more likely to cure versus how much should be invested in medical trials that treat prevalent conditions among the elderly. However, as explained in Bauer et al. (2023, Section 2.4), referencing foundational studies in welfare economics (Harsanyi, 1955; Fleurbaey, 2010), there are disadvantages to either measure, with tradeoffs with regards to efficiency vs. equity when using one over the other. Here, we consider both measures.

For aggregate surplus (i), we aggregate individual willingness to pay. By equation (8), marginal value of life extension of an intervention characterized by the perturbation δ_{ij}

[7]The net savings term in the VSL presented above arises only because those expressions are evaluated at time $t = 0$, when the annuity is purchased. The term disappears when evaluating VSL at $t > 0$—or, equivalently, in a setting with life-cycle income but no opportunity to purchase an annuity—because survival changes occurring after the purchase of the annuity do not affect its price. Philipson and Becker (1998) argue that this "moral hazard" effect induces excessive longevity because individuals do not internalize the costs to annuity programs of their increased lifespan.

[8]Remaining wealth at time 0, $W_i(0)$, is zero under full annuitization, which implies $W_0 = (1+\xi)\,\overline{a}\,a(0,i)$.

is given by:

$$
\begin{aligned}
&\left.\frac{\partial V/\partial \varepsilon}{\partial V/\partial W}\right|_{\varepsilon=0} \\
&= \int_0^T \tilde{S}(i,t)\Bigg\{\left(\int_0^t \sum_{j>i}\delta_{ij}(s)ds\right)\Bigg[\left(\frac{e^{-\rho t}u(c_i(t),q_i(t))+\sum_{j>i}\lambda_{ij}(t)V(t,W_i(t),j,\bar{a})}{u_c(c_i(0),q_i(0))}\right) \\
&\qquad -(1+\xi)\bar{a}e^{-rt}\left(1+\sum_{j>i}\lambda_{ij}(t)a(t,j)\right)\Bigg] \\
&\qquad -\sum_{j>i}\delta_{ij}(t)\left(\frac{V(t,W_i(t),j,\bar{a})}{u_c(c_i(0),q_i(0))}-(1+\xi)\bar{a}\,e^{-rt}a(t,j)\right)\Bigg\}dt.
\end{aligned}
\tag{10}
$$

By aggregating across the entire population by going over all individuals of age a in health state i, $f(a,i)$, it follows that the aggregate willingness to pay for life extension from the intervention is:

$$
Aggregate\ VLE = \sum_a \sum_{i=1}^n MVLE(a,i,\delta)f(a,i). \tag{11}
$$

If the number of health states is one (i.e., health is deterministic), then a medical advance only perturbates mortality probabilities (i.e., $\delta_{ij}(t)=0\ \forall\ j\neq n+1$) and equation (10) reduces to:

$$
\left.\frac{\partial V/\partial \varepsilon}{\partial V/\partial W}\right|_{\varepsilon=0} = \int_0^T S(t)\left(\int_0^t \delta(s)ds\right)\left[\frac{e^{-\rho t}u(c(t),q(t))}{u_c(c(0),q(0))}-(1+\xi)\bar{a}e^{-rt}\right]dt. \tag{12}
$$

While measure (i) focuses on efficiency and allocation of resources, it is often criticized for equity concerns. That is, irrespective of wealth status, each dollar of surplus is valued equally, see the discussion above. This is overcome by a utilitarian perspective. In our context, we can aggregate marginal improvements to utility across different individuals via:

$$
Aggregate\ VLE = \frac{\sum_a \sum_{i=1}^n\ V_\varepsilon(a,i,\delta)f(a,i)}{w}, \tag{13}
$$

where w is (a uniform) normalization across all ages a and states i.

We note that the distinction between the two measures is not material in a deterministic, complete setting as in Murphy and Topel (2006), since marginal utilities are equated. However, it matters for a stochastic approach. In order to not take a stance, we show results under both measures—one emphasizing efficiency and the other prioritizing equity. There are, of course, many alternatives to our extreme case of equal weights that intermediate the trade-off between efficiency and equity. We do not aim to address the longstanding debates in welfare economics.

3 Numerical Solution Algorithm

Proposition 2 give an analytical solution for the optimal consumption path:

$$C_i^*(t) = C(t, W_i(t), i, \overline{a}) \ \forall \ i \text{ and } t.$$

By varying initial wealth and annuity level, in principle this solution allows us to determine optimal consumption for any state-variable configuration in the stochastic problem (1). However, the solution in Proposition 2 depends on the value function in higher states and the "kink point" T_i^*, $i = 1, 2, ...n$, which in turn depends on initial wealth and annuity level.

In what follows, we present a numerical solution algorithm that first solves the optimal consumption paths in state n—along with the associated "kink point"—and then those in states $n-1, n-2, \cdots, 1$ in the backward order.

State n problem

Since the utility of death is normalized to 0, from Proposition 2, we have:

$$\begin{aligned} C_n^*(t) &= u_c^{-1}\left(e^{\rho t}\frac{1}{\tilde{S}(t,n)}\exp\left\{-\rho T_n^* + \int_t^{T_n^*} r_n(s)ds\right\}\tilde{S}(T_n^*, n)\, u_c(m_n(T_n^*) + \bar{a})\right) \\ &= u_c^{-1}\left(e^{\rho(t-T_n^*)}\exp\left\{-\int_t^{T_n^*}(\mu_n(s) - r_n(s))ds\right\}u_c(m_n(T_n^*) + \bar{a})\right). \end{aligned} \tag{14}$$

Assume that an individual starts in health state n with initial wealth $W_n(0) = W_0$. Again from Proposition 2, T_n^* is the greatest lower bound such that $W_n(t) = 0 \ \forall \ t \in [T_n^*, T]$, indicating that consumption from time 0 to T_n^* should exhaust wealth and income from time 0 to T_n^*. Mathematically:

$$W_0 + \int_0^{T_n^*} e^{-\int_0^t r_n(s)ds} m_n(t) + \bar{a}\, dt = \int_0^{T_n^*} e^{-\int_0^t r_n(s)ds} C_n^*(t)\, dt,$$

which provides a wealth constraint. Plugging $C_n^*(t)$ into the wealth constraint, we can solve for $T_n^*(W_0)$ and hence $C_n^*(t)$ for any W_0 analytically given tractable assumptions on the utility function, rate of time preference, force of mortality, quality of life, interest rate, income and annuity in state n.

Problem for state $n-1$, $n-2$, ...

The challenge for solving the optimal consumption paths in health states other than state n resides in finding the marginal utility of wealth following a health shock, $\frac{\partial V(t,W_i(t),j)}{\partial W_i(t)}$. By the first-order condition of the HJB equation (4), $\frac{\partial V(t,W_i(t),i)}{\partial W_i(t)} = u_c(C_i^*(t))$, $\forall\ i \in \{1,\cdots,n\}$. Then for any $j > i$, we can determine $\frac{\partial V(t,W_i(t),j)}{\partial W_i(t)}$ by $u_c(C_j^*(t)) = \frac{\partial V(t,W_j(t),j)}{\partial W_j(t)}$, where $C_j^*(t)$ is determined by the $W_j(0)$ such that $W_j(t) = W_i(t)$.

We start with state $n-1$. Because $W_{n-1}(t)$ is unknown for any $t < T_{n-1}^*$, we use numerical approximation to obtain a quasi-analytical solution for $C_{n-1}^*(t)$. With a first difference approximation, the wealth dynamic becomes:

$$\frac{W_{n-1}(T_{n-1}^* - (k-1)\delta_t) - W_{n-1}(T_{n-1}^* - k\delta_t)}{\delta_t} = r_{n-1}(T_{n-1}^*)W_{n-1}(T_{n-1}^* - k\delta_t) - C_{n-1}^*(T_{n-1}^* - k\delta_t) + m_{n-1}(T_{n-1}^* - k\delta_t) + \bar{a}, \quad (15)$$

for some δ_t small and $k = 1,\cdots,T_{n-1}^*/\delta_t$.

By Proposition 2, rearranging Equation (15) gives, $\forall\ k \in \left\{1,\cdots,\frac{T_{n-1}^*}{\delta_t}\right\}$:

$$\begin{aligned}\frac{W_{n-1}(T_{n-1}^* - (k-1)\delta_t) - W_{n-1}(T_{n-1}^* - k\delta_t)}{\delta_t} &= m_{n-1}(T_{n-1}^* - k\delta_t) + \bar{a} + r_{n-1}(T_{n-1}^* - k\delta_t)W_{n-1}(T_{n-1}^* - k\delta_t) \\ &- u_c^{-1}\Bigg(e^{-\rho k\delta_t}\frac{\tilde{S}(T_{n-1}^*, n-1)}{\tilde{S}(T_{n-1}^* - k\delta_t, n-1)}\exp\left\{\int_{T_{n-1}^*-k\delta_t}^{T_{n-1}^*} r_{n-1}(s)ds\right\}u_c(m_{n-1}(T_{n-1}^*) + \bar{a}) \\ + e^{\rho(T_{n-1}^*-k\delta_t)}&\frac{\exp\left\{-\int_0^{T_{n-1}^*-k\delta_t} r_{n-1}(s)ds\right\}}{\tilde{S}(T_{n-1}^* - k\delta_t, n-1)}\int_{T_{n-1}^*-k\delta_t}^{T_{n-1}^*} e^{\rho s + \int_{T_{n-1}^*-k\delta_t}^{s} r_{n-1}(s)ds}\tilde{S}(s, n-1)\lambda_{n-1,n}u_c(C_n^*(s))ds\Bigg), \quad (16)\end{aligned}$$

where $C_n^*(s)$ corresponds to the $W_n(0)$ such that $W_n(s) = W_{n-1}(s)$ and the integral can be approximated by the Trapezoidal rule. Given a specific T_{n-1}^*, the only unknown in equation (16) is $W_{n-1}(T_{n-1}^* - k\delta_t)$, which can be solved numerically. Sequentially, we get the initial wealth in state $n+1$, $W_{n-1}(0)$, for any given T_{n-1}^*. We then obtain the optimal consumption path on a discretized time grid, $C_{n-1}^*(T_{n-1}^* - k\delta_t)$, by equation (15) and a continuous path, $C_{n-1}^*(t)$, by linear interpolation.

Now since $C_n^*(t)$ and $C_{n-1}^*(t)$ are known for any given $W_n(0)$ and $W_{n-1}(0)$, respectively, $C_{n-2}^*(t)$ can be solved with the same algorithm, using solutions from higher-order states as inputs. Repeatedly, we obtain $C_i^*(t)$ for all health states i.

A key benefit in pursuing a closed-form solution lies in the generalization to more complex models with a high-dimensional state space. While our closed-form solution is implicit and therefore requires numerical solution of a set of integral equations, we can solve the problem iteratively state-by-state. Unlike conventional numerical methods used in the optimal control literature such as value iteration that searches the entire space and approaches the exact solution only when the number of iterations goes to infinity, the effort to solve higher-dimensional problems with this algorithm grows linear in the number of states, so that solving for high-dimensional problems becomes computationally feasible.

4 Data and Calibration

In this section, we introduce the data and the calibration we use when applying our model. The first part provides an introduction to the Future Elderly Model (FEM), which is where we draw transition probabilities, mortality probabilities, quality of life estimates, and medical spending. We then describe the specification of preferences and the calibration of the remaining parameters not originating from the FEM.

4.1 Future Elderly Model (FEM)

We obtain individual-level data on mortality, disease incidence (transition rates), quality of life, and medical spending from the FEM. At its core, the FEM operates as an economic-demographic micro-simulation model that combines data from nationally representative sources. Drawing from the Health and Retirement Study (HRS), Panel Study of Income Dynamics (PSID), National Health Interview Survey (NHIS), Medicare Current Beneficiary Survey (MCBS), and the Medical Expenditure Panel Survey (MEPS), the FEM provides a uniquely rich set of information about the U.S. elderly.

In the context of our study, the FEM produces estimates for individuals aged 65-110 with different comorbidities. More specifically, it accounts for six chronic conditions (cancer, diabetes, heart disease, hypertension, chronic lung disease, and stroke) and six impaired activities of daily living (bathing, eating, dressing, walking, getting into or out of bed, and using the toilet). We divide the health space into $n = 6$ states: state 1 corresponds to healthy with no chronic conditions or impaired activities of daily living (ADL); state 2 corresponds to 1 chronic condition or 1 ADL; state 3 corresponds to 1 chronic condition and 1 ADL; state 4 corresponds to multiple (2 or more) chronic conditions or multiple ADLs; state 5 corresponds to multiple chronic conditions and multiple ADLs; and state 6 corresponds to death.

Health State	Life Expectancy	Prob. stay in state i after 5 yrs	Survival prob. for 5 yrs	Survival prob. for 15 yrs
1	19.86	0.533	0.963	0.743
2	17.93	0.549	0.944	0.652
3	16.33	0.547	0.906	0.573
4	15.07	0.792	0.887	0.501
5	9.21	0.655	0.655	0.209

Table 1: Summary Statistics for 65-year-olds in the FEM

Table 1 shows the summary statistics on life expectancy and survival probabilities for 65-year-olds in our model setting. At age 65, life expectancy ranges from 19.86 years for healthy individuals to 9.21 years for sicker people with multiple chronic conditions and multiple ADLs. The probability of surviving to age 80 ranges from 74.3% to 20.9%, indicating substantial variations in longevity prospects across individuals in different health states.

4.2 Preferences and Financial Market Parameters

To calculate the optimal consumption paths and to quantify the value of health and longevity improvements, we need to specify the utility function. Following the life-cycle literature, we assume utility from consumption takes the form of an isoelastic function that implies constant relative risk aversion (CRRA). Since our utility function also incorporates quality of life, following Bauer et al. (2023), we assume:

$$u(c,q) = q\left(\frac{c^{1-\gamma} - \underline{c}^{1-\gamma}}{1-\gamma}\right),$$

where γ denotes degree of risk aversion, and $\underline{c}$ denotes subsistence level. The quality of life measure has non-negative values of $q \leq 1$, where $q = 1$ indexes perfect health, and is taken from the FEM. As explained in Bauer et al. (2023), utility is positive when non-medical consumption, c, exceeds the subsistence level, $\underline{c}$. The multiplicative relationship between quality of life and consumption utility implies the marginal utility of non-medical consumption increases with the health-related quality of life (negative state dependence), in line with the evidence in Viscusi and Evans (1990); Sloan et al. (1998); Finkelstein et al. (2013), and consistent with the assumption in Murphy and Topel (2006), although the evidence in the literature is mixed. This gives us four remaining parameters to calibrate: risk aversion γ, subsistence level $\underline{c}$, rate of time preference ρ, and rate of interest $r_i(t)$.

Following convention, we use exponential discounting with constant rate of interest and constant rate of time preference, where $r_i(t) = r = 0.03$ (Siegel, 1992) and $\rho = 0.05$ (Hurd, 1989). Consistent with the parametrization in Murphy and Topel (2006) and Bauer et al. (2023), we choose a moderate risk-aversion parameter $\gamma = 1.2$ and set the subsistence level at \$5,000.

Since our analysis focuses on the retired population aged 65 and above, we make a simplifying assumption that labor income is zero throughout, i.e., $m_i(t) = 0 \ \forall i$ and $\forall t$, although individuals receive (guaranteed, flat) annuity income from social security and possible defined benefit retirement plans. We further assume that the maximum possible lifetime is 50, or age 115, that mortality follows Gompertz-Makeham law with $\mu_i(t) = a_i + b_i e^{g_i t}$, and that transition follows Gompertz law with $\lambda_{ij}(t) = a_{ij} e^{b_{ij} t}$. We estimate the parameters a_i, b_i, g_i, a_{ij}, and b_{ij} using mortality and transition data from the FEM.

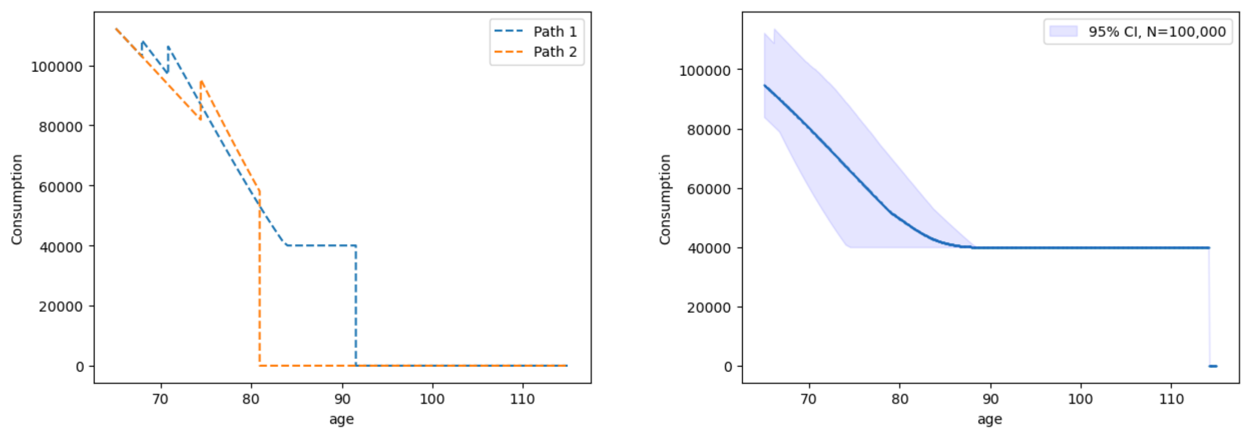
5 Application Results

This section presents the results from our quantitative analysis, using our calibrated model. Section 5.1 investigates the optimal consumption and optimal annuity choices. Section 5.2 studies the complementarity between annuity incomes from public programs and investments in health and longevity. And Section 5.3 measures the aggregate value of health and longevity improvements from future medical advances against life-threatening disease, and discusses the impacts of incomplete annuity markets and stochastic health risks on the results.

5.1 Optimal Consumption and Annuity Choices

To illustrate that our model can generate a variety of consumption profiles, we start with showing optimal consumption paths for two random trajectories, where we assume a given annuity income of \$40,000. Path 1 in Figure 1a corresponds to a healthy 65-year-old who experiences a minor health shock at age 68, where she develops one chronic condition or one ADL. The individual experiences another health shock at age 71, suffering multiple chronic conditions or multiple ADLs until she dies at age 92. Path 2 in Figure 1a corresponds to an individual that also starts healthy at age 65, then inflicts multiple chronic conditions or multiple ADLs from a health shock at age 74, and then dies at age 81. At age 65, both individuals have \$669,250 in non-annuitized liquid wealth, which is the average wealth for healthy 65-year-olds in the FEM. As shown by the dashed lines in

Figure 1a, consumption jumps up whenever there is a negative health shock. While the two individuals start with the same health conditions and the same wealth level, their optimal consumption paths vary significantly because they face distinct health shocks, which is a result from stochastic health risks. Note that as a consequence, their wealth trajectories will also be different. Consumer 1 dies without remaining liquid wealth and finances consumption solely from annuity income over the last years of her life; consumer 2 dies with some remaining, unspent wealth.



(a) Sample optimal consumption paths

(b) Monte Carlo simulation: optimal consumption paths

Figure 1: Optimal consumption paths

To illustrate the variation in optimal consumption choices from stochastic health risks among the retirees more generally, we conduct a Monte Carlo exercise with 100,000 individuals at age 65, where the proportion of 65-year-olds in each health state and their wealth levels follow information from the FEM. Again we assume annuity income of \$40,000. Figure 1b shows the mean (blue solid line) and the 95% confidence interval (shaded area) for the life-cycle optimal consumption choices of these 100,000 individuals. We observe substantial variation in their consumption paths. This is because each individual faces stochastic health shocks and follows a distinct health path as they grow older. Although some of the dispersion in consumption comes from differences in their initial health conditions and wealth amounts, individual-level health shocks play a central role in generating variations in consumption, where figure 1a shows one example. We note that there is also substantial dispersion in the times when the individuals first solely rely on annuity income to finance future consumption, i.e., when consumption hits \$40,000 (this corresponds to the "kink point" in the optimization path evaluated above).

We next investigate annuity choices based on our model. We start with a hypothetical example, where the solution is known. More specifically, we consider a healthy individual at age 65 and we assume constant quality of life, constant lifetime income, $r = \rho$, and zero mark-up rate for annuities. Under these assumptions, Yaari's famous full annuitization result (Yaari, 1965) holds (see Davidoff et al. (2005) for a formal argument).

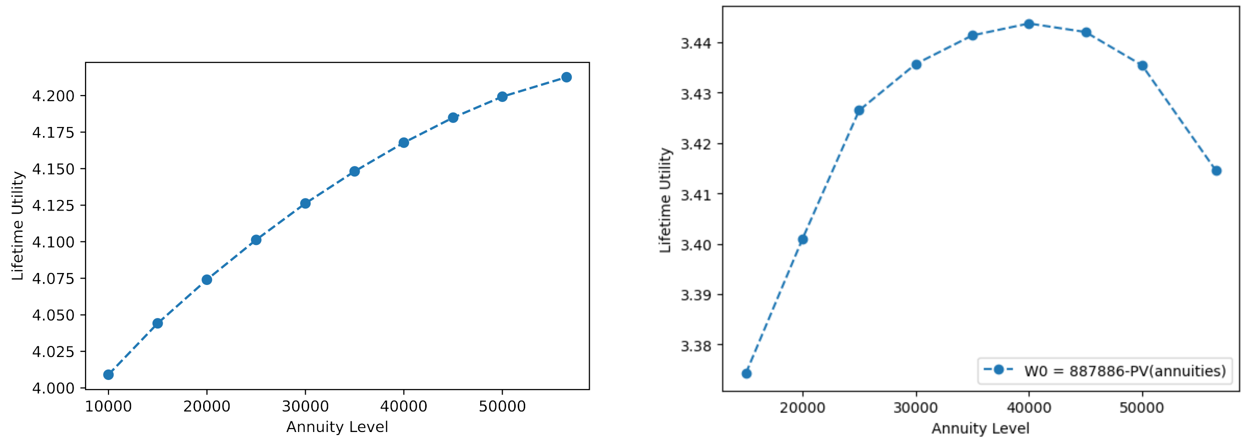
Figure 2a shows how lifetime utility varies in this situation with the annuitization level. On the x-axis are annual annuity payouts, ranging from \$10,000 to \$56,000, and we determine (liquid, remaining) wealth as a common initial wealth minus the present value of the annuity, where the initial wealth is equal to the present value of the highest amount of annuity payouts depicted on the x-axis. As is evident from the figure and consistent with Yaari's result that people without bequest motives should fully annuitize, lifetime utility at the highest annuity level (the point farthest to the right) corresponds to full annuitization.

We show next that the full annuitization result breaks under more general assumptions, when there is a limited menu of retirement products. We again determine how lifetime utility varies with annuity income, but we account for three features absent in 2a. First, we integrate the quality of life profiles from the FEM, which imply that consumption at different age/state combinations is valued differently. Second, we allow for out-of-pocket medical spending. Incorporating medical spending directly in the wealth dynamic, we have:

$$W_i'(t) = r_i(t)W_i(t) - C_i(t) + m_i(t) + \bar{a} - h_i(t),$$

where $m_i(t) + \bar{a}$ is lifetime income and $h_i(t)$ is medical spending in state i at time t. We use data on individual out-of-pocket (OOP) medical spending by age and health state from the FEM. To align with our continuous-time setup, we fit health expenditure by age in any health state using an exponential specification. Because OOP medical spending data for ages above 100 is noisy for all health states, we assume OOP medical spending is constant at ten thousand dollars from age 100 onward for all states. That is, $h_i(t) = a_i e^{b_i t}$ for $t < 35$ and $h_i(t) = 10000 \ \forall \ i$ for $t \geq 35$, where a_i and b_i are constants. Third, we consider a loading factor for private-market annuities. Mitchell et al. (1999) document that the expected present value of annuity payouts per dollar of annuity premium is between 80 to 85 cents. Relying on their estimates, we assume that 65-year-old retirees get annuities at a fair rate up to \$25,000 per year from public annuity programs, while any supplementary annuity payouts from the private market are subject to a 15% loading. We choose \$25,000 because this is close to the average social security benefits for the retired.

Figure 2b shows the lifetime utility for healthy 65-year-olds under different annuitization levels and these three modifications (quality of life profiles, OOP medical spending, and a loading for private annuity products). The optimal annuity is now partial. This is



(a) Yaari's result: special conditions

(b) with OOP medical spending and mark-up rate

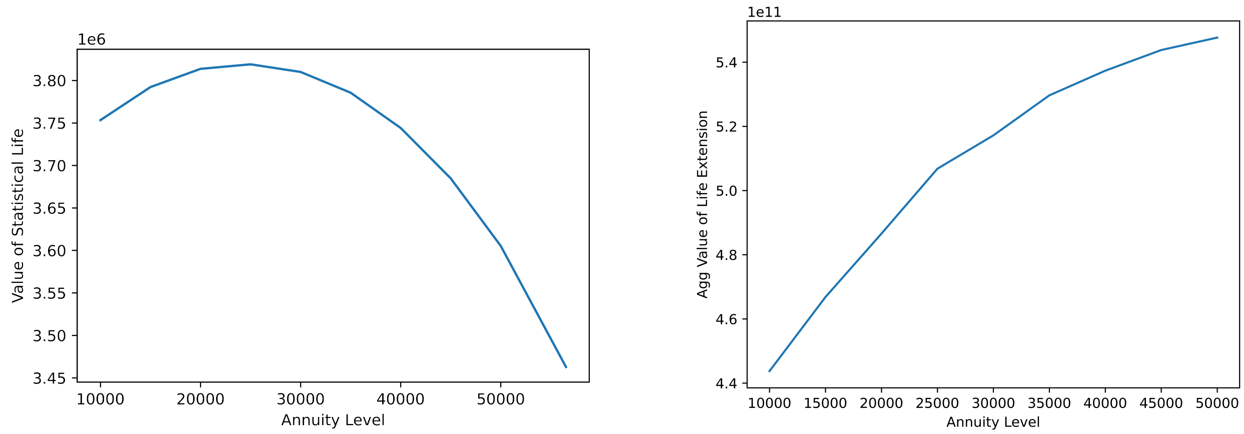
Figure 2: Lifetime utility versus annuity level for healthy individuals at age 65.

consistent with results from Reichling and Smetters (2015), who show that in a framework where medical spending and mortality are correlated, consumer demand for flat life annuities is significantly reduced. We find that the optimal annuity level reduces to about $40,000. This indicates that a rational healthy retiree at age 65 would choose to purchase a life annuity with $15,000 annual payout from the private market. Bequest motives and other aspects would further reduce the optimal annuity level, so that low observed annuitization rates seem rational in this context (see also Inkmann et al., 2011; Lockwood, 2012, for similar observations).

5.2 Public Annuity Programs and Health Investments

In what follows, when not mentioned otherwise, we use FEM quality of life profile for calculations. Figure 3a shows the VSL defined in equation (9) for healthy individuals at age 65 under different annuity levels. VSL is the highest at an annuity level of roughly $25,000. As consumers hold larger fractions of their initial wealth in flat life annuities, their willingness to pay for a marginal improvement in longevity decreases, with the lowest value at full annuitization. Interestingly, the maximal willingness to pay generally is higher for annuity levels below the optimal annuitization level, which originates from differences in the marginal utility of consumption. Nonetheless, we observe that public annuity income from social security increases VSL, illustrating the complimentarity of public annuity programs and public spending to improve health outcomes.

Figure 3a is cast for a specific health intervention—VSL corresponding to avoiding immediate death—and a specific cohort of individuals—65 year-olds in health state 1. To



(a) VSL for healthy individuals at age 65

(b) Value of health improvements for all retirees

Figure 3: Willingness to pay for health and longevity improvements versus annuity level.

appraise how annuitization levels affect a more general health intervention, we use aggregate measures defined in Section 2.3 to estimate the social value of health and longevity improvements from a particular health intervention.

For defining the health intervention, we again turn to the FEM. We use a medical advance against cancer as an example, where we label the post-intervention population as *the cancer cohort*. Table 2 shows the changes in life expectancy for 65-year-olds and 75-year-olds following this medical intervention that reduces the likelihood of cancer onset as well as cancer mortality. In the cancer cohort, life expectancy at age 65 (75) increases by 1.58% (2.95%) for healthy individuals while life expectancy at age 65 (75) increases by 5.13% (11.81%) for individuals with multiple chronic conditions and multiple ADLs. Since individuals with cancer and the elderly will benefit most from the cancer intervention, it is not surprising to see that this medical advance against cancer has larger impact on sicker people and on older people.

State	ΔLE(65)	%ΔLE(65)	ΔLE(75)	%ΔLE(75)
1	0.314	1.58%	0.305	2.95%
2	0.40	2.25%	0.38	4.36%
3	0.48	2.96%	0.42	5.67%
4	0.52	3.46%	0.44	6.91%
5	0.47	5.13%	0.31	11.81%

Table 2: Summary Statistics for the Cancer Cohort

Because the FEM provides a new set of information on transition and mortality rates, the perturbation $\delta_{ij}(t)$ resulting from this intervention is

$$\delta_{ij}(t) = \lambda^c_{ij}(t) - \lambda^b_{ij}(t),$$

where $\lambda^b_{ij}(t)$ and $\lambda^c_{ij}(t)$ are the transition rates before and after the medical advance. We can then rely on equations (10) and (11) to calculate the aggregate value of the intervention, using the 2010 U.S. elderly population to defined the relevant weights $f(a,i)$.[9]

Figure 3b shows how the value of health and longevity improvements from this intervention against cancer among retirees varies by the annuity level. Within the current public annuity system, the median social security income and the median pension income for retirees add up to $35,000, under which the value of health and longevity improvements is $0.53 trillion. Absent any public annuity program but guarantee people with lifetime income at the subsistence level, the value is $0.418 trillion. Hence, there is a 27% increase in the value of health and longevity improvements associated with a medical advance against cancer given the current public annuity programs. In other words, public annuity programs provide a strong complement to this health intervention.

Increases in social security would further boost this value, although this boost is not uniform across consumers and conditions. Figure 3a shows one example: VSL for a healthy and young retiree is close to maximal under an annual annuity payout of $25,000. This indicates that the increase in the value of health and longevity improvements from an increase in social security benefits mostly originate from sicker and older people's higher willingness to pay for living a little longer and better.

5.3 Value of Health and Longevity Improvements

Our focus in the previous section was how annuity income, e.g. from public annuity programs, affect the social value of health interventions. In this section we investigate to what extent values based on our model differ from the prior estimates, which were cast under the assumption of complete annuitization and determinstic health.

For a benchmark, we determine the value for the health intervention introduced in the previous section using the analysis in Murphy and Topel (2006) under complete markets and deterministic health. The value of a medical advance α in Murphy and Topel (2006) is defined as:

$$V_\alpha = \int_a^\infty v(t)S(t,a)\Gamma_\alpha(t,a)dt + \int_a^\infty \frac{H'_\alpha(t)}{H(t)} \frac{u(c(t),l(t))}{u_c} S(t,a)dt, \tag{17}$$

[9]More details on the calculation steps are provided in the next section.

where $v(t)$ is the value of a life-year, $S(t,a)\Gamma_\alpha(t,a)$ is the discounted impact on survival probabilities from factor α, $H(t)$ is quality of life, and $l(t)$ is leisure. Because our model does not include leisure and our utility function incorporates quality of life, we abstract from $H(t)$ and $l(t)$. Thus, equation (17) reduces to:

$$V_\alpha = \int_a^\infty v(t)S(t,a)\Gamma_\alpha(t,a)dt = \int_a^\infty v(t)e^{-r(t-a)}\tilde{S}'_\alpha(t,a)dt, \tag{18}$$

where $v(t) = y(t) + \frac{u(c(t),q(t))}{u_c(c(t),q(t))} - c(t)$ and $\tilde{S}'_\alpha(t,a)$ is the difference in survival probabilities following α. Since we focus on the retired population, it is reasonable to assume income $y(t)$ is zero.

In line with the calibrations in Murphy and Topel (2006), we use a wealth level for age 65 so that the estimated value of statistical life for age 70 is \$2 million. We then follow Yaari's framework (Yaari, 1965) to solve for optimal consumption choices $c(t)$, where single-state quality of life and single-state survival probabilities are obtained by simulating from our FEM data. More specifically, we construct a life table (for people with ages 65 and older) by simulating one million health paths with a pre-specified proportion starting in each health state at age 65. Again using the 2010 U.S. elderly population and the cancer intervention described in the previous section, via equation (18), the estimate for the value of health and longevity improvements among retirees is \$1.18 trillion, indicated by the red solid line in figure 4. Murphy and Topel (2006) find that the social value of a 10% reduction in cancer mortality is estimated at \$4.7 trillion for the entire U.S. population. Our estimate of \$1.18 trillion aligns in magnitude with this number, considering the distinct nature of the intervention and our focus on the retired population exclusively. We denote this framework by scenario (i).

Next, using equation (11), we estimate the value of health and longevity improvements from the cancer intervention under our framework, in which the annuity market is incomplete and health is stochastic. More specifically, we simulate 100,000 health paths starting at age 65 following the initial health distribution. To keep results from different frameworks comparable, we control for the total wealth (i.e., liquid assets plus the present value of annuities). In this case, we calculate the multiplier m such that m times the weighted-average wealth at age 65 is the same as the wealth at age 65 in scenario (i). This weighted-average is calculated by the initial health distribution and wealth at age 65 in each state, all given by the FEM data. Once we know m, we know the calibrated wealth at age 65 for each health state, and hence the entire wealth path and the marginal value of life extension by equation (10) for any simulated health path. Taking the average of the marginal values of life extension across 100,000 realizations for each age, we obtain an es-

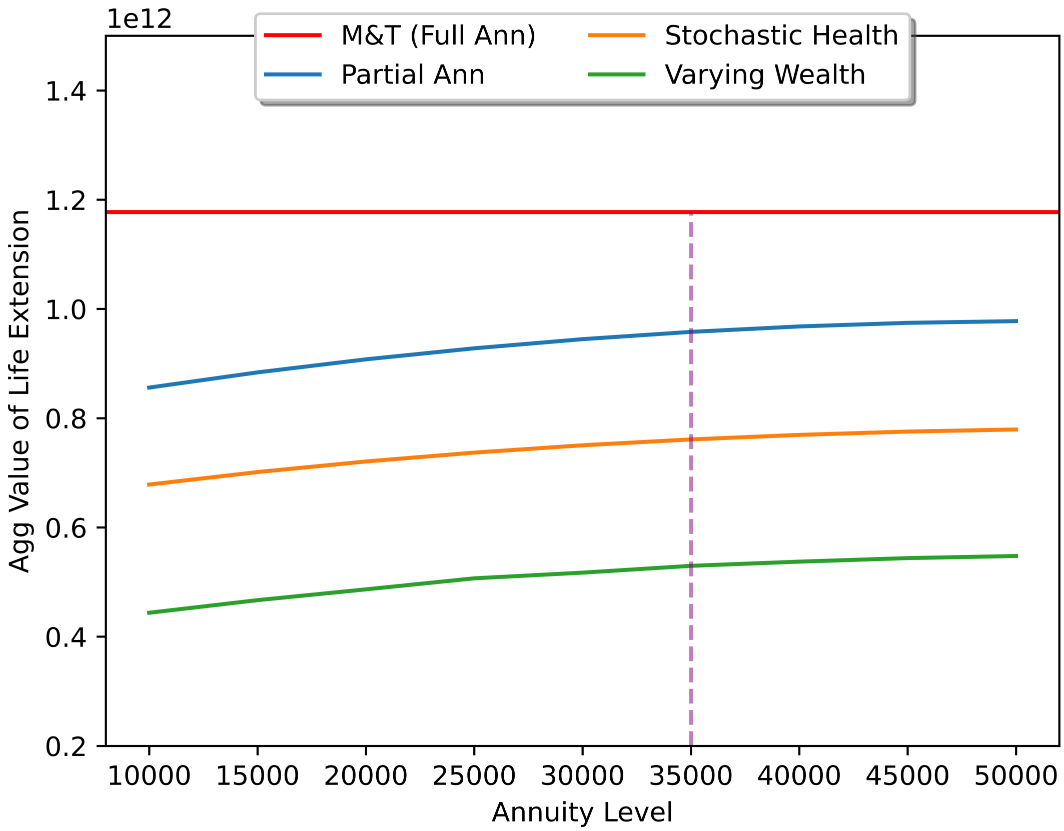


Figure 4: Aggregate willingness to pay for health and longevity improvements versus annuity level.

timate for $MVLE(a, i, \delta)$. By equation (11) and the 2010 U.S. elderly population, we get the estimates for the aggregate value of this medical advance under different annuity levels, as shown by the green solid line in Figure 4. In particular, with \$35,000 annual annuity income from public annuity programs, the value of health and longevity improvements from the cancer intervention among retirees is \$0.53 trillion, which is 55% lower than the comparable estimate, \$1.18 trillion, in the literature. For illustration purposes, we refer to this framework as scenario (iv).

To understand the reasons behind the 55% reduction, we study the value of health and longevity improvements under different frameworks, relaxing one assumption at a time. Denoted by scenario (ii), we allow the annuity market to be incomplete but assume health risk is deterministic. Similar to scenario (i), life-cycle mortality rates are deterministic; so, we use the same single-state quality of life and single-state survival probabilities as in scenario (i). What sets this scenario apart from the conventional framework is the incompleteness of the annuity market, in which case we use our analytical solution for state n to solve for optimal consumption choices instead of adopting Yaari's framework.

Holding total wealth for age 65 under all annuity levels constant at the calibrated wealth level for age 65 in scenario (i), we, by equations (11) and (12), obtain the estimates for the value of health and longevity improvements in a framework where the annuity market is incomplete and health is deterministic. The blue solid line in Figure 4 shows the results. With a \$35,000 annuity, the U.S. retirees are willing to pay \$0.96 trillion in aggregate for this intervention against cancer. Compared to scenario (i), retirees are willing to pay roughly 20% less. This reduction is due to the incompleteness of the annuity market, where consumers only hold a modest fraction of their wealth in annuities. As a result, their optimal consumption profile is decreasing in age, which in turn reduces their value of health and longevity improvements in old ages.

Next, we allow health risk to be stochastic, in addition to relaxing the complete-market assumption. To clearly identify the effect of stochastic health, we set wealth at any age x to be constant across all health states and to be equal to wealth at age x in scenario (ii). This eliminates any impact on the value of health and longevity improvements from variations in wealth, either across scenarios or across health states. We denote this framework as scenario (iii). Similar to scenario (iv), we use equations (10) and (11) to quantify the value of health and longevity improvements, the estimates of which under different annuity levels are shown by the yellow solid line in Figure 4. With a \$35,000 annuity, this intervention against cancer is worth \$0.76 trillion, about 35% lower than the value in scenario (i). The additional 15% reduction in the value of health and longevity improvements is due to people's ability to adjust their consumption decisions following negative health shocks. Under the stochastic-health and incomplete-market framework, any rational individual with lifetime income and no bequest motives will rationally choose to spend more after experiencing a negative health shock, as shown in Figure 1a. Intuitively, this is because people expect themselves to live for shorter periods when their health conditions suddenly get worse, thus incentivizing them to spend down their wealth more quickly to avoid dying with unutilized wealth that could have been enjoyed earlier in their lifetime. Hence, the ability to adjust consumption decisions, which exists in stochastic-health settings only, tempers the adverse effect of negative health shocks on lifetime utility, which in turn reduces people's willingness to pay to avoid or alleviate negative health shocks.

The remaining 20% reduction in the value of health and longevity improvements stems from wealth variations across individuals, the only difference between scenarios (iii) and (iv). More specifically, empirical evidence suggests healthier people are on average wealthier upon retirement. Using wealth data by health states at age 65 from the FEM, we find that the value of health and longevity improvements is 30% lower than that when wealth is constant across health states. Since individuals with poorer health conditions benefit more from this intervention (as shown by Table 2), reducing their wealth upon retirement lowers the aggregate willingness to pay for health and longevity improvements. As of now, we do not yet have information about wealth distributions by health state for later ages, so we rely on simulation to estimate the marginal value of life extension for each age for each state, $MVLE(a, i, \delta)$.

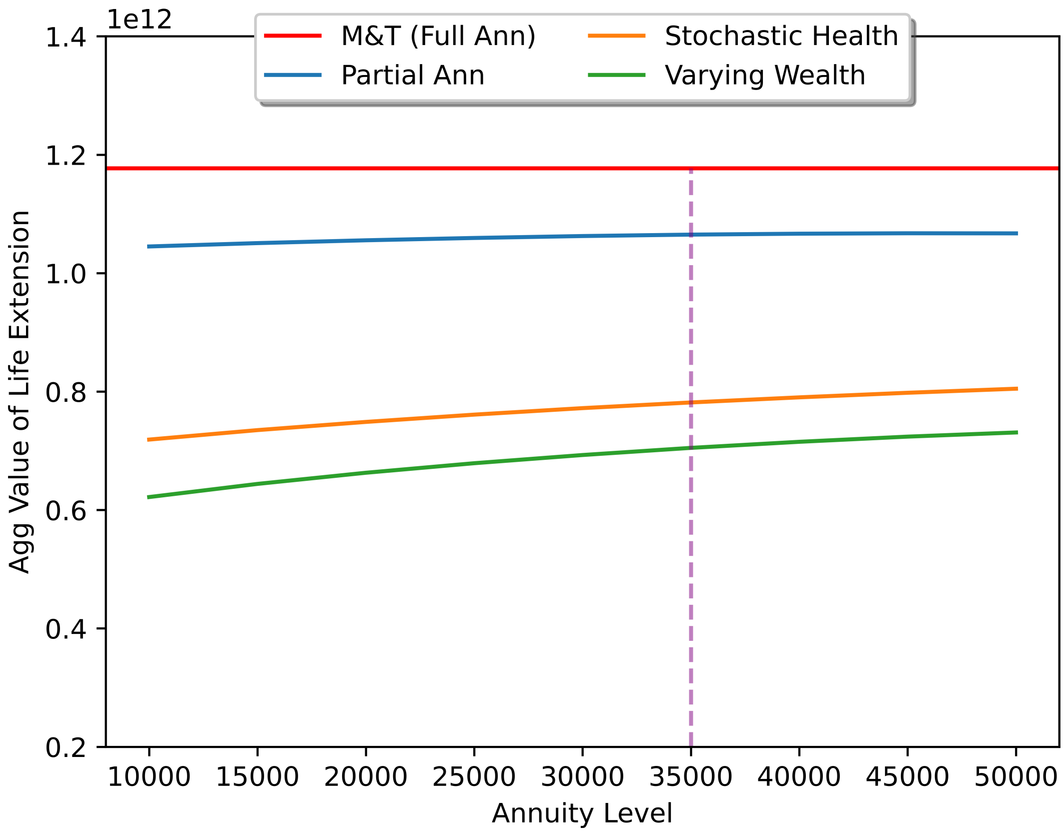


Figure 5: Aggregate utility of health and longevity improvements with equal weights versus annuity level.

Figure 5 shows results for the value of health and longevity improvements under the same frameworks, scenarios (i) - (iv), using aggregate utility with equal weights for all individuals, or equation (13), as the measure. Table 3 summarizes the values of health and longevity improvements under an annual annuity payout of \$35,000 within scenarios (i) - (iv) and two different measures. Without loss of generality, we choose the weights to be the marginal utility of consumption at age 65 within the conventional framework (scenario (i)) for all individuals, when converting utility to numeraire in consumption goods. Under the utilitarian approach, the reduction in the value of health and longevity improvements due to the first effect, incompleteness of annuity markets, and the third ef-

Scenario	Aggregate WTP (trillion)	Aggregate Utility (trillion)
(i)	1.18	1.18
(ii)	0.96	1.07
(iii)	0.76	0.78
(iv)	0.53	0.70

Table 3: Estimates for the value of health and longevity improvements under $35,000 annuity level

fect, wealth variations by health state, are both smaller, whereas the reduction due to the second effect, ability to adjust consumption decisions following health shocks, is larger. Under the willingness-to-pay measure, we divide the marginal utility of life extension by the marginal utility of wealth, which is equal to the marginal utility of consumption, $u_c(C_i(t), q_i(t))$. This term is smaller for individuals with worse health conditions since their consumption levels are higher. In other words, sicker people and older people are given more weights within the willingness-to-pay measure. When we switch to the utilitarian world where we give everyone the same weight, the effects that are primarily driven by sicker and/or older individuals, less consumption smoothing due to incomplete markets and less wealth due to poorer health conditions upon retirement, become smaller. The effect due to the ability to increase consumption after some negative health shocks, on the other hand, is driven more by the healthier ones as they are more likely to experience a negative health shock.

6 Conclusion

We develop a life-cycle framework to analyze optimal consumption and annuity choices among retirees, considering stochastic health risks and incomplete annuity markets. We use this framework to estimate the value of health and longevity improvements. Our main finding is that, compared to the estimates from the prevailing method in the literature that assumes deterministic health and full annuitization, our estimates associated with a medical advance in cancer for the retired population is substantially lower—55% reduction under an aggregate willingness-to-pay measure and over 40% reduction under an aggregate utility measure.

This reduction is attributed to the interplay of three key factors. First, the incompleteness of annuity markets results in decreasing optimal consumption patterns, diminishing the value of health and longevity improvements in older ages. Second, in the context of stochastic health, individuals can adjust their consumption decisions following a negative health shock, mitigating some of the impact of the health shock on lifetime utility and, consequently, the willingness to pay to avoid it. Third, individuals with poorer health conditions, who stand to benefit the most from health improvements, on average possess fewer assets upon retirements, leading to lower aggregate willingness-to-pay.

As a second result, this paper finds a strong complementary relationship between annuity income from public programs such as social security and investments in health and longevity improvements. The current public annuity programs boost the value of health and longevity improvements from a medical advance against cancer by 27%, and increasing public annuity payouts further boosts people's aggregate willingness-to-pay for improvements in health and longevity. This boost, however, is not uniform across individuals but stems from those with poorer health conditions.

Our findings are relevant for policymakers considering investments in health research and development as well as the payment strategies of public annuity programs. While our estimates of the value of health interventions is lower than that in prior literature, our figures are still large, in part due to public annuity programs.

References

Banks, J., R. Blundell, and S. Tanner (1998). Is there a retirement-savings puzzle? *American Economic Review 88*(4), 769–788.

Bauer, D., D. Lakdawalla, and J. Reif (2023). Health risk and the value of life. *NBER Working Paper*.

Bertsekas, D. (2005). *Dynamic Programming and Optimal Control*. Athena Scientific.

Carroll, C. D. and L. H. Summers (1991). *Consumption growth parallels income growth: Some new evidence*, pp. 305–348. University of Chicago Press.

Davidoff, T., J. R. Brown, and P. A. Diamond (2005). Annuities and individual welfare. *American Economic Review 95*(5), 1573–1590.

Fernandez-Villaverde, J. and D. Krueger (2007). Consumption over the life cycle: Facts from consumer expenditure survey data. *The Review of Economics and Statistics 89*(3), 552–565.

Finkelstein, A., E. F. Luttmer, and M. J. Notowidigdo (2013). What good is wealth without health? The effect of health on the marginal utility of consumption. *Journal of the European Economic Association 11*(s1), 221–258.

Fleurbaey, M. (2010). Assessing risky social situations. *Journal of Political Economy 118*(4), 649–680.

Goldman, D. P., D. Cutler, J. W. Rowe, P.-C. Michaud, J. Sullivan, D. Peneva, and S. J. Olshansky (2013). Substantial health and economic returns from delayed aging may warrant a new focus for medical research. *Health Affairs 32*(10), 1698–1705.

Goldman, D. P., P.-C. Michaud, D. Lakdawalla, Y. Zheng, A. Gailey, and I. Vaynman (2010). The fiscal consequences of trends in population health. *National Tax Journal 63*(2), 307–330.

Harsanyi, J. C. (1955). Cardinal welfare, individualistic ethics, and interpersonal comparisons of utility. *Journal of Political Economy 63*(4), 309–321.

Hartl, R. F. (1984). *A Survey of the Optimality Conditions for Optimal Control Problems with State Variable Inequality Constraints*. New York: Elsevier.

Hurd, M. D. (1989). Mortality risk and bequests. *Econometrica 57*(4), 779–813.

Inkmann, J., P. Lopes, and A. Michaelides (2011). How deep is the annuity market participation puzzle? *The Review of Financial Studies 24*(1), 279–319.

Leung, S. F. (1994). Uncertain lifetime, the theory of the consumer, and the life cycle hypothesis. *Econometrica 62*(5), 1233–1239.

Lockwood, L. M. (2012). Bequest motives and the annuity puzzle. *Review of economic dynamics 15*(2), 226–243.

Mitchell, O. S., J. M. Poterba, M. J. Warshawsky, and J. R. Brown (1999). New evidence on the money's worth of individual annuities. *American Economic Review 89*(5), 1299–1318.

Murphy, K. M. and R. H. Topel (2006). The value of health and longevity. *Journal of Political Economy 114*(5), 871–904.

Nunn, R., J. Parsons, and J. Shambaugh (2020). A dozen facts about the economics of the us health-care system. Technical report, The Hamilton Project, Economic Facts.

Parpas, P. and M. Webster (2013). A stochastic minimum principle and an adaptive pathwise algorithm for stochastic optimal control. *Automatica 49*(6), 1663–1671.

Philipson, T. J. and G. S. Becker (1998). Old-age longevity and mortality-contingent claims. *Journal of Political Economy 106*(3), 551–573.

Reichling, F. and K. Smetters (2015). Optimal annuitization with stochastic mortality and correlated medical costs. *American Economic Review 105*(11), 3273–3320.

Reif, J., H. Heun-Johnson, B. Tysinger, and D. Lakdawalla (2021). Measuring the covid-19 mortality burden in the united states: A microsimulation study. *Annals of Internal Medicine 174*(12), 1700–1709.

Research-America (2022). Us investments in medical and health research and development 2016–2020. Technical report, Research America.

Rosen, S. (1988). The value of changes in life expectancy. *Journal of Risk and Uncertainty 1*(3), 285–304.

Siegel, J. J. (1992). The real rate of interest from 1800–1990: A study of the us and the uk. *Journal of Monetary Economics 29*(2), 227–252.

Sloan, F. A., W. Kip Viscusi, H. W. Chesson, C. J. Conover, and K. Whetten-Goldstein (1998). Alternative approaches to valuing intangible health losses: the evidence for multiple sclerosis. *Journal of Health Economics 17*(4), 475–497.

Viscusi, W. K. and W. N. Evans (1990). Utility functions that depend on health status: estimates and economic implications. *American Economic Review 80*(3), 353–374.

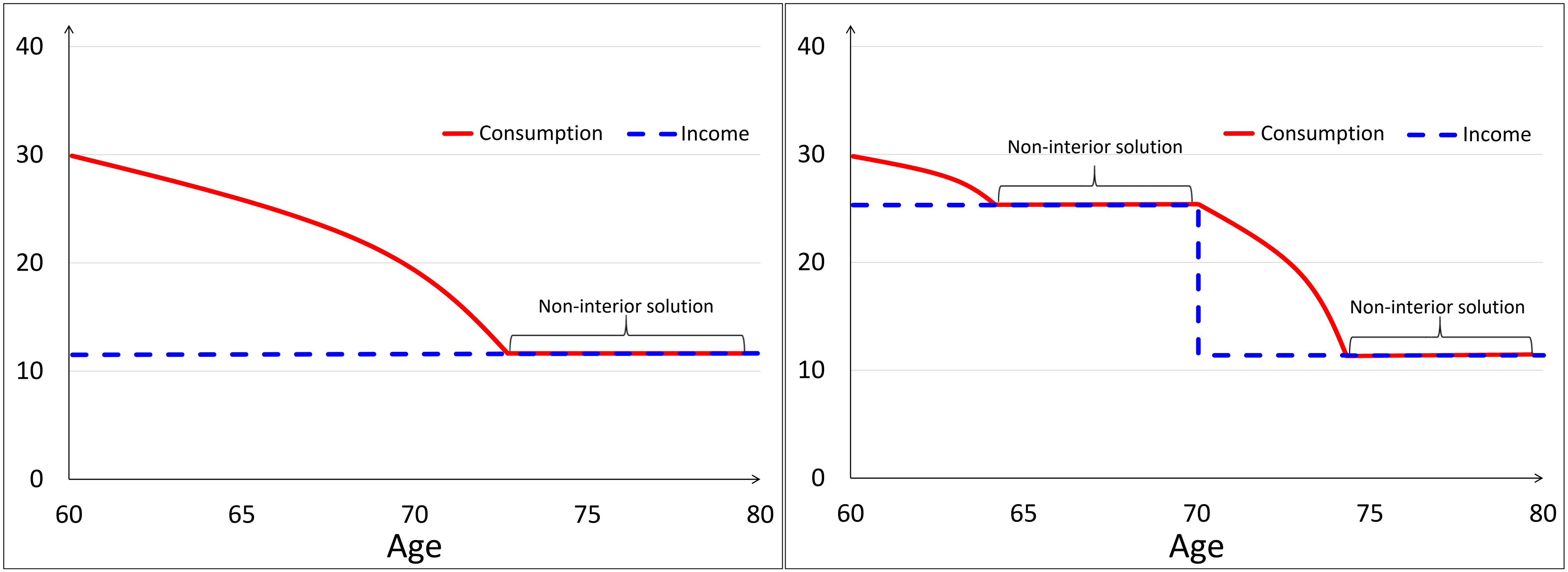
Webster, P. (2023). Big tech companies invest billions in health research. *Nature medicine 29*(5), 1034–1037.

Yaari, M. E. (1965). Uncertain lifetime, life insurance, and the theory of the consumer. *Review of Economic Studies 32*(2), 137–150.

Figure 6: Illustrative example: survival-contingent income can generate non-interior solutions

(a) One set of non-interior solutions

(b) Two sets of non-interior solutions



Notes: The solution to the consumer's maximization problem may be non-interior in the presence of survival-contingent income. Panel (a) gives an example where there is one set of non-interior solutions. Panel (b) gives an example where there are two sets of non-interior solutions. Income, illustrated by the dashed blue line, includes both labor income and annuity income.

Online Appendix

"The Value of Health and Longevity with Stochastic Health Risk and Partial Annuitization"

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A Mathematical Proofs

Proof of Lemma 1. We derive the first-order necessary conditions for a maximun corresponding to the Hamiltonian in equation (5). By the maximum principle,

$$\frac{\partial H(W_i(t), C_i(t), p_i(t), \eta_i(t))}{\partial C_i(t)} = 0,$$

where $\frac{\partial H(W_i(t),C_i(t),p_i(t),\eta_i(t))}{\partial C_i(t)} = e^{-\rho t}\tilde{S}(t,i)\frac{\partial u(C_i(t),q_i(t))}{\partial C_i(t)} - p_i(t)$. This gives condition (i).

The costate variable $p_i(t)$ should satisfy

$$p_i'(t) = -\frac{\partial H(W_i(t), C_i(t), p_i(t), \eta_i(t))}{\partial W_i(t)},$$

where $\frac{\partial H(W_i(t),C_i(t),p_i(t),\eta_i(t))}{\partial W_i(t)} = r_i(t)p_i(t)+\eta_i(t)+e^{-\rho t}\tilde{S}(t,i)\sum_{j>i}\lambda_{ij}(t)\frac{\partial}{\partial W_i(t)}V(t, W_i(t), \overline{a}, j)$. This results in condition (ii).

By the non-borrowing constraint $W_i(t) \geq 0$, we obtain condition (iii). Conditions (iv) and (v) correspond to boundary conditions and ensure that optimal solution exists. ■

Proof of Proposition 2. Following Proposition 1 in Leung (1994), one can show the following: the Hamiltonian is regular on $[0, T)$, so optimal consumption $C_i(t)$ is everywhere continuous; the state-variable inequality constraint is of first-order, so $p_t^{(i)}$ is everywhere continuous; and optimal consumption $C_i(t)$ is continuously differentiable when $W_i(t) > 0$ (i.e., when the wealth constraint is not binding).

First, consider the case when $W_i(t) > 0$. Define the elasticity of intertemporal substitution, σ, as

$$\frac{1}{\sigma} \equiv -\frac{u_{cc}c}{u_c}$$

and define the elasticity of quality of life with respect to the marginal utility of consumption as

$$\eta \equiv \frac{u_{cq}q}{u_c}$$

Differentiating the first-order condition for consumption with respect to t, plugging in the result for the costate equation and its solution, and then rearranging yields the rate of change in life-cycle consumption

$$\frac{\dot{c}_i}{c_i} = \sigma(r-\rho) + \sigma\eta\frac{\dot{q}_i}{q_i} - \sigma\lambda_{i,n+1}(t) - \sigma\sum_{j=i+1}^{n}\lambda_{ij}(t)\left[1 - \frac{u_c\Big(c\big(t, W_i(t), j\big), q_j(t)\Big)}{u_c\big(c(t, W_i(t), i), q_i(t)\big)}\right],$$

which is weakly declining by assumption.

The presence of life-cycle earnings introduces the possibility of multiple sets of non-interior solutions (e.g., right panel of Figure 6). Modeling these scenarios is possible, but cumbersome. As discussed in the main text, we therefore restrict ourselves to considering the case with a single set of non-interior solutions (i.e., a single "kink point", see left panel of Figure 6). A sufficient (but not necessary) assumption is that consumption growth is weakly declining. We employ that assumption in the following Lemma, which establishes the existence of a single kink point, T_i^*, where the consumer runs out of wealth.

Lemma A.1. *Let $m_i^*(t) = m_i(t) + \bar{a}$. Assume $m_i(t)$ is non-decreasing. Then there must exist a T_i^* such that (1) $W_i(t) = 0$ and $C_i(t) = m_i^*(t)$ for $t \geq T_i^*$; and (2) $C_i(t) > m_i^*(t)$ for $t < T_i^*$. The solution to the costate equation on $\left[0, T_i^*\right]$ is thus:*

$$p_t^{(i)} = \left[\int_t^{T_i^*} e^{(r-\rho)s}\tilde{S}(i,s)\sum_{j>i}\lambda_{ij}(s)\frac{\partial V(s, W_i(s), \bar{a}, j)}{\partial W_i(s)}\,ds\right]e^{-rt} + \theta^{(i)}e^{-rt}$$

where $\theta^{(i)} > 0$ is a constant.

Proof. By assumption, $\frac{\dot{c}_i}{c_i} < 0$ whenever $W_i(t) > 0$. Following the same argument as in Proposition 2 of Leung (1994), there is a smallest T_i^* such that $W_i(t) = 0$ on $\left[T_i^*, T\right]$ and, thus, $C_i(t) = m_i(t)$ on $\left[T_i^*, T\right]$. Since this is the smallest such T_i^*, there exists an interval $\left(\underline{T}_i, T_i^*\right)$ such that $W_i(t) > 0$ and $C_i(t_0) > m_i^*(t_0)$ for a t_0 in the vicinity of T_i^*. Now assume $W_i(\underline{T}_i) = 0$. Then there exists a t_1 in the vicinity of $\underline{T}_i$ such that $c_i(t_1) < m_i^*(t_1)$. This is a contradiction, since $m_i^*(t)$ is non-decreasing (because $m_i(t)$ is non-decreasing and $\bar{a}$ is a constant) and $C_i(t)$ is decreasing whenever $W_i(t) > 0$. Hence $W_i(t) > 0$ on $[0, T_i^*)$ and $C_i(t) > m_i^*(t)$ for $t \in [0, T_i^*)$. ■

It follows that $W_i(0) + \int_0^{T_i^*} e^{-\int_0^s r_i(s)ds}(m_i(t) + \bar{a})dt = \int_0^{T_i^*} e^{-\int_0^s r_i(s)ds}C_i^*(t)dt$. That is, consumption from time 0 to T_i^* should exhaust wealth and income from 0 to T_i^*. Since $W_i(t) > 0$ for all $t \in [0, T_i^*)$, by condition (iii) we have $\eta_i(t) = 0$ for all t in $[0, T_i^*)$.

Then (ii) becomes $-p_i'(t) = r_i(t)p_i(t) + e^{-\rho t}\tilde{S}(t,i)\sum_{j>i}\lambda_{ij}(t)\frac{\partial}{\partial W_i(t)}V(t, W_i(t), \bar{a}, j)$. Solving for $p_i(t)$ we have

$$p_i(t) = e^{-\int_0^t r_i(s)ds}\int_t^{T_i^*}\exp\left\{-\rho s + \int_0^s r_i(u)du\right\}\tilde{S}(s,i)\sum_{j>i}\lambda_{ij}(s)\frac{\partial}{\partial W_i(s)}V(s, W_i(s), \bar{a}, j)ds + \beta_i e^{-\int_0^t r_i(s)ds},$$

where $\beta_i \in \mathbb{R}$ is a constant. Condition (i) then becomes

$$\begin{aligned} e^{-\rho t}\tilde{S}(t,i)u_c(C_i(t)) = {} & \beta_i e^{-\int_0^t r_i(s)ds} \\ & + e^{-\int_0^t r_i(s)ds}\int_t^{T_i^*}\exp\left\{-\rho s + \int_0^s r_i(u)du\right\}\tilde{S}(s,i)\sum_{j>i}\lambda_{ij}(s)\frac{\partial}{\partial W_i(s)}V(s, W_i(s), \bar{a}, j)ds. \end{aligned}$$

This implies that

$$C_i^*(t) = u_c^{-1}\Bigg((e^{\rho t}\frac{1}{\tilde{S}(t,i)}\beta_i e^{-\int_0^t r_i(s)ds} + e^{-\int_0^t r_i(s)ds}\int_t^{T_i^*}\exp\left\{-\rho s+\int_0^s r_i(u)du\right\}\tilde{S}(s,i)\sum_{j>i}\lambda_{ij}(s)\frac{\partial}{\partial W_i(s)}V(s,W_i(s),\bar{a},j)ds\Bigg), \tag{A.1}$$

with boundary condition

$$C_i^*(T_i^*) = u_c^{-1}\left(e^{\rho T_i^*}\frac{1}{\tilde{S}(T_i^*,i)}\beta_i e^{-\int_0^{T_i^*} r_i(s)ds}\right) = m_i(T_i^*)+\bar{a}.$$

Solving for β_i we have

$$\beta_i = e^{-\rho T_i^* + \int_0^{T_i^*} r_i(s)ds}\tilde{S}(T_i^*,i)u_c(m_i(T_i^*)+\bar{a}).$$

Plugging β_i into Equation A.1, we obtain

$$C_i^*(t) = u_c^{-1}\Bigg(e^{\rho t}\frac{1}{\tilde{S}(t,i)}\Bigg[\exp\left\{-\rho T_i^* + \int_t^{T_i^*} r_i(s)ds\right\}\tilde{S}(T_i^*,i)u_c(m_i(T_i^*)+\bar{a}) + e^{-\int_0^t r_i(s)ds}\int_t^{T_i^*}\exp\left\{-\rho s+\int_0^s r_i(u)du\right\}\tilde{S}(s,i)\sum_{j>i}\lambda_{ij}(s)\frac{\partial}{\partial W_i(s)}V(s,W_i(s),\bar{a},j)\,ds\Bigg]\Bigg).$$

■

Proof of Proposition 3 and Corollary 4. Our goal is to derive expressions for VSL when annuity markets are incomplete and the consumer is endowed with state-dependent life-cycle income. We first consider in part (i) the case with life-cycle earnings only. This part also provides expressions for the incomplete markets case at time $t > 0$, because after a flat annuity has been purchased it is equivalent to adding a constant to life-cycle earnings. Part (ii) considers the optimal purchase of the annuity and provides expressions for VSL at time $t = 0$.

(i) No annuity markets

Denote the consumer's earnings in state i at time t as $m_i(t)$. The consumer's maximization problem is again equation (1), but the law of motion for wealth now excludes annuity:

$$W(0) = W_0,$$
$$W(t) \geq 0,$$
$$\frac{\partial W(t)}{\partial t} = rW(t) + m_{Y_t}(t) - C(t)$$

Once again, we solve this stochastic finite-horizon optimization problem by reformulating it as a deterministic optimization problem. Specifically, we consider equation (3), subject to:

$$W_i(0) = W_0,$$
$$W_i(t) \geq 0,$$
$$\frac{\partial W_i(t)}{\partial t} = rW_i(t) + m_i(t) - C_i(t)$$

Let $\delta_{ij}(t)$ be a perturbation on the transition rate, and consider the impact on survival as described by equation (7). From equation (3), we obtain:

$$\begin{aligned}
\left.\frac{\partial V}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon}\left[\int_0^{T_i(\varepsilon)} e^{-\rho t}\tilde{S}^{\varepsilon}(i,t)\left(u\left(C_i^{\varepsilon}(t), q_i(t)\right) + \sum_{j>i}\left(\lambda_{ij}(t) - \varepsilon\delta_{ij}(t)\right)V(t, W_i^{\varepsilon}(t), j)\right)dt + \int_{T_i(\varepsilon)}^{T} e^{-\rho t}\tilde{S}^{\varepsilon}(i,t)\left(u\left(m_i(t), q_i(t)\right) + \sum_{j>i}\left(\lambda_{ij}(t) - \varepsilon\delta_{ij}(t)\right)V(t,0,j)\right)dt\right]\Bigg|_{\varepsilon=0} \\
&= \int_0^T e^{-\rho t}\tilde{S}(i,t)\left[\left(\int_0^t \sum_{j>i}\delta_{ij}(s)ds\right)\left(u(C_i(t), q_i(t)) + \sum_{j>i}\lambda_{ij}(t)V(t, W_i(t), j)\right) - \sum_{j>i}\delta_{ij}(t)V(t, W_i(t), j)\right]dt \\
&\quad + \underbrace{\int_0^{T_i} e^{-\rho t}\tilde{S}(i,t)\left(u_c(C_i(t), q_i(t))\left.\frac{\partial C_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0} + \sum_{j>i}\lambda_{ij}(t)\frac{\partial V(t, W_i(t), j)}{\partial W_i(t)}\left.\frac{\partial W_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0}\right)dt}_{=0}
\end{aligned}$$

where the second term in the last equality is equal to 0:

$$\begin{aligned}
&\int_0^{T_i} e^{-\rho t}\tilde{S}(i,t)\left(u_c(c_i(t), q_i(t))\left.\frac{\partial c_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0} + \sum_{j>i}\lambda_{ij}(t)\frac{\partial V(t, W_i(t), j)}{\partial W_i(t)}\left.\frac{\partial W_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0}\right)dt \\
&= \int_0^{T_i} p_t^{(i)}\left.\frac{\partial c_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0} + e^{-\rho t}\tilde{S}(i,t)\sum_{j>i}\lambda_{ij}(t)\frac{\partial V(t, W_i(t), j)}{\partial W_i(t)}\left[-\int_0^t e^{r(t-s)}\left.\frac{\partial c_i^{\varepsilon}(s)}{\partial \varepsilon}\right|_{\varepsilon=0} ds\right]dt \\
&= \int_0^{T_i} \theta^{(i)}e^{-rt}\left.\frac{\partial c_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0} dt + \int_0^{T_i}\int_t^{T_i} e^{(r-\rho)s}\tilde{S}(i,s)\sum_{j>i}\lambda_{ij}(s)\frac{\partial V(s, W_i(s), j)}{\partial W_i(s)}ds e^{-rt}\left.\frac{\partial c_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0} dt \\
&\quad - \int_0^{T_i}\int_t^{T_i} e^{-\rho s}\tilde{S}(i,s)\sum_{j>i}\lambda_{ij}(s)\frac{\partial V(s, W_i(s), j)}{\partial W_i(s)}ds e^{rs}e^{-rt}\left.\frac{\partial c_i^{\varepsilon}(t)}{\partial \varepsilon}\right|_{\varepsilon=0} dt \\
&= \theta^{(i)}\frac{\partial}{\partial \varepsilon}\int_0^{T_i} e^{-rt}c_i^{\varepsilon}(t)dt\Bigg|_{\varepsilon=0} \\
&= 0
\end{aligned}$$

The final equality follows because $W_i(T_i) = 0$ (by definition), which in turn implies $0 = W_0 + \int_0^{T_i} e^{-rt}m_i(t)dt - \int_0^{T_i} e^{-rt}c_i^{\varepsilon}(t)dt$, so that differentiation yields zero. Thus we obtain:

$$\left.\frac{\partial V}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^T e^{-\rho t}\tilde{S}(i,t)\left[\left(\int_0^t \sum_{j>i}\delta_{ij}(s)ds\right)\left(u(c_i(t), q_i(t)) + \sum_{j>i}\lambda_{ij}(t)V(t, W_i(t), j)\right) - \sum_{j>i}\delta_{ij}(t)V(t, W_i(t), j)\right]dt \tag{A.2}$$

Dividing by the marginal utility of wealth yields the value of life-extension. Choosing the Dirac delta function for $\delta_{i,n+1}(t)$ yields VSL:

$$VSL(i) = \frac{V(0, W(0), i)}{u_c(c_i(0), q_i(0))} \tag{A.3}$$

(ii) Incomplete annuity markets

Now, we introduce a one-time opportunity at time $t = 0$ to purchase a flat lifetime annuity at a level $\overline{a}_{Y_0} \geq 0$ with a price markup $\xi \geq 0$. Let $a(t,i) = \mathbb{E}\left[\int_t^T e^{-r(s-t)} \exp\left\{-\int_t^s \mu(u)\,du\right\} ds \,\middle|\, Y_t = i\right]$ be the expected value of a one-dollar annuity purchased at time t in state i. Note that for any given annuity, $\overline{a}_i$, the consumer's problem can be mapped to the no-annuity case in part (i) above by setting the constraints equal to:

$$W_i(0) = W_0 - (1+\xi)\overline{a}_i\; a(0,i),$$
$$\frac{\partial W_i(t)}{\partial t} = rW_i(t) + m_i(t) + \overline{a}_i - c_i(t)$$

Solving for the optimal fixed annuity then becomes a straightforward static optimization problem:

$$\overline{a}_i^* = \arg\max_{\overline{a}_i} V\left(0, W_i(0), \overline{a}_i, i\right)$$

The optimal annuity must satisfy the necessary first-order condition:

$$\frac{\partial V\left(0, W_i(0), \overline{a}_i, i\right)}{\partial \overline{a}_i} = \frac{\partial V\left(0, W_i(0), \overline{a}_i, i\right)}{\partial W(0)}(1+\xi)a(0,i) \tag{A.4}$$

Because the consumer may favor a non-flat optimal consumption profile, the optimal level of annuitization is likely to be partial even if the markup ξ is equal to zero. However, full annuitization is optimal when $\xi = 0$, $r = \rho$, and quality of life and income are constant.[1]

The value of an annuity depends on a consumer's expected future survival. Life-extension affects the value and cost of a given annuity, and may also affect the level of the optimal annuity. Thus, the effect of the mortality rate perturbation on the marginal utility of life-extension is:

$$\left.\frac{\partial V\left(0, W_i^\varepsilon(0), \overline{a}_i^\varepsilon, i\right)}{\partial \varepsilon}\right|_{\varepsilon=0} = (A.2) + \frac{\partial V}{\partial \overline{a}_i}\left.\frac{\partial \overline{a}_i^\varepsilon(0)}{\partial \varepsilon}\right|_{\varepsilon=0} + \frac{\partial V}{\partial W_i(0)}\left.\frac{\partial W_i^\varepsilon(0)}{\partial \varepsilon}\right|_{\varepsilon=0}$$

where the first term on the right-hand side is equal to equation (A.2) derived in part (i) above for the case with life-cycle earnings but no annuity. Note that:

$$\begin{aligned}
\left.\frac{\partial W_i^\varepsilon(0)}{\partial \varepsilon}\right|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon}\left(-(1+\xi)\overline{a}_i^\varepsilon \int_0^T \tilde{S}^\varepsilon(i,t)e^{-rt}\left[1 + \sum_{j>i}\left(\lambda_{ij}(t) - \varepsilon\delta_{ij}(t)\right)a(t,j)\right]dt\right) \\
&= -(1+\xi)\left.\frac{\partial \overline{a}_i^\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} a(0,i) - (1+\xi)\overline{a}_i \int_0^T e^{-rt}\tilde{S}(i,t)\left[\left(\int_0^t \sum_{j>i}\delta_{ij}(s)ds\right)\left(1 + \sum_{j>i}\lambda_{ij}(t)a(t,j)\right) - \sum_{j>i}\delta_{ij}(t)a(t,j)\right]dt
\end{aligned}$$

Combining this with the first-order condition (A.4) implies that:

$$\frac{\partial V}{\partial \overline{a}_i}\left.\frac{\partial \overline{a}_i^\varepsilon(0)}{\partial \varepsilon}\right|_{\varepsilon=0} + \frac{\partial V}{\partial W_i(0)}\left.\frac{\partial W_i^\varepsilon(0)}{\partial \varepsilon}\right|_{\varepsilon=0} = -\frac{\partial V}{\partial W_i(0)}(1+\xi)\overline{a}_i \int_0^T e^{-rt}\tilde{S}(i,t)\left[\left(\int_0^t \sum_{j>i}\delta_{ij}(s)\,ds\right)\left(1 + \sum_{j>i}\lambda_{ij}(t)a(t,j)\right) - \sum_{j>i}\delta_{ij}(t)a(t,j)\right]dt$$

[1] Even in the case of full annuitization, the first-order condition (A.4) holds with strict equality since the consumer is indifferent between an increase in the annuity level or a proportionate increase in baseline wealth.

Thus the marginal utility of life-extension is equal to:

$$\left.\frac{\partial V}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_0^T e^{-\rho t}\tilde{S}(i,t)\left[\left(\int_0^t \sum_{j>i}\delta_{ij}(s)\,ds\right)\left(u(c_i(t),q_i(t)) + \sum_{j>i}\lambda_{ij}(t)V(t,W_i(t),\overline{a}_i,j)\right) - \sum_{j>i}\delta_{ij}(t)V(t,W_i(t),\overline{a}_i,j)\right]dt$$
$$- \frac{\partial V}{\partial W_i(0)}(1+\xi)\overline{a}_i\int_0^T e^{-rt}\tilde{S}(i,t)\left[\left(\int_0^t \sum_{j>i}\delta_{ij}(s)\,ds\right)\left(1+\sum_{j>i}\lambda_{ij}(t)a(t,j)\right) - \sum_{j>i}\delta_{ij}(t)a(t,j)\right]dt$$

The marginal utility of wealth, $\partial V/\partial W_i(0)$, is equal to $u_c(c_i(0),q_i(0))$ when the solution is interior. Dividing by the marginal utility of wealth and rearranging yields the marginal value of life-extension:

$$\left.\frac{\partial V/\partial \varepsilon}{\partial V/\partial W}\right|_{\varepsilon=0}$$
$$= \int_0^T \tilde{S}(i,t)\left\{\left(\int_0^t \sum_{j>i}\delta_{ij}(s)ds\right)\left[\left(\frac{e^{-\rho t}u(c_i(t),q_i(t)) + \sum_{j>i}\lambda_{ij}(t)V(t,W_i(t),\overline{a}_i,j)}{u_c(c_i(0),q_i(0))}\right) - (1+\xi)\overline{a}_i\,e^{-rt}\left(1+\sum_{j>i}\lambda_{ij}(t)a(t,j)\right)\right]\right.$$
$$\left. - \sum_{j>i}\delta_{ij}(t)\left(\frac{V(t,W_i(t),\overline{a}_i,j)}{u_c(c_i(0),q_i(0))} - (1+\xi)\overline{a}_i e^{-rt}a(t,j)\right)\right\}dt$$

Choosing the Dirac delta function for $\delta_{i,n+1}(t)$ yields:

$$VSL(i) = \frac{V(0,W_i(0),\overline{a}_i,i)}{u_c(c_i(0),q_i(0))} - (1+\xi)\overline{a}_i\int_0^T \tilde{S}(i,t)e^{-rt}\left(1+\sum_{j>i}\lambda_{ij}(s)a(t,j)\right)dt$$
$$= \frac{V(0,W_i(0),\overline{a}_i,i)}{u_c(c_i(0),q_i(0))} - (1+\xi)\overline{a}_i a(0,i)$$

■